Fundamental Journal of Mathematics and Applications, 3 (1) (2020) 61-69
Research Article

Fundamental Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/fujma
ISSN: 2645-8845
doi: 10.33401/fujma.691602

Pseudoblocks of Finite Dimensional Algebras

Afaf S. Alharthi[1*] and Ahmed A. Khammash[1]

[1] *Department of Mathematical Sciences, Umm Al-Qura University, Makkah, Saudi Arabia*
[*] *Corresponding author*

Article Info

Keywords: *Endomorphism algebra, Extensions, Hecke algebras, Modules, Pseudoblocks, Representations, Tensor algebras*
2010 AMS: *20C30*
Received: *21 February 2020*
Accepted: *27 May 2020*
Available online: *10 June 2020*

Abstract

The notion of pseudoblocks is borrowed from [1] and introduced to finite dimensional algebras. We determine the pseudoblocks for several known algebras such as the triangular algebra and the cyclic group algebra. Also, we determine the pseudoblocks for the group algebra of the special linear group $SL(2,p)$ in the natural characteristic being the only finite group of Lie type of finite representation type.

1. Introduction

In [1], the concept of pseudoblocks of the endomorphism algebra of a module over an algebra was introduced and shown to have a control on the (Brauer) block distribution of the simple modules for the endomorphism algebra in the light of the Brauer-Fitting correspondence. In this paper, we borrow the concept of pseudoblock from [1] to introduce it to finite dimensional (not only endomorphism) algebras. We investigate the pseudoblocks for several known algebras such as the triangular algebra and the cyclic group algebra. Towards the end, we investigate the pseudoblock distribution for the group algebra of the special linear group $SL(2,p)$ in the natural characteristic being the only finite group of Lie type of finite representation type.

2. The pseudoblocks

The Brauer-Fitting correspondence relates the isomorphism classes of indecomposable direct summands of a module to the projective indecomposable modules for its endomorphism algebra. This correspondence is shown in [1] to be incompatible with the (Brauer) block distribution of modules in both sides. Instead, the concept of the pseudoblock of an endomorphism algebra of a module over an algebra was introduced to ensure such compatibility. Here, we borrow this notion and introduce it for any finite dimensional algebra. Let A be a finite dimensional algebra over an algebraically closed field F, $modA$ denotes the category of finitely generated A-modules, and we write $IndA$ for the class of indecomposable A-modules. We also write $(X,Y)_A$ for the A-homomorphism space $Hom_A(X,Y)$ between two modules $X,Y \in modA$. The **pseudoblock linkage relation** $\underset{PSA}{\approx}$ is an equivalence relation defined on $IndA$ in terms of the homomorphism space.

Definition 2.1. *If $X,Y \in IndA$, then $X \underset{PSA}{\approx} Y$ iff there is a sequence of modules $X = X_1, X_2, ..., X_t = Y$ in $IndA$ such that for all $i \in \{1,2,...,t\}$ either $(X_i, X_{i+1})_A \neq 0$ or $(X_{i+1}, X_i)_A \neq 0$.*

Email addresses and ORCID numbers: **s43880619@st.uqu.edu.sa, afaf.s@tu.edu.sa, afaf.saad.1438@hotmail.com, 0000-0001-5313-5919 (A. S. Alharthi), aakhammash@uqu.edu.sa, prof.khammash@gmail.com, 0000-0001-9404-1732 (A. A. Khammash).**

Clearly, $\underset{PSA}{\approx}$ is an equivalence relation on IndA. We call the equivalence classes $IndA/\underset{PSA}{\approx}$ are called ***pseudoblocks*** *of the algebra A.*

3. Connection with the Brauer blocks

The following shows that the pseudoblock linkage principle $\underset{PSA}{\approx}$ is stronger than the Brauer linkage principle $\underset{A}{\approx}$ relating indecomposable modules which belong to the same block.

Lemma 3.1. *If $X,Y \in IndA$ and $X \underset{PSA}{\approx} Y$, then $X \underset{A}{\approx} Y$.*

Proof. If $X \underset{PSA}{\approx} Y$, then there is a sequence of modules $X = X_1, X_2, \ldots, X_t = Y$ in $IndA$ such that for all $i \in \{1,2,\ldots,t\}$ either $(X_i, X_{i+1})_A \neq 0$ or $(X_{i+1}, X_i)_A \neq 0$. But this implies (see [2], p.93) that for all $i \in \{1,2,\ldots,t\}$ either $X_i \underset{A}{\approx} X_{i+1}$ or $X_{i+1} \underset{A}{\approx} X_i$, and so $X \underset{A}{\approx} Y$. □

Remark 3.2. *The converse of lemma 3.1 does not hold. If we take $A = FSL(2,4)$ and $CharF = 2$, then A has four simple modules namely $1, 2_1, 2_2, 4$ (the latter being the Steinberg module) distributed into two Brauer blocks $\underbrace{1, 2_1, 2_2}_{B_1}, \underbrace{4}_{B_2}$. The two indecomposable modules 1, $\begin{matrix}2_1\\1\\2_2\end{matrix} \in IndA$ belong to the same (Brauer) block, but they lie in a two different pseudoblocks of A. To see this,*

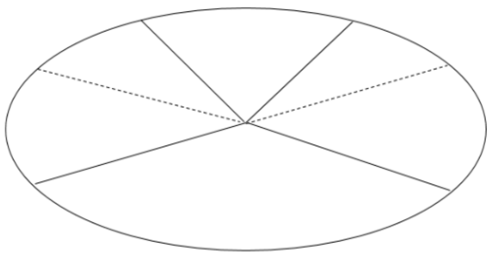


Figure 3.1: Some blocks in *IndA* split into union of pseudoblocks

It follows that, in principle, some (Brauer) blocks of A split into a union of pseudoblocks, and so we have $|IndA/\underset{A}{\approx}| \leqslant |IndA/\underset{PSA}{\approx}|$.

Motivation 3.3. *If we take $Y \in modA$ (not necessary indecomposable) and write $Inds(Y)$ for the isomorphism class of indecomposable A-summands of Y, then applying the linkage relation $\underset{PSA}{\approx}$ on $Inds(Y)$, it was shown in [1] that the (Brauer) block distribution of the simple modules of the endomorphism algebra $E(Y) = End_A(Y)$ is controlled by the pseudoblocks distribution of $Inds(Y)$; that is if $Y_i, Y_j \in Inds(Y)$ and $S_i, S_j \in Irr(E(Y))$ are the corresponding simple $E(Y)$-modules under the Brauer-Fitting correspondence, then $S_i \underset{E(Y)}{\approx} S_j \Leftrightarrow Y_i \underset{PSA}{\approx} Y_j$.*

A Useful Criterion 3.4. *The pseudoblock equivalence relation $\underset{PSA}{\approx}$ is defined in terms of the homomorphism space $(X,Y)_A$. If $X,Y \in IndA$, then $(X,Y)_A \neq 0$ if and only if $\exists K \leq_A X : X/K \cong$ submodule of Y. For, if $0 \neq f \in (X,Y)_A$, then $K = kerf \lneqq X$ and $X/K \cong Imf \leqslant_A Y$. Conversely, if $\exists K \leqslant_A X : X/K \cong T \leqslant_A Y$, then composing the map $X/K \cong T \longrightarrow Y$ with the natural map $X \to X/K$ we get a nonzero map $\theta : X \to Y$. Therefore, we have the figure 3.2*

Lemma 3.5. *$(X,Y)_A \neq 0$ if and only if $\exists K \leqslant_A X$: $X/K \cong$ a submodule of Y.*

4. Connection with tensor algebras

Suppose that A_1, A_2 are two finite dimensional F-algebras. If $X_i \in Ind(A_i); i = 1,2$, then it is known (by considering endomorphism algebras) that $X_1 \otimes X_2 \in Ind(A_1 \otimes A_2)$. The following theorem shows that the concept of pseudo-blocks is compatible with tensor operation of modules.

Theorem 4.1. *[3]. If $X_i, X_i' \in Ind(A_i); i = 1,2$, then $X_1 \otimes X_2 \underset{PS(A_1 \otimes A_2)}{\approx} X_1' \otimes X_2'$ if and only if $X_1 \underset{PSA_1}{\approx} X_1' \quad \wedge \quad X_2 \underset{PSA_2}{\approx} X_2'$.*

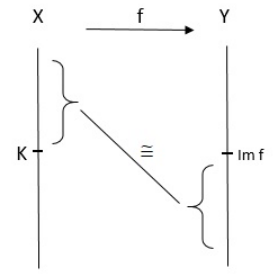


Figure 3.2

Proof. Since $X_1 \underset{PSA_1}{\approx} X_1'$, there is a sequence $X_1 = U_1, U_2, \ldots, U_t = X_1'$ in $IndA_1$ such that for all $j \in \{1,2,\ldots,t\}$ either $(U_j,U_{j+1})_{A_1} \neq 0$ or $(U_{j+1},U_j)_{A_1} \neq 0$. Similarly, since $X_2 \underset{PSA_2}{\approx} X_2'$, there is a sequence $X_2 = V_1, V_2, \ldots, V_t = X_2'$ in $IndA_2$ such that for all $j \in \{1,2,\ldots,t\}$ either $(V_j,V_{j+1})_{A_2} \neq 0$ or $(V_{j+1},V_j)_{A_2} \neq 0$ if and only if we have a sequence (with refining sequences if necessary) $X_1 \otimes X_2 = U_1 \otimes V_1, U_2 \otimes V_2, \ldots, U_t \otimes V_t = X_1' \otimes X_2'$ such that for all $j \in \{1,2,\ldots,t\}$ either

$$(U_j \otimes V_j, U_{j+1} \otimes V_{j+1})_{A_1 \otimes A_2} \neq 0 \quad \text{or} \quad (U_{j+1} \otimes V_{j+1}, U_j \otimes V_j)_{A_1 \otimes A_2} \neq 0$$

(by taking the tensor homomorphisms). Therefore, $X_1 \otimes X_2 \underset{PS(A_1 \otimes A_2)}{\approx} X_1' \otimes X_2'$.

□

5. The pseudoblocks of certain finite dimensional algebras

Here, we determine the pseudoblocks for some finite dimensional algebras. It turns out that the two concepts; blocks and pseudo-blocks, coincide for all.

5.1. Semisimple algebras

It is clear that the two notions; blocks and pseudoblocks, coincide for any finite dimensional semisimple algebra A; that is $IndA/\underset{PSA}{\approx} = IndA/\underset{A}{\approx}$. □

5.2. The symmetric group algebra FS_3

Let $A = FS_3$.

1. If $CharF \nmid |S_3|$, then $A = FS_3$ is semisimple, and so $IndA/\underset{PSA}{\approx} = IndA/\underset{A}{\approx}$ as shown above.
2. If $CharF = 2$, then A has two simple module 1, 2 and $IndA$ (consists of three indecomposable modules) has the following block distribution: $\underbrace{1, \begin{matrix}1\\1\end{matrix}}_{B_1}, \underbrace{2}_{B_2}$ which clearly coincides with the pseudoblock distribution.
3. If $CharF = 3$, then A has two simple modules both of dimension 1; S_0 (the trivial module) and S_1 (the sign module), and $IndA$ consists of six indecomposable modules all lie in one Brauer block and are connected by the following sequence of A-maps

$$S_1 \to \begin{matrix}S_0\\S_1\end{matrix} \to \begin{matrix}S_1\\S_0\\S_1\end{matrix} \to \begin{matrix}S_1\\S_0\end{matrix} \to \begin{matrix}S_0\\S_1\\S_0\end{matrix} \to \begin{matrix}S_0\\S_1\end{matrix} \to S_0.$$

Hence, $A = FS_3$ has a single pseudoblock in this case. Therefore, we have the following

Theorem 5.1. *For $A = FS_3$ and in all characteristic of F, we have $IndA/\underset{PSA}{\approx} = IndA/\underset{A}{\approx}$.* □

5.3. The triangular algebra

Now take

$$A=\{(a_{ij})\in M_n(F)|a_{ij}=0;\forall i>j\}=\left\{a=\begin{pmatrix}a_{11}&a_{12}&\dots&a_{1n}\\ &a_{22}&\dots&a_{2n}\\ &&\ddots&\vdots\\ &&&a_{nn}\end{pmatrix};a_{ij}\in F\right\};$$

the algebra of $n\times n$ upper triangular matrices (which is isomorphic to the algebra of lower triangular matrices). Then, A is isomorphic to the path algebra of an equi-oriented quiver of type A_n. By Gabriel's theorem (see [4, Chapter11]), this quiver has $n(n+1)/2$ indecomposable modules corresponding to the positive roots of Lie algebra of type A_n. In fact, A acts on the space of column vectors $U=F^n$ by matrix multiplication and

$$N=\left\{\begin{pmatrix}0&a_{12}&\dots&\dots&a_{1n}\\ &0&a_{23}&\dots&a_{2n}\\ &&\ddots&\ddots&\vdots\\ &&&0&a_{n-1n}\\ &&&&0\end{pmatrix}\right\}=J(A);$$

the Jacobson radical of A, and consequently A has n simple (1-dimensional) representations $\psi_v:A\longrightarrow F\quad(a\longmapsto a_{vv});v=1,2,\dots,n$ (ψ_v is an algebra map $\psi_v=\psi_\mu\Leftrightarrow v=\mu$). We also have

$NU=\left\{\begin{pmatrix}v_1\\v_2\\\vdots\\v_{n-1}\\0\end{pmatrix}:v_i\in F\right\}$, and $N^iU=\left\{\begin{pmatrix}v_1\\\vdots\\v_{n-i}\\0\\0\end{pmatrix}:v_i\in F\right\}$, so $U\supset NU\supset N^2U\supset\dots\supset N^{n-1}U\supset 0$ is a composition series with $dimN^{i-1}U/N^iU=1;\forall i=1,2,\dots,n$ and $N^{i-1}U/N^iU\cong\psi_{n-i+1}$. Therefore, as A-module, $U=F^n$ has the following (unique) composition series

$$U\supset NU\supset N^2U\supset\dots\supset N^{n-1}U\supset 0$$

$$\psi_n\quad\psi_{n-1}\quad\psi_{n-2}\dots\psi_2\quad\psi_1.$$

It follows that the quotient module $U_{i,\alpha}=N^{n-i}U/N^{n-i+\alpha}U$ is a uniserial (hence indecomposable) with the following (unique) composition series

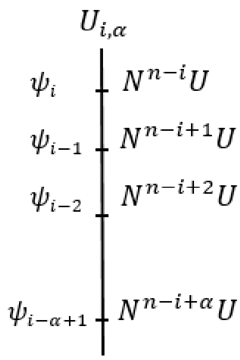


Figure 5.1

and hence $U_{i,\alpha}=N^{n-i}U/N^{n-i+\alpha}U$; $(i=1,2,\dots,n\ and\ \alpha=1,2,\dots,i)$ give a complete set of indecomposable A-modules. Not that $U_{i,\alpha}\cong U_{j,\beta}\Leftrightarrow i=j\wedge\alpha=\beta$ and $U_{i,1}=\psi_i$. The modules $U_{1,1},U_{2,2},\dots,U_{n,n}$ give a complete set of projective indecomposable A-modules. In fact, it is clear that $U_{v,v}=L_v=\left\{\begin{pmatrix}0&0&a_{1v}&\dots&0\\0&0&a_{2v}&\dots&0\\\vdots&\vdots&\vdots&\vdots&\vdots\\\vdots&\vdots&a_{vv}&\vdots&\vdots\\0&0&0&0&0\end{pmatrix}:a_{iv}\in F,i\le v;v=1,2,\dots,n\right\}\lhd A$. Note that the composition factors of $U_{v,v}=N^{n-v}U/N^nU=N^{n-1}U$ are as follows:

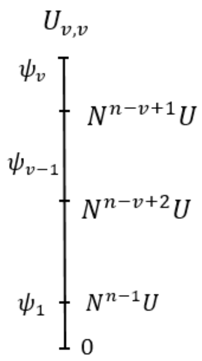


Figure 5.2

The triangular algebra A is not semisimple ($J(A) = N \neq 0$), hence it has a nontrivial block theory. In fact, $A = \sum^{\oplus}_{1\leq v\leq n} U_{v,v}$ (projective indecomposable A-modules PIM decomposition) is known to be connected; i.e. it has a single non-zero central idempotent, namely I_n, and so it has a single block. On the other hand, from the structure of the objects $U_{i,\alpha} = N^{n-i}U/N^{n-i+\alpha}U$ ($i = 1,2,\ldots,n$ *and* $\alpha = 1,2,\ldots,i$) of $Ind(A)$, the objects of the class $Ind(A)$ can be connected by a series of A-maps as follows:

$$\begin{array}{ccccccccccc}
 & & & & & & & & & & U_{n,1} \\
 & & & & & & & & & & \uparrow \\
 & & & & & & & & & & \vdots \\
 & & & & & & & & & & \uparrow \\
 & & & & & & U_{4,1} & \to & \cdots & \to & U_{n,n-3} \\
 & & & & & & \uparrow & & & & \uparrow \\
 & & & & U_{3,1} & \to & U_{4,2} & \to & \cdots & \to & U_{n,n-2} \\
 & & & & \uparrow & & \uparrow & & & & \uparrow \\
 & & U_{2,1} & \to & U_{3,2} & \to & U_{4,3} & \to & \cdots & \to & U_{n,n-1} \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \uparrow \\
U_{1,1} & \to & U_{2,2} & \to & U_{3,3} & \to & U_{4,4} & \to & \cdots & \to & U_{n,n}
\end{array}$$

$$\underbrace{U_{1,1} \to U_{2,2}}_{n=2}, \quad \underbrace{U_{1,1} \to U_{2,2} \to U_{3,3}}_{n=3}, \quad \underbrace{U_{1,1} \to U_{2,2} \to U_{3,3} \to U_{4,4}}_{n=4}$$

Therefore, A has a single pseudo-block, and so we have:

Theorem 5.2. *For the triangular algebra A over a field F, we have* $IndA/\underset{PSA}{\approx} = IndA/\underset{A}{\approx}$. □

5.4. The group algebra of cylic groups

We now consider the group algebra of cyclic group $A = FC_n; n = p^a e; p \nmid e$ over a field of characteristic p. It is known (see [2], p.34) that $A = FC_n$ has e simple (all are 1-dimensional) modules $\{S_\lambda | \lambda$ is an e-th root of 1$\}$, where $S_\lambda = F$ on which C_n acts by multiplication with λ. It is also known that $A = FC_n$ has a total of $n = p^a e$ indecomposable modules. For each integer $1 \leq m \leq p^a$, there is a uniserial module $L_{\lambda,m}$ of dimension m with all composition factors are isomorphic to S_λ (note that $L_{\lambda,1} = S_\lambda$). The set $\{L_{\lambda,m} | \lambda, m\}$ gives a complete set of $n = p^a e$ indecomposable FC_n-modules. Clearly, **PIM**$= \{L_{\lambda,p^a} | \lambda\}$ ($L_{\lambda,p^a} = P(S_\lambda)$ is the projective cover of S_λ), and $FC_n = \sum^{\oplus}_{\lambda} L_{\lambda,p^a}$. The group algebra FC_n has e blocks $\{B_\lambda | \lambda\}$, where $B_\lambda = \{L_{\lambda,m} | 1 \leq m \leq p^a\}$. It is clear from the structure of $L_{\lambda,m}$ that FC_n has e pseudo-blocks.

Theorem 5.3. *For the group algebra* FC_n *over a field F,* $IndFC_n/\underset{PSFC_n}{\approx} = IndFC_n/\underset{FC_n}{\approx}$. □

5.5. p-group algebra in characteristic p

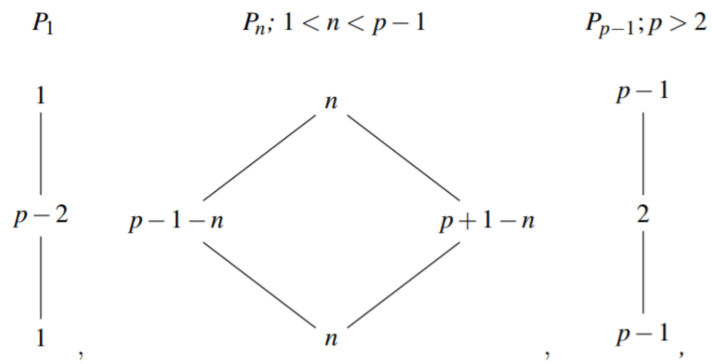
The group algebra FG of a finite p-group over a field F of characteristic p is known to be indecomposable and has a single simple module, namely the trivial module $1 = F_G$, and hence has a single block. All indecomposable FG-modules are uniserial with all of its composition factors are isomorphic to F_G. Hence, $IndFG$ forms a single pseudo-block of FG.

Theorem 5.4. *For the group algebra FG of a finite p-group over a field F of characteristic p,* $IndFG/\underset{PSFG}{\approx} = IndFG/\underset{FG}{\approx}$. □

6. The special linear group $SL(2,p)$

We now consider the group algebra $A = FSL(2,p)$ in characteristic odd prime number p. It is known that $SL(2,p)$ is the only finite group of Lie type which is of finite representation type in the natural characteristic (see [5, Chapter1]). It is known that $SL(2,p)$ has p (p-regular) conjugacy classes and (hence) p isomorphism classes of simple $FSL(2,p)$-modules of dimensions $1,2,3,\dots,p$ distributed in three blocks B_1,B_2,B_3 (see [6], p.469). We refer to each simple module by its dimension; hence 1 is the natural representation of $SL(2,p)$ and p is the Steinberg representation. There are $p^2 - p + 1$ indecomposable $FSL(2,p)$-modules of which $2p-1$ of them are either simple or projective (The Steinberg representation is both simple and projective). The number of remaining indecomposable (non-simple non-projective) $FSL(2,p)$-modules is $(p-1)(p-2)$. Denote by $P_i; 1 \le i \le p$, the projective cover of the simple $FSL(2,p)$-module i. The following theorem describes the structure of the projective indecomposable modules.

Theorem 6.1. *[2]. The projective indecomposable $FSL(2,p)$-modules have the following structures:*



, , , *and P_p.*

Figure 6.1

The structures of the other indecomposable (non-simple, non-projective) $FSL(2,p)$-modules are explained in the following theorem

Theorem 6.2. *[7]. Every (non-simple,non-projective) indecomposable $FSL(2,p)$-module M has two socle layers. The socle of M consists of the modules $i, i+2,\dots,j (i \leqslant j)$, and the top consists of the modules $p-j+\varepsilon, p-j+\varepsilon+2,\dots,p-i+\delta$, where $\varepsilon,\delta = \pm 1$.*

The following theorem shows that, the compatiblity between the pseudoblock of $FSL(2,p)$ and block theory.

Theorem 6.3. *For the group algebra $A = FG; G = SL(2,p)$ over a field F of characteristic prime number p,*

$$IndA/\underset{PSA}{\approx} \quad = \quad IndA/\underset{A}{\approx}.$$

Proof. **<u>First:</u>** The block B_3 (which contains the Steinberg module $p \cong P_p$) is clearly pseudoblock.

<u>Second:</u> Since B_1 contains all odd-dimensional simple A-modules except p, let P_m, P_i be projective indecomposable A-modules, let m,i be simple A-modules; for all $m,i \in \{1,3,\dots,p-2\}$, and let $M_{i'}$ be non-simple, non-projective, indecomposable A-modules; $i' = \{1,2,\dots,r\}$; in which P_m, P_i, m, i and $M_{i'}$ in B_1 for all m,i,i'. Let $P_i = i/p-1-i, p+1-i/i$, $P_m = m/p-1-m, p+1-m/m$, $M_1 = i/p-1-i, p+1-i$, $M_2 = p+1-i/i$, $M_3 = p+1-m, p-1-m/m$, $M_4 = p+1-m, p-1-m/i,m$, $M_5 = m/p-1-m, p+1-m$, $M_6 = p+1-i, p-1-i/i$.

Then, we have six cases as follows:

1. Let i,m be any two simple A-modules. Hence,
$$i \to M_2 \to m.$$
Then, all odd-dimensional simple A-modules are connected either ways by a sequence of A-module homomorphisms.
2. Let i,m be simple A-modules, and let P_i, P_m be projective indecomposable A-modules. Hence,
$$P_i \to M_1 \to i, \quad m \to M_3 \to P_m.$$

Then, all odd-dimensional simple A-modules and all projective indecomposable A-modules are connected either ways by a sequence of A-module homomorphisms.

3. Let $M_{i^{\backprime}};i^{\backprime}=\{1,2,3,5\}$ be any non-simple, non-projective, indecomposable A-modules, and let i,m be any two simple A-modules. Hence,
$$M_1 \to i, \quad M_2 \to m, \quad M_3 \to P_m \to M_5 \to m.$$
Then, all odd-dimensional simple A-modules and all non-simple, non-projective, indecomposable A-modules $M_{i^{\backprime}};i^{\backprime}=\{1,2,\ldots,r\}$ are connected either ways by a sequence of A-module homomorphisms.
4. Let P_i,P_m be any two projective indecomposable A-modules. Hence,
$$P_i \to M_1 \to i \to M_2 \to p+1-i \to M_3 \to P_m.$$
Then, all projective indecomposable A-modules $P_m, \forall m=\{1,3,\ldots,p-2\}$ are connected either ways by a sequence of A-module homomorphisms.
5. Let P_i,P_m be any two projective indecomposable A-modules, and let M_1,M_3,M_5,M_6 be non-simple, non-projective, indecomposable A-modules. Hence,
$$P_i \to M_1, \quad P_m \to M_5.$$
Also,
$$M_6 \to P_i, \quad M_3 \to P_m.$$
Then, all projective indecomposable A-modules $P_m, \forall m=\{1,3,\ldots,p-2\}$ and all non-simple, non-projective, indecomposable A-modules $M_{i^{\backprime}};i^{\backprime}=\{1,2,\ldots,r\}$ are connected either ways by a sequence of A-module homomorphisms.
6. Let M_1,M_2,M_3,M_4,M_5,M_6 be any non-simple, non-projective, indecomposable A-modules. Hence,
$$M_6 \to P_i \to M_1,$$
$$M_1 \to i \to M_2,$$
$$M_3 \to P_m \to M_5,$$
and
$$M_4 \to M_3.$$
Then, all non-simple, non-projective, indecomposable A-modules are connected either ways by a sequence of A-module homomorphisms.

The previous six cases are enough without loss of generality. So, all indecomposable A-modules in B_1 are connected either ways by a sequence of A-module homomorphisms as follows:
$$P_i \to M_{i^{\backprime}} \to i \to \ldots \leftarrow M^{\backprime}_{i^{\backprime}} \leftarrow m \leftarrow M^{\backprime\backprime}_{i^{\backprime}} \leftarrow P_m;$$
for all $i,m \in \{1,3,5,\ldots,p-2\}$ and $i^{\backprime}=\{1,2,\ldots,r\}$.

Thus, the block B_1 does not split into union of pseudoblocks. So, B_1 is one pseudoblock.

<u>Third:</u> Similarly, since the block B_2 contains all even-dimensional simple A-modules.
Let P_e,P_j be projective indecomposable A-modules, let e,j be simple A-modules; for all $j,e \in \{2,4,\ldots,p-1\}$, and let $N_{j^{\backprime}}$ be non-simple, non-projective, indecomposable A-modules; $j^{\backprime}=\{1,2,\ldots,r\}$; in which P_e,P_j, e,j, and $N_{j^{\backprime}}$ in B_2 for all $e,j,j^{\backprime}$. Let $P_j=j/p-1-j,p+1-j/j$, $\quad P_e=e/p-1-e,p+1-e/e$, $\quad N_1=j/p-1-j,p+1-j$, $\quad N_2=p+1-j/j$, $N_3=p-1-e,p+1-e/e$, $\quad N_4=p-1-e,p+1-e/e,j$, $\quad N_5=e/p-1-e,p+1-e$, $\quad N_6=p-1-j,p+1-j/j$.

Then, we have six cases as follows:

1. Let j,e be any two simple A-modules. Hence,
$$j \to N_2 \to e.$$
Then, all even-dimensional simple A-modules are connected either ways by a sequence of A-module homomorphisms.
2. Let j,e be simple A-modules, and let P_j,P_e be projective indecomposable A-modules. Hence,
$$P_j \to N_1 \to j, \quad e \to N_3 \to P_e.$$
Then, all even-dimensional simple A-modules and all projective indecomposable A-modules are connected either ways by a sequence of A-module homomorphisms.

3. Let $N_{j'}; j' = \{1,2,3,5\}$ be any non-simple, non-projective, indecomposable A-modules, and let j, e be any two simple A-modules. Hence,
$$N_1 \to j, \quad N_2 \to e, \quad N_3 \to P_e \to N_5 \to e.$$
Then, all even-dimensional simple A-modules and all non-simple, non-projective, indecomposable A-modules $N_{j'}; j' = \{1,2,\ldots,r\}$ are connected either ways by a sequence of A-module homomorphisms.
4. Let P_j, P_e be any two projective indecomposable A-modules. Hence,
$$P_j \to N_1 \to j \to N_2 \to p+1-j \to N_3 \to P_e.$$
Then, all projective indecomposable A-modules $P_e, \forall e = \{2,4,\ldots,p-1\}$ are connected either ways by a sequence of A-module homomorphisms.
5. Let P_j, P_e be any two projective indecomposable A-modules, and let N_1, N_3, N_5, N_6 be non-simple, non-projective, indecomposable A-modules. Hence,
$$P_j \to N_1, \quad P_e \to N_5.$$
Also,
$$N_6 \to P_j, \quad N_3 \to P_e.$$
Then, all projective indecomposable A-modules $P_e; \forall e = \{2,4,\ldots,p-1\}$ and all non-simple, non-projective, indecomposable A-modules $N_{j'}; j' = \{1,2,\ldots,r\}$ are connected either ways by a sequence of A-module homomorphisms.
6. Let $N_1, N_2, N_3, N_4, N_5, N_6$ be any non-simple, non-projective, indecomposable A-modules. Hence,
$$N_6 \to P_j \to N_1,$$
$$N_1 \to j \to N_2,$$
$$N_3 \to P_e \to N_5,$$
and
$$N_4 \to N_3.$$
Then, all non-simple, non-projective, indecomposable A-modules are connected either ways by a sequence of A-module homomorphisms.

The previous six cases are enough without loss of generality. So, all indecomposable A-modules in B_2 are connected either ways by a sequence of A-module homomorphisms as follows:
$$P_j \to N_{j'} \to j \to \ldots \leftarrow N'_{j'} \leftarrow e \leftarrow N''_{j'} \leftarrow P_e;$$
for all $j, e \in \{2,4,\ldots,p-1\}$ and $j' = \{1,2,\ldots,r\}$.

Thus, the block B_2 does not split into union of pseudoblocks. So, B_2 is one pseudoblock.

Thus, for group algebra $FSL(2,p)$ in characteristic odd prime p the two notions blocks and pseudoblocks coincide.

Example 6.4. *If $p = 2$, then the representations of $SL(2,2) \cong S_3$ in characteristic 2; hence the two notions blocks and pseudoblocks coincide as stated in section 5.*

If $p = 7$, then the following are the indecomposable $FSL(2,7)$-modules:

- *The simple $FSL(2,7)$-modules are:* $\underbrace{1,3,5,}_{B_1}\ \underbrace{2,4,6,}_{B_2}\ \underbrace{7}_{B_3}.$
- *The projective indecomposable $FSL(2,7)$-modules are:*
$$1/5/1, \quad 3/3,5/3, \quad 5/1,3/5, \quad 4/2,4/4, \quad 2/4,6/2, \quad 6/2/6, \quad 7.$$
- *The (non-projective non-simple) indecomposable $FSL(2,7)$-modules are:*
5/1, 1/5, 3/5, 5/3, 3/3, 3,5/3, 3/3,5 1,3/5, 5/1,3, 3,5/1,3,5,
1,3,5/3,5, 1,3,5/1,3,5, 3,5/1,3, 3,5/3,5, 1,3/3,5. *(in B_1)*

2/6, 6/2, 4/2, 2/4, 4/4, 4,6/2, 2/4,6, 2,4/4, 4/2,4, 2,4/2,4,6,
2,4,6/2,4, 2,4,6/2,4,6, 2,4/2,4, 2,4/4,6, 4,6/2,4. *(in B_2)*

The total number of indecomposable modules is $43 = 7^2 - 7 + 1$, *where* $Ext_{FSL(2,7)}(i, j)$ *are* 1*-dimension for all indecomposable* $FSL(2,7)$*-modules as stated in ([5], p.117).*

The indecomposable $FSL(2,7)$*-modules in* B_1 *forms a single pseudoblock via the following sequence of homomorphisms:*
$3/3 \to 1,3,5/1,3,5 \to 1,3,5/3,5 \to 1,3/3,5 \to 1,3/5 \to 5/1,3/5 \to 5/1,3 \to 3,5/1,3 \to 3,5/1,3,5 \to 3,5/3,5 \to 3,5/3 \to 3/3,5/3 \to 3/3,5 \to 3/5 \to 3 \to 5/3 \to 5 \to 1/5 \to 1 \to 5/1 \to 1/5/1.$

The indecomposable $FSL(2,7)$*-modules in* B_2 *forms a single pseudoblock via the following sequence of homomorphisms:*
$4/4 \to 2,4,6/2,4,6 \to 2,4,6/2,4 \to 4,6/2,4 \to 4,6/2 \to 2/4,6/2 \to 2/4,6 \to 2,4/4,6 \to 2,4/2,4,6 \to 2,4/2,4 \to 2,4/4 \to 4/2,4/4 \to 4/2,4 \to 4/2 \to 4 \to 2/4 \to 2 \to 6/2 \to 6 \to 2/6 \to 6/2/6.$

□

Acknowledgement

The main results of this paper are taken from a dissertation written by the first author as a part of a master degree fulfillments from Umm Al-Qura University. The first author thanks her supervisor Prof. Ahmed A. Khammash for his help and guidance and Taif University for their generous grant.

References

[1] A. Khammash, *The pseudoblocks of endomorphism algebras*, Int. Math. Forum, **4**(48) (2009), 2363-2368.
[2] J. Alperin, *Local Representation Theory: Modular Representations as an Introduction to the Local Representation Theory of Finite Groups*, Cambridge University Press, 1986.
[3] A. Khammash, *Brauer-fitting correspondence on tensor algebra,* Int. J. Algebra, **8**(19) (2014), 895-901.
[4] K. Erdmann, T. Holm, *Algebras and Representation Theory*, Springer, 2018.
[5] J. Humphreys, *Modular Representations of Finite Groups of Lie Type*, Cambridge University Press, 2006.
[6] L. Dornhoff, *Group Representation Theory: Modular Representation Theory*, M. Dekker, 1972.
[7] D. Craven, *Maximal* psl_2 *subgroups of exceptional groups of lie type*, (2019), arXiv:1610.07469.