



Gauss Map and Local Approach of Isoparametric Surfaces in Lorentz and Euclidean Space

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Abstract

In this study, we determine the isoparametric surfaces and we give the Gauss map of these surfaces by semi symmetric matrix, in Lorentz space. Also we define any chord property and we show that the surfaces which have the chord property corresponds to isoparametric surfaces. Moreover, we consider the chord property locally and we give some examples in the Euclidean space.

1. Introduction and preliminaries

Isoparametric surfaces, surfaces with constant principal curvatures, are studied in [1]-[3] in terms of the chord property and helical points of the surface in the Euclidean space. In [4], the unit disk characterized by the following:

Lemma 1.1. *The only bounded, smooth and simply-connected plane region whose Szegő kernel coincides with the Cauchy kernel is the disc.*

Kerzman and Stein [4] used complex analysis technics related with the chord of the curve and they proved the Lemma above. Then, Boas [5] extended this idea to n - dimensional Euclidean space. Boas gave the following theorem, by the help of Bochner-Martinelli kernel:

Theorem 1.2. *Ball is the only bounded C^1 domain in \mathbb{R}^m such that given any two points of the boundary, the chord joining them meets the normals at the two endpoints with equal angles.*

Thus, in ([5], Proof of Theorem 2, pp. 277-278), the chord property idea of [4] extended to the hyperspheres. Moreover, Boas [2], extended his study [5], to all isoparametric surfaces in the Euclidean space. He gave such a local characterization theorem for hyperspheres and spherical cylinders and proved that these surfaces satisfy

$$\langle x - y, \vec{\nabla} f(x) \rangle = \langle y - x, \vec{\nabla} f(y) \rangle \quad (1.1)$$

where x, y are points on surface and $\vec{\nabla} f$ is the unit normal (gradient) vector field. Wegner [6], gave the short proof of ([2], Local characterization theorem, p.120). In [1], in the light of [2, 5], the equation (1.1) considered on a hypersurface such that a unit normal vector field G is naturally defined on the surface. Such G is called the Gauss map of surface. For any hypersphere, the chord joining any two points on it meets the sphere at the same angle at the two points, that is, the sphere satisfies

$$\langle y - x, G(x) + G(y) \rangle = 0 \quad (1.2)$$

In [1], the following question considered:

What are the hypersurfaces of Euclidean space that satisfy the (1.2)?

They used algebraic approaches and stated that Gauss map of surfaces which satisfy (1.2) is written as $G(x) = Ax + b$ where A is constant symmetric matrix, b is column vector. In [7], some special curves are defined and relations between these curves and isoparametric surfaces are given in Lorentz-Minkowski space. In this study, we are looking for answers of the followings:

What are the hypersurfaces of Lorentz space that satisfy the (1.2)?

and

What are the hypersurfaces of Euclidean space that satisfy the (1.2) locally?

In Lorentz space, vectors have different causal characters such as if $\langle u, u \rangle > 0$ or $u = 0$, $\langle u, u \rangle < 0$ and $\langle u, u \rangle = 0$ ($u \neq 0$) then u is called by spacelike, timelike and lightlike (or null) vector respectively. The number of timelike vectors of the orthonormal basis of the vector space is called the index of space and usually denoted by ν . Through the [8], we give the followings:

Definition 1.3. Let $X = (x_1, x_2, \dots, x_n)$ and $Y = (y_1, y_2, \dots, y_n)$ be a two vector distinct from zero in n -dimensional real vector space \mathbb{R}^n . Following inner product,

$$\langle X, Y \rangle = -x_1 y_1 + \sum_{i=2}^n x_i y_i$$

is called by Lorentzian inner product of X and Y , and $\langle \cdot, \cdot \rangle$ is called metric tensor of vector space. $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ is called Lorentz space and denoted by \mathbb{L}^n or \mathbb{R}_1^n . If $\langle u, v \rangle = 0$ implies that $u=0$ for all v where $u, v \in T_p \mathbb{R}^n$, then $\langle \cdot, \cdot \rangle$ is called canonical non-degenerated inner product with arbitrary index.

Norm of the vector $u \in \mathbb{R}_1^n$ is given by $\|u\| = \sqrt{|\langle u, u \rangle|}$. Let the index of n -dimensional non-degenerated inner product space of V be $1 \leq \nu \leq n$ and its orthonormal base be $\{e_1, e_2, \dots, e_n\}$. Then $\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_\nu = -1$ and $\varepsilon_{\nu+1} = \varepsilon_{\nu+2} = \dots = \varepsilon_n = 1$, where $\varepsilon_i = \langle e_i, e_i \rangle$. Therefore, the diagonal matrix $(\delta_{ij} \varepsilon_j)$ is called by the sign matrix of V such that

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, \quad 1 \leq i, j \leq n$$

is Kronecker delta.

Definition 1.4. [8] Let $n \geq 2$ and $0 \leq \nu \leq n$,

(1) The pseudosphere of radius $r > 0$ in \mathbb{R}_ν^{n+1} is the hyperquadric

$$\mathbb{S}_\nu^n(r) = \{P \in \mathbb{R}_\nu^{n+1} \mid \langle P, P \rangle = r^2\}$$

with dimension n and index ν .

(2) The pseudohyperbolic space of radius $r > 0$ in $\mathbb{R}_{\nu+1}^{n+1}$ is the hyperquadric

$$\mathbb{H}_\nu^n(r) = \{P \in \mathbb{R}_{\nu+1}^{n+1} \mid \langle P, P \rangle = -r^2\}$$

with dimension n and index ν .

Definition 1.5. Let M be a hypersurface in the Minkowski space and \vec{n} be a unit normal vector field of M . If $\langle \vec{n}, \vec{n} \rangle < 0$, ($\langle \vec{n}, \vec{n} \rangle > 0$) then M is said to be spacelike (timelike) surface.

Lemma 1.6. [8] Let S be a shape operator (Weingarten map) and ν be a tangent vector on M . $S(\nu) = -\nabla_\nu \vec{n}$ and for all $P \in M$, linear operator of S is self-adjoint on $T_p M$. Here ∇ is Levi-Civita connection on M in \mathbb{R}_1^n space.

2. Linear operators and isoparametric surfaces

An integral operator in complex space \mathbb{C}^n is given by

$$\zeta(f, x) = \int_a^b f(x) K(t, x) dt$$

such that $K(t, x)$ is continuous according to parameter t and it is called the kernel of the operator ζ . Suppose that \mathcal{B} is a bounded smooth domain in the complex plane. The Cauchy kernel represents holomorphic functions f in \mathcal{B} in terms of the boundary values on γ . Here γ is the boundary of the domain \mathcal{B} . Cauchy integral operator on any \mathcal{B} domain whose bounded by the curve γ in complex plane is given by

$$\zeta(z, w) = \frac{1}{2\pi i} \int_{z \in \gamma} \frac{f(z)}{z-w} \dot{\gamma}(z) d\sigma(z)$$

where $w \in \mathbb{C}$ is on γ . Here $dz = \dot{\gamma}(z) d\sigma$, $z = \gamma(s)$ unit speed curve and $d\sigma$ is Lebesgue measure (arc length). Hence, Cauchy kernel of the ζ operator is $\frac{1}{2\pi i} \frac{1}{z-w} \dot{\gamma}(z)$. Similarly the $\mathcal{S}(z, w)$ Szegő integral operator is given by

$$\zeta(z, w) = \frac{1}{2\pi i} \int_{z \in \gamma} f(z) \mathcal{S}(z, w) d\sigma(z)$$

where the kernel is considered the orthogonal projection of $\mathbb{S} : \mathcal{L}^2(\gamma, d\sigma) \rightarrow \mathcal{H}^2(\gamma)$. Here $\mathcal{H}^2(\gamma)$ is closed subspace of $\mathcal{L}^2(\gamma, d\sigma)$ of boundary values of holomorphic functions in \mathcal{B} . It is easy to see that $\mathbb{S}\zeta = \zeta$ holds identically and the curve γ satisfy

$$\langle \gamma(t) - \gamma(s), T(t) - T(s) \rangle = 0 \tag{2.1}$$

where T is the unit tangent normal vector field of the curve. It follows from the definitions above and (2.1) that γ is non-null hyperbolic curve in Lorentz plane.

Now we extend the chord idea to the surfaces in the high dimensions and we give some characterizations about these surfaces by the help of Gauss map itself, in terms of [1] and [7], in Lorentz space. Throughout this chapter, the metric tensor will be considered as a Lorentzian unless otherwise mentioned. Let us give the following definition first.

Definition 2.1. Let M be a non-null hypersurface and G is Gauss map of M . If

$$\langle Q - P, G(P) + G(Q) \rangle = 0$$

for all $P, Q \in M$ then, M is called by **G**-hypersurface.

Theorem 2.2. Let M be a **G**-hypersurface. Gauss map of this surface is given by

$$G(x) = Ax + b$$

where A is the semi-symmetric matrix and $b \in E_1^n$ column vector.

Proof. Let the hypersurface M fully lies in space and consider the points y_0, y_1, \dots, y_n on M such that $\{y_j - y_0 \mid 1 \leq j \leq n\}$ spans \mathbb{R}_1^n . Similar to [1], we find

$$A^T = \varepsilon \left(B_j A_j^{-1} \right) \varepsilon \tag{2.2}$$

where $\varepsilon = \text{diag}(-1, 1, \dots, 1)$ is the sign matrix, A_j and B_j are the $n \times n$ matrices that accepts the $y_j - y_0$ and $G(y_j) - G(y_0)$ as j -column respectively. Also $b = \sum_{k=1}^n b_k \alpha_k$ such that b_k is given by

$$\begin{pmatrix} \langle \alpha_1, \alpha_1 \rangle & \langle \alpha_1, \alpha_2 \rangle & \dots & \langle \alpha_1, \alpha_n \rangle \\ \langle \alpha_2, \alpha_1 \rangle & \ddots & & \langle \alpha_2, \alpha_n \rangle \\ \vdots & & \ddots & \vdots \\ \langle \alpha_n, \alpha_1 \rangle & \langle \alpha_n, \alpha_2 \rangle & \dots & \langle \alpha_n, \alpha_n \rangle \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

where $\alpha_j = y_j - y_0$ and $c_j = \langle G(y_0), y_0 \rangle - \langle G(y_j), y_j \rangle$. Hence we write

$$c_j = \langle \alpha_j, \alpha_1 \rangle b_1 + \langle \alpha_j, \alpha_2 \rangle b_2 + \dots + \langle \alpha_j, \alpha_n \rangle b_n. \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$G(x) = Ax + b.$$

We note that A is constant matrix (see [1]). Now we prove that A is semi-symmetric matrix. Let $X, Y \in T_x M$ be a tangent vector. Due to Lemma 1.6

$$\nabla_X G(X) = AX = -S(X) \quad (2.4)$$

and

$$\langle S(X), Y \rangle = \langle X, S(Y) \rangle. \quad (2.5)$$

It follows from (2.4) and (2.5) that

$$\begin{aligned} \langle -AX, Y \rangle = \langle X, -AY \rangle &\Leftrightarrow \langle AX, Y \rangle = \langle X, AY \rangle \\ &\Leftrightarrow \langle X, (\varepsilon A^T \varepsilon) Y \rangle = \langle X, AY \rangle \\ &\Leftrightarrow A = \varepsilon A^T \varepsilon \end{aligned}$$

which is intended. \square

Theorem 2.3. *Let us assume that hypersurface M have diagonalized shape operator. For non-null hypersurface M , the following statements are equivalent:*

i) M is the \mathbf{G} -hypersurface.

ii) M is an isoparametric surface.

iii) M is the open part of non-null hyperplane, pseudosphere, pseudohyperbolic space, pseudospherical cylinder or pseudohyperbolic cylinder.

Proof. Let M be a \mathbf{G} -hypersurface and $\{E_1, E_2, \dots, E_{n-1}\}$ orthonormal frame on surface such that $E_i, 1 \leq i \leq n-1$ are characteristic vectors corresponding to characteristic values μ_i of the shape operator. Hence, $S(E_i) = \mu_i E_i$ for all i . Due to Theorem 2.2, $G(x) = Ax + b$ where A is semi-symmetric matrix. It follows from (2.4) that

$$AE_j(x) = -S(E_j(x)) = -\mu_j(x)E_j(x)$$

and

$$(A + \mu_j I)E_j(x) = 0.$$

In order to the existence of non-zero characteristic vectors

$$\det(A + \mu_j I) = 0. \quad (2.6)$$

From equation (2.6), it is obvious that μ_i is constant. Therefore, M is an isoparametric surface.

Let M be an isoparametric surface. Let us define

$$f(x) = \langle Ax + b, Ax + b \rangle \quad (2.7)$$

where $f: \mathbb{R}^n \rightarrow \mathbb{R}$ and $M \subset f^{-1}(\pm 1)$. It follows from (2.7) that

$$f(x) = \langle x, A^2 x \rangle + 2 \langle x, Ab \rangle + \langle b, b \rangle.$$

By straightforward calculations we get $\overrightarrow{\nabla} f(x) = 2A(Ax + b)$. Since the gradient of f and Gauss map G is linear dependent

$$A(Ax + b) = \lambda(x)(Ax + b), \quad x \in M \quad (2.8)$$

for some real valued $\lambda(x)$ functions. It follows from (2.8) that $(A - \lambda(x)I)(Ax + b) = 0$ and $\det(A - \lambda(x)I) = 0$. Obviously $\lambda(x)$ is constant. Let us consider $V = \{Ax + b \mid x \in M\}$ as a characteristic space that corresponding to $\lambda(x) = \lambda$ and $Sp\{V\}$ is normal space at $x \in M$. Let us determine the surface M depends on the norm of V .

a) Let $\|V\| = 1$. In this case $G(x) = b$ is constant and M is the open part of non-null hyperplane.

b) If $\|V\| = n$ then $V = Im(A)$. For some $r > 0$ and $\lambda = \pm \frac{1}{r}$ we have $A|_V = \pm \frac{1}{r}I$ such that $G(x) = \pm \frac{1}{r}x + b$. Depends on causal character of $Sp\{V\}$, we get pseudosphere of S_1^{n-1} or pseudohyperbolic space of \mathbb{H}_1^{n-1} .

e) Let $\|V\| = p, 2 \leq p \leq n - 1$. Dimension of V^\perp orthogonal complement is $n - p$ and $V^\perp \subseteq T_x M$. In the neighborhood of $x_0 \in M$, we choose $\{E_1(x), E_2(x), \dots, E_{n-1}(x)\}$ orthonormal frame such that E_1, E_2, \dots, E_{n-p} are constant in V^\perp , we write

$$V = Sp \{E_{n-p+1}(x), E_{n-p+2}(x), \dots, E_{n-1}(x), G(x)\}$$

The tangent subspace spanned by $\{E_{n-p+1}(x), E_{n-p+2}(x), \dots, E_{n-1}(x)\}$ is integrable and integral submanifold of M_1 through the point x_0 is given as $M_1 = M \cap (x_0 + V)$. Hence $\mathbb{R}_1^{n-p} = V^\perp$ and $M = M_1 \times \mathbb{R}_1^{n-p}$ where M_1 is hypersurface in \mathbb{R}_1^p and Gauss map of M_1 satisfy the $G_1(x) = G(x)$. Besides, $A_1 = A|_V$ satisfy

$$G_1(x) = A_1 x + b. \tag{2.9}$$

It follows from (2.8) and (2.9) that $A_1 = \pm \frac{1}{r} I$. So M_1 is pseudosphere of $\mathbb{S}_1^{p-1}(r)$ or pseudohyperbolic space of $\mathbb{H}_1^{p-1}(r)$. Hence M is the open part of $\mathbb{S}_1^{p-1}(r) \times \mathbb{R}^{n-p}$ or $\mathbb{H}_1^{p-1}(r) \times \mathbb{R}^{n-p}$, respectively.

Let us consider isometric immersion $f_1 : M \rightarrow \mathbb{R}_1^n$ with respect to $[x_1, x_2, \dots, x_n]$ rectangular coordinate system. Let us give

$$f_1(x_1, x_2, \dots, x_n) = a_1 x_1 + a_2 x_2 + \dots + a_n x_n = c$$

where $a_i, 1 \leq i \leq n$ are constant coefficients, $x = (x_1, x_2, \dots, x_n) \in M$ and c is real number. The Gauss map of this immersion is given by

$$G(P) := \frac{\vec{\nabla} f_1}{\|\vec{\nabla} f_1\|} \Big|_P = \frac{1}{m} (-a_1, a_2, \dots, a_n) \Big|_P$$

where $P \in M$ and $m = \sqrt{|-a_1^2 + a_2^2 + \dots + a_n^2|}$. It can be easily seen that

$$\langle Q - P, G(P) + G(Q) \rangle = 0$$

where $P, Q \in f_1$. By Definition 1.4, pseudosphere with center x_0 and radius r is given by

$$\mathbb{S}_1^{n-1}(r) = \{x \in \mathbb{R}_1^n : \langle x - x_0, x - x_0 \rangle = r^2\}.$$

Without loss of generality, we can consider $x_0 = (0, 0, \dots, 0)$ and $x = (x_1, x_2, \dots, x_n)$. In this case,

$$f_2(x_1, x_2, \dots, x_n) = -x_1^2 + x_2^2 + \dots + x_n^2 = r^2$$

where $f_2 : \mathbb{S}_1^{n-1} \rightarrow \mathbb{R}_1^n$ is an isometric immersion. The Gauss map of this immersion is given by

$$G(P) := \frac{\vec{\nabla} f_2}{\|\vec{\nabla} f_2\|} \Big|_P = \frac{1}{r} (x_1, x_2, \dots, x_n) \Big|_P.$$

By straightforward calculations we get

$$\langle Q - P, G(P) + G(Q) \rangle = 0$$

where $P, Q \in f_2$. Therefore, $\mathbb{S}_1^{n-1}(r)$ is \mathbf{G} -hypersurface (similarly $\mathbb{H}_1^{n-1}(r)$ is \mathbf{G} -hypersurface). Moreover, we consider

$$-x_1^2 + x_2^2 + \dots + x_p^2 = r^2, x_{p+1} = u_{p+1}, x_{p+2} = u_{p+2}, \dots, x_n = u_n$$

where u_i are real variables such that $p + 1 \leq i \leq n$. By the help of natural Lorentz projection $\pi_L : \mathbb{R}_1^p \times \mathbb{R}_1^{n-p} \rightarrow \mathbb{R}_1^p \hookrightarrow \mathbb{R}_1^n$ onto \mathbb{S}_1^{p-1} we get

$$G(P) = \frac{1}{r} (x_1, x_2, \dots, x_p, 0, \dots, 0) \Big|_P.$$

Hence,

$$\langle Q - P, G(P) + G(Q) \rangle = 0$$

where $P, Q \in \mathbb{S}_1^{p-1}(r) \times \mathbb{R}^{n-p}$. Proof is similar for $\mathbb{H}_1^{p-1}(r) \times \mathbb{R}^{n-p}$. □

3. Local isoparametric surfaces in Euclidean space

Chord property of isoparametric surfaces are examined locally and globally in [2, 5]. Our point of view to localization is totally different from previous studies. Let us give the following definition and results.

Definition 3.1. Let M be a hypersurface in Euclidean space and $\langle \cdot, \cdot \rangle$ be the metric tensor of the space. If some points such $P, Q \in M$ satisfy

$$\langle Q - P, G(P) + G(Q) \rangle = 0$$

then, M is called as local isoparametric surface.

Theorem 3.2. The helicoid surface given by

$$\Phi(s, t) = (0, 0, bs) + t(\cos s, \sin s, 0), \quad b \neq 0$$

is local isoparametric surface if and only if

$$p_1 = q_1 \quad \text{or} \quad p_2 = -q_2$$

where $\varphi(p_1, p_2) = P$, $\varphi(q_1, q_2) = Q$, $\varphi : U \subseteq \mathbb{R}^2 \rightarrow \Phi$ and $(p_1, p_2), (q_1, q_2) \in U$.

Proof. Let us consider the surface $\varphi(U) = \Phi$ and $\varphi : U \rightarrow \Phi$ differentiable map where $U \subseteq \mathbb{R}^2$. Let $P, Q \in \Phi$ two points on surface such that $\varphi(p_1, p_2) = P$ and $\varphi(q_1, q_2) = Q$. By straightforward calculations we get

$$\Phi_s(s, t) = (-t \sin s, t \cos s, b)$$

and

$$\Phi_t(s, t) = (\cos s, \sin s, 0)$$

Unit normal vector field Z of helicoid is given by

$$Z \circ \varphi = \frac{\Phi_s(s, t) \times \Phi_t(s, t)}{\|\Phi_s(s, t) \times \Phi_t(s, t)\|} = \left(-\frac{b}{\sqrt{b^2 + t^2}} \sin s, \frac{b}{\sqrt{b^2 + t^2}} \cos s, -\frac{1}{\sqrt{b^2 + t^2}} t \right).$$

Therefore,

$$G(P) + G(Q) = (\alpha_1, \alpha_2, \alpha_3) \tag{3.1}$$

where

$$\begin{aligned} \alpha_1 &= -\frac{b}{\sqrt{b^2 + p_2^2}} \sin p_1 - \frac{b}{\sqrt{b^2 + q_2^2}} \sin q_1 \\ \alpha_2 &= \frac{b}{\sqrt{b^2 + p_2^2}} \cos p_1 + \frac{b}{\sqrt{b^2 + q_2^2}} \cos q_1 \\ \alpha_3 &= -\frac{b}{\sqrt{b^2 + p_2^2}} p_2 - \frac{1}{\sqrt{b^2 + q_2^2}} q_2. \end{aligned}$$

Besides,

$$Q - P = (q_2 \cos q_1 - p_2 \cos p_1, q_2 \sin q_1 - p_2 \sin p_1, bq_1 - bp_1). \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\langle Q - P, G(P) + G(Q) \rangle = b \left(\frac{p_2}{\sqrt{b^2 + p_2^2}} + \frac{q_2}{\sqrt{b^2 + q_2^2}} \right) (\sin(q_1 - p_1) + (p_1 - q_1)).$$

Obviously $\langle Q - P, G(P) + G(Q) \rangle = 0$ if and only if $p_1 = q_1$ or $p_2 = -q_2$. By Definition 3.1, Φ is local isoparametric surface. \square

Theorem 3.3. *The hyperbolic paraboloid surface given by*

$$\Phi(u, v) = (u, v, \frac{v^2}{b^2} - \frac{u^2}{a^2}), \quad a, b \in \mathbb{R} \setminus \{0\}$$

is local isoparametric surface if and only if

$$\frac{a}{b} = \left| \frac{p_1 - q_1}{p_2 - q_2} \right|$$

where $\varphi(p_1, p_2) = P, \varphi(q_1, q_2) = Q, \varphi : U \subseteq \mathbb{R}^2 \rightarrow \Phi$ and $(p_1, p_2), (q_1, q_2) \in U$.

Proof. By straightforward calculations we get

$$Z \circ \varphi = \left(-\frac{2u}{a^2 \sqrt{1 + \frac{4u^2}{a^4} + \frac{4v^2}{b^4}}}, \frac{2v}{b^2 \sqrt{1 + \frac{4u^2}{a^4} + \frac{4v^2}{b^4}}}, -\frac{1}{\sqrt{1 + \frac{4u^2}{a^4} + \frac{4v^2}{b^4}}} \right)$$

and

$$\langle Q - P, G(P) + G(Q) \rangle = \frac{(-b^2(p_1 - q_1)^2 + a^2(p_2 - q_2)^2) \left(\sqrt{1 + \frac{4p_1^2}{a^4} + \frac{4p_2^2}{b^4}} - \sqrt{1 + \frac{4q_1^2}{a^4} + \frac{4q_2^2}{b^4}} \right)}{a^2 b^2 \sqrt{1 + \frac{4p_1^2}{a^4} + \frac{4p_2^2}{b^4}} \sqrt{1 + \frac{4q_1^2}{a^4} + \frac{4q_2^2}{b^4}}}$$

Hence $\langle Q - P, G(P) + G(Q) \rangle = 0$ if and only if

$$\frac{a}{b} = \left| \frac{p_1 - q_1}{p_2 - q_2} \right|$$

which is intended. □

Let us give the following surface, in the light of [9].

Theorem 3.4. *The Viviani ruled surface given by*

$$\Phi(u, v) = \left(\frac{5}{2} + \frac{5}{2} \cos u, \frac{5}{2} \sin u, 5 \sin \frac{u}{2} \right) + 4v(1 + \cos u, \sin u, 2 \sin \frac{u}{2})$$

is local isoparametric surface if and only if

$$p_1 = q_1 + 4k\pi, \quad k \in \mathbb{Z}$$

where $\varphi(p_1, p_2) = P, \varphi(q_1, q_2) = Q, \varphi : U \subseteq \mathbb{R}^2 \rightarrow \Phi$ and $(p_1, p_2), (q_1, q_2) \in U$.

Proof. Similar to previous operations we get

$$Z \circ \varphi = \left(\frac{\sqrt{2} \sin^3 \frac{u}{2}}{\sqrt{3 + \cos u}}, \frac{-5 \cos \frac{u}{2} + \cos \frac{3u}{2}}{2\sqrt{2}\sqrt{3 + \cos u}}, \frac{\sqrt{2} \cos^2 \frac{u}{2}}{\sqrt{3 + \cos u}} \right)$$

and

$$\langle Q - P, G(P) + G(Q) \rangle = \alpha \sin^2 \left(\frac{p_1 - q_1}{4} \right)$$

where α is non-zero constant. Hence $\langle Q - P, G(P) + G(Q) \rangle = 0$ if and only if

$$p_1 = q_1 + 4k\pi, \quad k \in \mathbb{Z}$$

and this completes the proof. □

4. Conclusion

In this study, we showed that the Gauss map of isoparametric surfaces is written by $G(x) = Ax + b$ where A is semi-symmetric matrix and b is column vector, in Lorentz space. Moreover, in the Euclidean space; we gave the definition of local isoparametric surface, and we examined the some of them such as helicoid, hyperbolic paraboloid and Viviani ruled surface, by different point of view from the previous studies.

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