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THE PATHWAY INTEGRAL OPERATOR INVOLVING EXTENSION OF K-BESSEL-MAITLAND FUNCTION

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Abstract. In the present paper, we establish generalized extension of k-Bessel-Maitland function involving pathway integral operator. We obtain certain composition formulas with pathway fractional integral operators. Further more, Some interesting special cases involving Bessel functions, generalized Bessel functions, generalized Mittag-Leffer functions, generalized k-Mittag-Leffer functions are deduced.

1. Introduction

The study of special functions play an important role in Mathematics, Physics, Chemistry, Biology, Engineering and applied Sciences. It has a wide application of almost all branches of Science and technology. The Bessel-Maitland function [10, 28] is denoted by $J_\nu^\mu(z)$ and is defined as follows:

$$\mathbf{J}_\nu^\mu(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(n\mu+\nu+1)}. \tag{1.1}$$

The theory of Bessel functions is intimately connected with the theory of certain types of differential equations. A detail account of applications of Bessel functions are given in the book of Watson [27].

Now, Singh *et al.* [25] introduced and investigate of the following generalization of Bessel-Maitland function as follows:

$$\mathbf{J}_{\nu,\tau}^{\mu,q}(z) = \sum_{n=0}^{\infty} \frac{(\tau)_{qn}(-z)^n}{n!\Gamma(n\mu+\nu+1)}, \tag{1.2}$$

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where $\mu, \nu, \tau \in \mathbb{C}; \Re(\mu) \geq 0, \Re(\nu) \geq -1, \Re(\tau) \geq 0$, and $q \in (0,1) \bigcup \mathbb{N}$ and $(\tau)_{qn} = \frac{\Gamma(\tau+qn)}{\Gamma(\tau)}$ denotes the generalized Pochhammer symbol (see Rainville [21]).

Furthermore, Ghayasuddin *et al.* [7] investigate a new extension of Bessel-Maitland function as follows:

$$\mathbf{J}_{\nu,\tau,\zeta}^{\mu,q,p}(z) = \sum_{n=0}^{\infty} \frac{(\tau)_{qn}(-z)^n}{\Gamma(n\mu+\nu+1)(\zeta)_{pn}}, \tag{1.3}$$

where $\mu, \nu, \tau, \zeta \in \mathbb{C}; \Re(\mu) \geq 0, \Re(\nu) \geq -1, \Re(\tau) \geq 0, \Re(\zeta) \geq 0; p, q > 0$, and $q < \Re(\alpha) + p$.

Recently, Khan *et al.* [9] consider a new generalized Bessel-Maitland function which is defined as:

$$\mathbf{J}_{\alpha,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}(z) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}(\tau)_{qn}(-z)^n}{\Gamma(n\beta+\alpha+1)(\zeta)_{pn}(\nu)_{n\sigma}}, \tag{1.4}$$

where $\alpha, \beta, \mu, \rho, \nu, \tau, \zeta \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\rho) > 0, \Re(\mu) \geq 0, \Re(\nu) \geq -1, \Re(\tau) \geq 0, \Re(\zeta) \geq 0; p, q > 0$, and $q < \Re(\alpha) + p$.

In this paper, we consider a new extension of generalized k-Bessel-Maitland function which is defined as:

$$\mathbf{J}_{k,\alpha,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}(z) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n,k}(\tau)_{qn,k}(-z)^n}{\Gamma_k(n\beta+\alpha+1)(\delta)_{pn,k}(\nu)_{n\sigma,k}}, \tag{1.5}$$

where $k, \alpha, \beta, \mu, \rho, \nu, \tau, \zeta \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\rho) > 0, \Re(\mu) \geq 0, \Re(\nu) \geq -1, \Re(\gamma) \geq 0, \Re(\delta) \geq 0; p, q > 0$, and $q < \Re(\alpha) + p$.

1.1. **Relation with Mittag-Leffler function.**

(1) If we put α by $\alpha - 1$ in (1.5), we get the following result

$$\mathbf{J}_{k,\alpha-1,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}(-x) = E_{k,\alpha,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}(x), \tag{1.6}$$

where $E_{k,\alpha,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}(x)$ is the Mittag-Leffler function defined by Khan and Ahmad [8].

(2) If we put $\mu = \nu = \sigma = \rho = k = 1$ and replacing α by $\alpha - 1$ in (1.5), we get

$$\mathbf{J}_{\alpha-1,\beta,1,1,\delta,p}^{1,1,\gamma,q}(-x) = E_{\alpha,\beta,p}^{\zeta,\tau,q}(x), \tag{1.7}$$

where $E_{\alpha,\beta,p}^{\zeta,\tau,q}(x)$ is the Mittag-Leffler function defined by Salim and Faraz [23].

(3) If we put $\mu = \nu = \sigma = \rho = \zeta = p = 1$ and replacing α by $\alpha - 1$ in (1.5), we get

$$\mathbf{J}_{k,\alpha-1,\beta,1,1,1,1}^{1,1,\zeta,q}(-x) = E_{k,\alpha,\beta}^{\tau,q}(x), \tag{1.8}$$

where $E_{k,\alpha,\beta}^{\tau,q}(x)$ is the k-Mittag-Leffler function defined by Chand *et al.* [4].

(4) If we put $\mu = \nu = \sigma = \rho = \zeta = p = k = 1$ and replacing α by $\alpha - 1$ in (1.5), we get
$$\mathbf{J}^{1,1,\gamma,q}_{\alpha-1,\beta,1,1,1,1}(-x) = E^{\tau,q}_{\alpha,\beta}(x), \tag{1.9}$$
where $E^{\tau,q}_{\alpha,\beta}(x)$ is the Mittag-Leffler function defined by Shukla and Prajapati [26].

(5) If we put $\mu = \nu = \sigma = \rho = \zeta = 1$ and replacing α by $\alpha - 1$ in (1.5), we get
$$\mathbf{J}^{1,1,\tau,\zeta}_{\alpha-1,\beta,1,1,1,1}(-x) = E^{\tau,\zeta}_{\alpha,\beta}(x), \tag{1.10}$$
where $E^{\tau,q}_{\alpha,\beta}(x)$ is the Mittag-Leffler function defined by Salim [24].

(6) If we put $\mu = \nu = \sigma = \rho = \zeta = p = q = 1$ and replacing α by $\alpha - 1$ in (1.5), we get
$$\mathbf{J}^{1,1,\tau}_{k,\alpha-1,\beta,1,1,1,1}(-x) = E^{\tau}_{k,\alpha,\beta}(x), \tag{1.11}$$
where $E^{\tau}_{k,\alpha,\beta}(x)$ is the k-Mittag-Leffler function defined by Dorrego and Cerutti [6].

(7) If we put $\mu = \nu = \sigma = \rho = \zeta = p = q = k = 1$ and replacing α by $\alpha - 1$ in (1.5), we get
$$\mathbf{J}^{1,1,\tau}_{\alpha-1,\beta,1,1,1,1}(-x) = E^{\tau}_{\alpha,\beta}(x), \tag{1.12}$$
where $E^{\tau}_{\alpha,\beta}(x)$ is the Mittag-Leffler function defined by Prabhakar [22].

(8) If we put $\mu = \nu = \sigma = \rho = \zeta = \tau = p = q = k = 1$ and replacing α by $\alpha - 1$ in (1.5), we get
$$\mathbf{J}^{1,1,1}_{\alpha-1,\beta,1,1,1,1}(-x) = E_{\alpha,\beta}(x), \tag{1.13}$$
where $E_{\alpha,\beta}(x)$ is the Mittag-Leffler function defined by Wiman [28].

(9) If we put $\mu = \nu = \sigma = \rho = \zeta = \tau = p = q = k = 1, \alpha = 0$ and replacing α by $\alpha - 1$ in (1.5), we get
$$\mathbf{J}^{1,1,1}_{0,\beta,1,1,1,1}(-x) = E_{\beta}(x), \tag{1.14}$$
where $E_{\beta}(x)$ is the Mittag-Leffler function defined by Mittag-Leffler [16].

We investigate some special cases of the generalized Bessel Maitland function (1.3) by particular values to the parameters $\mu, \nu, \delta, \gamma, p, q$.

Now, we recall the classical Beta function denoted by $B(a, b)$ and is defined as
$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, (\Re(a) > 0, \Re(b) > 0). \tag{1.15}$$
(see [21], and also see [10]). The integral representation of the k-Gamma function is given as:
$$\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma(\frac{z}{k}) = \int_0^\infty e^{\frac{-t^k}{k}} t^{z-1}dt, \tag{1.16}$$
$k \in \mathbb{R}, z \in \mathbb{C}$,

and k-Beta function is defined as:

$$B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, x>0, y>0. \tag{1.17}$$

The generalized Wright function represented as follows [29, 30, 31]:

$${}_p\Psi_q\left[\begin{array}{cc}(\alpha_1,A_1),...,(\alpha_p,A_p); & \\ & z \\ (\beta_1,B_1),...,(\beta_p,B_p); & \end{array}\right] = {}_p\Psi_q\left((\alpha_j,A_j)_{1,p};(\beta_j,B_j)_{1,q};z\right)$$

$$= \sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1+nA_1)...,\Gamma(\alpha_p+nA_p)}{\Gamma(\beta_1+nB_1)...,\Gamma(\beta_p+nB_p)}\frac{z^n}{n!}. \tag{1.18}$$

In 1961, MacRobert [11] investigate the following interesting result which is given below:

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}[at+b(1-t)]^{-\alpha-\beta}dt = \frac{1}{a^\alpha b^\beta}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{1.19}$$

where a and b are non zero constants such that the expression $at+b(1-t)$, for $0\le t\le 1$, is non zero, provided $\Re(\alpha)>0, \Re(\beta)>0$.

In this paper, we further apply the following useful result which is given below:

$$\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\beta}{k}-1}[at+b(1-t)]^{\frac{-\alpha-\beta}{k}}dt = \frac{1}{a^{\frac{\alpha}{k}}b^{\frac{\beta}{k}}}\frac{k\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)}, \tag{1.20}$$

where a and b are non zero constants such that the expression $at+b(1-t)$, for $0\le t\le 1$, is non zero, provided $\Re(\alpha)>0, \Re(\beta)>0$.
It is easy to see that for $k=1$ the equation (1.20) reduces to known result (1.19).

Recently, by using the pathway idea of Mathai [13] and developed further by Mathai and Haubold [14, 15], Nair [17], we introduce a pathway fractional integral operator which is given below.

Suppose $f(x)\in L(a,b), \eta\in\mathbb{C}, \Re(\eta)>0, a>0$ and the pathway parameter $\alpha<1$ as (cf. [2]), then

$$(P_{0+}^{(\eta,\alpha)}f)(x) = x^\eta\int_0^{[\frac{x}{a(1-\alpha)}]}\left[1-\frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{(1-\alpha)}}f(t)dt. \tag{1.21}$$

For a real scalar α, the pathway model for scalar random variables is represented by the following probability density function (p.d.f.):

$$f(x) = c|x|^{\gamma-1}\left[1-a(1-\alpha)|x|^\delta\right]^{\frac{\beta}{(1-\alpha)}}, \tag{1.22}$$

provided that $-\infty < x < \infty, \delta > 0, \beta \geq 0, \left[1 - a(1-\alpha)|x|^{\delta}\right] > 0$ and $\gamma > 0$, where c is the normalizing constant and α is called the pathway parameter,

$$c = \frac{1}{2}\frac{\delta\left(a(1-\alpha)\right)^{\frac{\gamma}{\delta}}\Gamma\left(\frac{\gamma}{\delta}+\frac{\beta}{(1-\alpha)}+1\right)}{\Gamma(\frac{\gamma}{\delta})\Gamma\left(\frac{\beta}{(1-\alpha)}+1\right)}, for\ \alpha < 1 \tag{1.23}$$

$$= \frac{1}{2}\frac{\delta\left(a(1-\alpha)\right)^{\frac{\gamma}{\delta}}\Gamma\left(\frac{\beta}{(1-\alpha)}\right)}{\Gamma(\frac{\gamma}{\delta})\Gamma\left(\frac{\beta}{(1-\alpha)}-\frac{\gamma}{\delta}\right)}, for \frac{1}{1-\alpha}-\frac{\gamma}{\delta} > 0,\ \alpha > 1 \tag{1.24}$$

$$= \frac{1}{2}\frac{(a\beta)^{\frac{\gamma}{\delta}}}{\Gamma(\frac{\gamma}{\delta})},\ \alpha \to 1. \tag{1.25}$$

For $\alpha < 1$, it is a finite range density with $\left[1 - a(1-\alpha)|x|^{\delta}\right] > 0$ and (1.21) remains in the extended generalized type-1 beta family. The Pathway density in (1.21), for $\alpha < 1$, includes the extended type-1 beta density, the triangular density, the uniform density and many other p.d.f's. [2]. For $\alpha > 1$,

$$f(x) = c|x|^{\gamma-1}\left[1 + a(1-\alpha)|x|^{\delta}\right]^{-\frac{\beta}{1-\alpha}}, \tag{1.26}$$

provided that $-\infty < x < \infty, \delta > 0, \beta \geq 0$ and $\alpha > 0$ which is extended generalized type-2 modal for real x. It includes the type-2 beta density, the F density, the student-t density, the cauchy density and many more. For instance, $\alpha > 1$, writing $(1-\alpha) = -(\alpha - 1)$ gives:

$$(P_{0+}^{(\eta,\alpha)}f)(x) = x^{\eta}\int_{0}^{\left[\frac{x}{-a(1-\alpha)}\right]}\left[1 + \frac{a(\alpha-1)t}{x}\right]^{-\frac{\eta}{(\alpha-1)}} f(t)dt. \tag{1.27}$$

For more basic details about pathway integral operator, one may refer [1, 2, 18, 19, 20].

2. Main Results

The pathway integral operator of k-Bessel-Maitland function is given in the following theorems.

Theorem 2.1. *Let* $k \in \mathcal{R}, \alpha, \beta, \tau, \zeta, \mu, \nu, \rho, \sigma \in \mathcal{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\tau) > 0, \Re(\zeta) > 0, \Re(\mu) > 0, \Re(\nu) > 0, \Re(\rho) > 0, \Re(\sigma) > 0, p, q > 0$ *and* $q \leq \Re(\alpha) + p, \eta \in C, \Re(\frac{\eta}{1-\xi}) > -1, \lambda > 1, w > R.$

$$P_{0+}^{(\eta,\lambda)}\left[t^{\frac{\beta}{k}-1}\mathbf{J}_{k,\alpha,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}(wt^{\frac{\alpha}{k}})\right](x) = \frac{x^{\eta+\frac{\beta}{k}}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda)^{\frac{\beta}{k}}k^{\frac{\alpha+1}{k}-1}}\mathbf{J}_{k,\alpha,\beta+k(\frac{\eta}{1-\lambda}),\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}\left(w(\frac{x}{a(1-\lambda)})^{\frac{\alpha}{k}}\right). \tag{2.1}$$

Proof. On taking L.H.S. of Theorem 2.1, and then expanding the definition of generalized k-Bessel-Maitland function $\mathbf{J}_{k,\alpha,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}(wt^{\frac{\alpha}{k}})$, by using (1.18) we obtain:

$$P_{0+}^{(\eta,\lambda)}\left[t^{\frac{\beta}{k}-1}J_{k,\alpha,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}(wt^{\frac{\alpha}{k}})\right](x)$$

$$= x^{\eta} \int_{0}^{[\frac{x}{a(1-\lambda)}]} t^{\frac{\beta}{k}-1} \left[1 - \frac{a(1-\lambda)t}{x}\right]^{\frac{\eta}{(1-\lambda)}} J^{\mu,\rho,\tau,q}_{k,\alpha,\beta,\nu,\sigma,\zeta,p}(wt^{\frac{\alpha}{k}})dt,$$

$$= x^{\eta} \int_{0}^{[\frac{x}{a(1-\lambda)}]} t^{\frac{\beta}{k}-1} \left[1 - \frac{a(1-\lambda)t}{x}\right]^{\frac{\eta}{(1-\lambda)}} \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n,k}(\tau)_{qn,k}(-wt^{\frac{\alpha}{k}})^n}{\Gamma_k(n\beta+\alpha+1)(\zeta)_{pn,k}(\nu)_{n\sigma,k}}dt,$$

Interchanging the integration and summation under the suitable convergence condition, we obtain

$$= x^{\eta} \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n,k}(\tau)_{qn,k}(-w)^n}{\Gamma_k(n\beta+\alpha+1)(\zeta)_{pn,k}(\nu)_{n\sigma,k}} \int_{0}^{[\frac{x}{a(1-\lambda)}]} t^{\frac{\beta}{k}+\frac{n\alpha}{k}-1} \left[1 - \frac{a(1-\lambda)t}{x}\right]^{\frac{\eta}{(1-\lambda)}} dt,$$

Now, interchanging the inner integral by beta function formula (1.12), we get

$$= x^{\eta} \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n,k}(\tau)_{qn,k}(-w)^n}{\Gamma_k(n\beta+\alpha+1)(\zeta)_{pn,k}(\nu)_{n\sigma,k}} \int_{0}^{1} u^{\frac{\beta}{k}+\frac{n\alpha}{k}-1}(1-u)^{\frac{\eta}{(1-\lambda)}} \left(\frac{x}{a(1-\lambda)}\right)$$
$$\times \left(\frac{x}{a(1-\lambda)}\right)^{\frac{\beta}{k}+\frac{n\alpha}{k}-1} du,$$

again applying the Beta function formula, we have

$$= \frac{x^{\eta+\frac{\beta}{k}}}{(a(1-\lambda))^{\frac{\beta}{k}}} \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n,k}(\tau)_{qn,k}(-w)^n x^{\frac{n\alpha}{k}}}{\Gamma_k(n\beta+\alpha+1)(\zeta)_{pn,k}(\nu)_{n\sigma,k}} \frac{\Gamma(\frac{\eta}{(1-\lambda)}+1)\Gamma(\frac{\beta}{k}+\frac{n\alpha}{k})}{\Gamma(\frac{\eta}{(1-\lambda)}+\frac{\beta}{k}+\frac{n\alpha}{k}+1)} \frac{1}{(a(1-\lambda))^{\frac{n\alpha}{k}}}.$$

Now, using the result,

$$\Gamma_k(\lambda) = k^{\frac{\lambda}{k}-1}\Gamma(\frac{\lambda}{k}), \tag{2.2}$$

we get,

$$= \frac{x^{\eta+\frac{\beta}{k}}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda))^{\frac{\beta}{k}}} \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n,k}(\tau)_{qn,k}(-w)^n x^{\frac{n\alpha}{k}}}{k^{\frac{n\beta+\alpha+1}{k}-1}\Gamma(\frac{n\beta+\alpha+1}{k})(\zeta)_{pn,k}(\nu)_{n\sigma,k}} \frac{\Gamma(\frac{\beta}{k}+\frac{n\alpha}{k})}{\Gamma(\frac{\eta}{(1-\lambda)}+\frac{\beta}{k}+\frac{n\alpha}{k}+1)} \frac{1}{(a(1-\lambda))^{\frac{n\alpha}{k}}},$$

$$= \frac{x^{\eta+\frac{\beta}{k}}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda))^{\frac{\beta}{k}}k^{\frac{\alpha+1}{k}-1}} \mathbf{J}^{\mu,\rho,\tau,q}_{k,\alpha,\beta+k(\frac{\eta}{1-\lambda}),\nu,\sigma,\zeta,p}\left(w(\frac{x}{a(1-\lambda)})^{\frac{\alpha}{k}}\right),$$

which is our desired result (2.1).
Thus, the proof of Theorem 2.1 is complete. □

Corollary 2.2. *If we put $\tau = q = 1, \nu = \sigma = p = 1$ in Theorem 2.1, then we get the result corresponding result of Nisar et al.* [19] *as:*

$$P_{0+}^{(\eta,\lambda)}\left[t^{\frac{\beta}{k}-1}\mathbf{J}_{k,\alpha,\beta,1,1,\zeta,1}^{\mu,\rho,1,1}(wt^{\frac{\alpha}{k}})\right](x) = \frac{x^{\eta+\frac{\beta}{k}}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda)^{\frac{\beta}{k}}k^{\frac{\alpha+1}{k}-1}}\mathbf{J}_{k,\alpha,\beta+k(\frac{\eta}{1-\lambda}),1,1,\zeta,1}^{\mu,\rho,1,1}\left(-w(\frac{x}{a(1-\lambda)})^{\frac{\alpha}{k}}\right). \tag{2.3}$$

Corollary 2.3. *If we put $\tau = q = 1, \nu = \sigma = p = \zeta = k = 1$ in Theorem 2.1, then we obtain the corresponding result of Nair* [17] *as:*

$$P_{0+}^{(\eta,\lambda)}\left[t^{\beta-1}\mathbf{J}_{1,\alpha,\beta,1,1,1,1}^{\mu,\rho,1,1}(wt^{\frac{\alpha}{k}})\right](x) = \frac{x^{\eta+\beta}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda))^{\beta}}\mathbf{J}_{1,\alpha,\beta+1(\frac{\eta}{1-\lambda}),1,1,1,1}^{\mu,\rho,1,1}\left(w(\frac{x}{a(1-\lambda)})^{\alpha}\right). \tag{2.4}$$

Theorem 2.4. *Let $k \in \mathcal{R}, \alpha, \beta, \tau, \zeta, \mu, \nu, \rho, \sigma \in \mathcal{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\tau) > 0, \Re(\zeta) > 0, \Re(\mu) > 0, \Re(\nu) > 0, \Re(\rho) > 0, \Re(\sigma) > 0, p, q > 0$ and $q \leq \Re(\alpha) + p, \eta \in C, \Re(\frac{\eta}{1-\xi}) > -1, \lambda > 1, w > R$.*

$$P_{0+}^{(\eta,\lambda)}\left[t^{\frac{\beta}{k}-1}\mathbf{J}_{k,\alpha,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}(wt^{\frac{\alpha}{k}})\right](x) = \frac{x^{\eta+\frac{\beta}{k}+1}\Gamma(1-\frac{\eta}{(\lambda-1)})}{(-a(1-\lambda)^{\frac{\beta}{k}}k^{\frac{\alpha+1}{k}-1}}\mathbf{J}_{k,\alpha,\beta+k(n\alpha+k-\frac{\eta}{\lambda-1}),\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}\left(w(\frac{x}{-a(\lambda-1)})^{\frac{\alpha}{k}}\right). \tag{2.5}$$

Proof. On taking L.H.S of (2.5) and applying the definition (1.5) and (1.24), we obtain

$$P_{0+}^{(\eta,\lambda)}\left[t^{\frac{\beta}{k}-1}\mathbf{J}_{k,\alpha,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}(wt^{\frac{\alpha}{k}})\right](x)$$

$$= x^{\eta}\int_{0}^{[\frac{x}{-a(1-\lambda)}]} t^{\frac{\beta}{k}-1}\left[1+\frac{a(\lambda-1)t}{x}\right]^{\frac{\eta}{-(\lambda-1)}} J_{k,\alpha,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}(wt^{\frac{\alpha}{k}})dt,$$

$$= x^{\eta}\int_{0}^{[\frac{x}{-a(1-\lambda)}]} t^{\frac{\beta}{k}-1}\left[1+\frac{a(\lambda-1)t}{x}\right]^{\frac{\eta}{-(\lambda-1)}} \sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,k}(\tau)_{qn,k}(-wt^{\frac{\alpha}{k}})^{n}}{\Gamma_k(n\beta+\alpha+1)(\zeta)_{pn,k}(\nu)_{n\sigma,k}}dt.$$

Interchanging the integration and summation under the suitable convergence condition, we obtain

$$= x^{\eta}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,k}(\tau)_{qn,k}(-w)^{n}}{\Gamma_k(n\beta+\alpha+1)(\zeta)_{pn,k}(\nu)_{n\sigma,k}}\int_{0}^{[\frac{x}{-a(1-\lambda)}]} t^{\frac{\beta}{k}+\frac{n\alpha}{k}-1}\left[1+\frac{a(\lambda-1)t}{x}\right]^{\frac{\eta}{-(\lambda-1)}} dt.$$

Now, interchanging the inner integral by beta function formula, we get

$$= x^{\eta}\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n,k}(\tau)_{qn,k}(-w)^{n}}{\Gamma_k(n\beta+\alpha+1)(\zeta)_{pn,k}(\nu)_{n\sigma,k}}\int_{0}^{1} u^{\frac{\beta}{k}+\frac{n\alpha}{k}-1}(1-u)^{\frac{\eta}{(1-\lambda)}}\left(\frac{x}{a(1-\lambda)}\right)$$

$$\times \left(\frac{x}{a(1-\lambda)}\right)^{\frac{\beta}{k}+\frac{n\alpha}{k}-1} du,$$

again applying the beta function formula, we have

$$= \frac{x^{\eta+\frac{\beta}{k}}}{(-a(1-\lambda)^{\frac{\beta}{k}}} \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n,k}(\tau)_{qn,k}(-w)^n x^{\frac{n\alpha}{k}}}{\Gamma_k(n\beta+\alpha+1)(\zeta)_{pn,k}(\nu)_{n\sigma,k}} \frac{\Gamma(1-\frac{\upsilon}{(\lambda-1)})\Gamma(\frac{\beta}{k}+\frac{n\alpha}{k})}{\Gamma(1-\frac{\upsilon}{(\lambda-1)}+\frac{\beta}{k}+\frac{n\alpha}{k})} \frac{1}{(-a(\lambda-1))^{\frac{n\alpha}{k}}}.$$

Now, using the result,

$$\Gamma_k(\lambda) = k^{\frac{\lambda}{k}-1}\Gamma(\frac{\lambda}{k}), \tag{2.6}$$

we obtain,

$$= \frac{x^{\eta+\frac{\beta}{k}+1}\Gamma(1-\frac{\eta}{(\lambda-1)})}{(-a(1-\lambda)^{\frac{\beta}{k}+1}} \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n,k}(\tau)_{qn,k}(-w)^n x^{\frac{n\alpha}{k}}}{k^{\frac{n\beta+\alpha+1}{k}-1}\Gamma(\frac{n\beta+\alpha+1}{k})(\zeta)_{pn,k}(\nu)_{n\sigma,k}} \frac{\Gamma(\frac{\beta}{k}+\frac{n\alpha}{k})}{\Gamma(1-\frac{\eta}{(1-\lambda)}+\frac{\beta}{k}+\frac{n\alpha}{k})} \frac{1}{(-a(1-\lambda))^{\frac{n\alpha}{k}}},$$

$$= \frac{x^{\eta+\frac{\beta}{k}+1}\Gamma(1-\frac{\eta}{(\lambda-1)})}{(-a(1-\lambda)^{\frac{\beta}{k}} k^{\frac{\alpha+1}{k}-1}} \mathbf{J}^{\mu,\rho,\tau,q}_{k,\alpha,\beta+k(n\alpha+k-\frac{\eta}{\lambda-1}),\nu,\sigma,\zeta,p} \left(w(\frac{x}{-a(\lambda-1)})^{\frac{\alpha}{k}}\right),$$

which is our desired result (2.5). □

Corollary 2.5. *If we put $\tau = q = 1, \nu = \sigma = p = 1$ in Theorem 2.4, then it reduces to the corresponding result of* [16]*:*

$$P_{0+}^{(\eta,\lambda)}\left[t^{\frac{\beta}{k}-1}\mathbf{J}^{\mu,\rho,1,1}_{k,\alpha,\beta,1,1,\zeta,1}(wt^{\frac{\alpha}{k}})\right](x) = \frac{x^{\eta+\frac{\beta}{k}+1}\Gamma(1-\frac{\eta}{(\lambda-1)})}{(-a(1-\lambda)^{\frac{\beta}{k}+1}k^{\frac{\alpha+1}{k}-1}} \mathbf{J}^{\mu,\rho,1,1}_{k,\alpha,\beta+k(n\alpha+k-\frac{\eta}{1-\lambda}),1,1,\zeta,1}\left(w(\frac{x}{-a(\lambda-1)})^{\frac{\alpha}{k}}\right). \tag{2.7}$$

Corollary 2.6. *If we put $\tau = q = 1, \nu = \sigma = p = \zeta = k = 1$ in Theorem 2.4, then it reduces to the following result of Nair* [17].

$$P_{0+}^{(\eta,\lambda)}\left[t^{\beta-1}\mathbf{J}^{\mu,\rho,1,1}_{1,\alpha,\beta,1,1,1,1}(wt^{\alpha})\right](x) = \frac{x^{\eta+\beta+1}\Gamma(1-\frac{\eta}{(1-\lambda)})}{(-a(1-\lambda))^{\beta+1}} \mathbf{J}^{\mu,\rho,1,1}_{1,\alpha,\beta+1(n\alpha+1-\frac{\eta}{\lambda-1}),1,1,1,1}\left(w(\frac{x}{a(1-\lambda)})^{\alpha}\right). \tag{2.8}$$

Theorem 2.7. *Let $k \in \mathcal{R}, \alpha, \beta, \upsilon, \zeta, \mu, \nu, \rho, \sigma, \lambda, \tau \in \mathcal{C}, \Re(\alpha) > -1, \Re(\beta) > 0, \Re(\upsilon) > 0, \Re(\zeta) > 0, \Re(\mu) > 0, \Re(\nu) > 0, \Re(\rho) > 0, \Re(\sigma) > 0, \Re(\lambda) > 0, \Re(\tau) > 0, p, q > 0$ and $q \leq \Re(\alpha) + p$.*

$$\int_0^1 t^{\frac{\upsilon}{k}-1}(1-t)^{\frac{\xi}{k}-1}[at+b(1-t)]^{\frac{-\upsilon-\xi}{k}} \mathbf{J}^{\mu,\rho,\tau,q}_{k,\alpha,\beta,\nu,\sigma,\zeta,p}\left[\frac{2abt(1-t)}{(at+b(1-t))^2}\right]^{\frac{1}{k}} dt$$
$$= \frac{\Gamma_k(\zeta)\Gamma_k(\mu)}{\Gamma_k(\tau)\Gamma_k(\mu)a^{\upsilon}b^{\lambda}} \sum_{s=0}^{\infty} \frac{\Gamma_k(\mu+s\rho)\Gamma_k(\gamma+sq)(-2)^{\frac{s}{k}}a^{\frac{s}{k}}b^{\frac{s}{k}}}{\Gamma_k(s\beta+\alpha+1)\Gamma_k(\zeta+ps)\Gamma_k(\nu+s\sigma)\Gamma} \frac{\Gamma_k(\upsilon+s)\Gamma_k(\lambda+s)}{\Gamma_k(\upsilon+\lambda+2s)}. \tag{2.9}$$

Proof. On taking L.H.S. of Theorem 2.7, using the definition of generalized k-Bessel-Maitland function (1.5) and (1.17), we obtain

$$\int_0^1 t^{\frac{\upsilon}{k}-1}(1-t)^{\frac{\xi}{k}-1}[at+b(1-t)]^{\frac{-\tau-\xi}{k}}\mathbf{J}^{\mu,\rho,\tau,q}_{k,\alpha,\beta,\nu,\sigma,\zeta,p}\left[\frac{2abt(1-t)}{(at+b(1-t))^2}\right]^{\frac{1}{k}}dt,$$

$$=\int_0^1 t^{\frac{\upsilon}{k}-1}(1-t)^{\frac{\xi}{k}-1}[at+b(1-t)]^{\frac{-\upsilon-\xi}{k}}\sum_{s=0}^{\infty}\frac{(\mu)_{\rho s,k}(\tau)_{qs,k}}{\Gamma_k(s\beta+\alpha+1)(\zeta)_{ps,k}(\nu)_{s\sigma,k}}\frac{(-2)^{\frac{s}{k}}(ab)^{\frac{s}{k}}t^{\frac{s}{k}}(1-t)^{\frac{s}{k}}}{(at+b(1-t))^{\frac{2s}{k}}}dt,$$

$$=\sum_{s=0}^{\infty}\frac{(\mu)_{\rho s,k}(\tau)_{qs,k}}{\Gamma_k(s\beta+\alpha+1)(\zeta)_{ps,k}(\nu)_{s\sigma,k}}(-2)^{\frac{s}{k}}(ab)^{\frac{s}{k}}\int_0^1 t^{\frac{\upsilon+s}{k}-1}(1-t)^{\frac{\xi+s}{k}-1}[at+b(1-t)]^{\frac{-\upsilon-\xi-2s}{k}}dt,$$

by using the integral (1.17), we obtain

$$=\sum_{s=0}^{\infty}\frac{(\mu)_{\rho s,k}(\tau)_{qs,k}}{\Gamma_k(s\beta+\alpha+1)(\zeta)_{ps,k}(\nu)_{s\sigma,k}}\frac{(-2)^{\frac{s}{k}}a^{\frac{s}{k}}b^{\frac{s}{k}}}{a^{\frac{\tau}{k}}b^{\frac{\lambda}{k}}}\frac{k\Gamma_k(\tau+s)\Gamma_k(\lambda+s)}{\Gamma_k(\upsilon+\lambda+2s)},$$

$$=\frac{\Gamma_k(\zeta)\Gamma_k(\mu)}{\Gamma_k(\tau)\Gamma_k(\mu)a^{\upsilon}b^{\lambda}}\sum_{s=0}^{\infty}\frac{\Gamma_k(\mu+s\rho)\Gamma_k(\tau+sq)(-2)^{\frac{s}{k}}a^{\frac{s}{k}}b^{\frac{s}{k}}}{\Gamma_k(s\beta+\alpha+1)\Gamma_k(\zeta+ps)\Gamma_k(\nu+s\sigma)\Gamma}\frac{\Gamma_k(\upsilon+s)\Gamma_k(\lambda+s)}{\Gamma_k(\upsilon+\lambda+2s)},$$

we derive required result.
Thus, the proof of Theorem 2.7 is established. □

3. Special Case

In this section, we establish the following potentially useful integral operators involving generalized k-Beta type functions as special cases of our main results:

(1) If we let α by $\alpha-1$ in Theorem 2.1, and then by using (1.6), we get:

$$P^{(\eta,\lambda)}_{0+}\left[t^{\frac{\beta}{k}-1}E^{\mu,\rho,\tau,q}_{k,\alpha,\beta,\nu,\sigma,\zeta,p}(wt^{\alpha})\right](x)=\frac{x^{\eta+\frac{\beta}{k}}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda)^{\frac{\beta}{k}}k^{\frac{\alpha}{k-1}}}E^{\mu,\rho,\tau,q}_{k,\alpha,\beta+k(\frac{\eta}{1-\lambda}),\nu,\sigma,\zeta,p}\left(w(\frac{x}{a(1-\lambda)})^{\frac{\alpha}{k}}\right) \tag{3.1}$$

(2) If we let $\mu=\nu=\sigma=\rho=k=1$ and replacing α by $\alpha-1$ in Theorem 2.1, and then by using (1.7), we obtain:

$$P^{(\eta,\lambda)}_{0+}\left[t^{\beta-1}E^{\zeta,\tau,q}_{\alpha,\beta,p}(wt^{\alpha})\right](x)=\frac{x^{\eta+\beta}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda))^{\frac{\beta}{k}}}E^{\zeta,\tau,q}_{\alpha,\beta+1(\frac{\eta}{1-\lambda}),p}\left(w(\frac{x}{a(1-\lambda)})^{\alpha}\right) \tag{3.2}$$

(3) If we let $\mu=\nu=\sigma=\rho=\zeta=p=1$ and replacing α by $\alpha-1$ in Theorem 2.1, and then by using (1.8), we obtain

$$P^{(\eta,\lambda)}_{0+}\left[t^{\frac{\beta}{k}-1}E^{\tau,q}_{k,\alpha,\beta}(wt^{\alpha})\right](x)=\frac{x^{\eta+\frac{\beta}{k}}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda)^{\frac{\beta}{k}}k^{\frac{\alpha}{k-1}}}E^{1,1,\tau,q}_{k,\alpha,\beta+k(\frac{\eta}{1-\lambda}),1,1,1,1}\left(w(\frac{x}{a(1-\lambda)})^{\frac{\alpha}{k}}\right) \tag{3.3}$$

(4) If we let $\mu=\nu=\sigma=\rho=\zeta=p=k=1$ and replacing α by $\alpha-1$ in Theorem 2.1, and then by using (1.9), we attain:

$$P^{(\eta,\lambda)}_{0+}\left[t^{\beta-1}E^{\tau,q}_{\alpha,\beta}(wt^{\alpha})\right](x)=\frac{x^{\eta+\beta}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda))^{\beta}}E^{\tau,q}_{\alpha,\beta+1(\frac{\eta}{1-\lambda})}\left(w(\frac{x}{a(1-\lambda)})^{\alpha}\right) \tag{3.4}$$

(5) If we let $\mu = \nu = \sigma = \rho = q = 1$ and replacing α by $\alpha - 1$ in Theorem 2.1, and then bu using (1.10), we get

$$P_{0+}^{(\eta,\lambda)}\left[t^{\beta-1}E_{\alpha,\beta}^{\tau,\zeta}(wt^{\alpha})\right](x) = \frac{x^{\eta+\beta}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda))^{\beta}}E_{\alpha,\beta+1(\frac{\eta}{1-\lambda})}^{\tau,\zeta}\left(w(\frac{x}{a(1-\lambda)})^{\alpha}\right) \tag{3.5}$$

(6) If we let $\mu = \nu = \sigma = \rho = \zeta = p = q = 1$ and replacing α by $\alpha - 1$ in Theorem 2.1, and then by using (1.11), we attain:

$$P_{0+}^{(\eta,\lambda)}\left[t^{\frac{\beta}{k}-1}E_{k,\alpha,\beta}^{\tau}(wt^{\alpha})\right](x) = \frac{x^{\eta+\frac{\beta}{k}}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda)^{\frac{\beta}{k}}k^{\frac{\alpha}{k-1}}}E_{k,\alpha,\beta+k(\frac{\eta}{1-\lambda})}^{\tau}\left(w(\frac{x}{a(1-\lambda)})^{\frac{\alpha}{k}}\right) \tag{3.6}$$

(7) If we let $\mu = \nu = \sigma = \rho = \zeta = p = q = k = 1$ and replacing α by $\alpha - 1$ in Theorem 2.1, and then by using (1.12), we obtain:

$$P_{0+}^{(\eta,\lambda)}\left[t^{\beta-1}E_{\alpha,\beta}^{\tau}(wt^{\alpha})\right](x) = \frac{x^{\eta+\beta}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda))^{\beta}}E_{\alpha,\beta+1(\frac{\eta}{1-\lambda})}^{\gamma}\left(w(\frac{x}{a(1-\lambda)})^{\alpha}\right) \tag{3.7}$$

(8) If we let $\mu = \nu = \sigma = \rho = \zeta = \tau = p = q = k = 1$ and replacing α by $\alpha - 1$ in Theorem 2.1, and then by using (1.13), we obtain:

$$P_{0+}^{(\eta,\lambda)}\left[t^{\beta-1}E_{\alpha,\beta}^{1}(wt^{\alpha})\right](x) = \frac{x^{\eta+\beta}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda))^{\beta}}E_{\alpha,\beta+1(\frac{\eta}{1-\lambda})}^{1}\left(w(\frac{x}{a(1-\lambda)})^{\alpha}\right) \tag{3.8}$$

(9) If we let $\mu = \nu = \sigma = \rho = \zeta = \tau = p = q = k = 1, \alpha = 0$ and replacing α by $\alpha - 1$ in Theorem 2.1, and then by using (1.14), we find:

$$P_{0+}^{(\eta,\lambda)}\left[t^{\beta-1}E_{\beta}^{1}(w)\right](x) = \frac{x^{\eta+\beta}\Gamma(\frac{\eta}{(1-\lambda)}+1)}{(a(1-\lambda))^{\beta}}E_{\beta+1(\frac{\eta}{1-\lambda})}^{1}\left(w(\frac{x}{a(1-\lambda)})\right) \tag{3.9}$$

(10) If we let α by $\alpha - 1$ in Theorem 2.4, and then by using (1.6), we get:

$$P_{0+}^{(\eta,\lambda)}\left[t^{\frac{\beta}{k}-1}E_{k,\alpha,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}(wt^{\frac{\alpha}{k}})\right](x) = \frac{x^{\eta+\frac{\beta}{k}+1}\Gamma(1-\frac{\eta}{(\lambda-1)})}{(-a(1-\lambda)^{\frac{\beta}{k}}k^{\frac{\alpha+1}{k}-1}}E_{k,\alpha,\beta+k(n\alpha+k-\frac{\eta}{\lambda-1}),\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}\left(w(\frac{x}{-a(\lambda-1)})^{\frac{\alpha}{k}}\right). \tag{3.10}$$

(11) If we let $\mu = \nu = \sigma = \rho = k = 1$ and replacing α by $\alpha - 1$ in Theorem 2.4, and then by using (1.7), we get:

$$P_{0+}^{(\eta,\lambda)}\left[t^{\beta-1}E_{\alpha,\beta,p}^{\tau,\zeta,q}(wt^{\alpha})\right](x) = \frac{x^{\eta+\beta+1}\Gamma(1-\frac{\eta}{(\lambda-1)})}{(-a(1-\lambda)^{\beta}}E_{\alpha,\beta+1(n\alpha+1-\frac{\eta}{\lambda-1}),p}^{\zeta,\gamma,q}\left(w(\frac{x}{-a(\lambda-1)})^{\alpha}\right). \tag{3.11}$$

(12) If we let $\mu = \nu = \sigma = \rho = \zeta = p = 1$ and replacing α by $\alpha - 1$ in Theorem 2.7, and then by using (1.8), we get:

$$P_{0+}^{(\eta,\lambda)}\left[t^{\frac{\beta}{k}-1}E_{k,\alpha,\beta}^{\tau,q}(wt^{\frac{\alpha}{k}})\right](x) = \frac{x^{\eta+\frac{\beta}{k}+1}\Gamma(1-\frac{\eta}{(\lambda-1)})}{(-a(1-\lambda)^{\frac{\beta}{k}}k^{\frac{\alpha+1}{k}-1}}E_{k,\alpha,\beta+k(n\alpha+k-\frac{\eta}{\lambda-1})}^{\gamma,q}\left(w(\frac{x}{-a(\lambda-1)})^{\frac{\alpha}{k}}\right). \tag{3.12}$$

(13) If we let $\mu=\nu=\sigma=\rho=\zeta=p=q=k=1$ and replacing α by $\alpha-1$ in Theorem 2.4, and then by using (1.9), we obtain:

$$P_{0+}^{(\eta,\lambda)}\left[t^{\beta-1}E_{\alpha,\beta}^{\tau,q}(wt^{\alpha})\right](x)=\frac{x^{\eta+\beta+1}\Gamma(1-\frac{\eta}{(\lambda-1)})}{(-a(1-\lambda)^{\beta}}E_{\alpha,\beta+1(n\alpha+1-\frac{\eta}{\lambda-1})}^{\tau,q}\left(w(\frac{x}{-a(\lambda-1)})^{\alpha}\right). \tag{3.13}$$

(14) If we let $\mu=\nu=\sigma=\rho=q=1$ and replacing α by $\alpha-1$ in Theorem 2.4, and then bu using (1.10), we get

$$P_{0+}^{(\eta,\lambda)}\left[t^{\beta-1}E_{\alpha,\beta}^{\tau,\zeta}(wt^{\alpha})\right](x)=\frac{x^{\eta+\beta+1}\Gamma(1-\frac{\eta}{(\lambda-1)})}{(-a(1-\lambda)^{\beta}}E_{\alpha,\beta+1(n\alpha+1-\frac{\eta}{\lambda-1})}^{\tau,\zeta}\left(w(\frac{x}{-a(\lambda-1)})^{\alpha}\right). \tag{3.14}$$

(15) If we let $\mu=\nu=\sigma=\rho=\zeta=p=q=1$ and replacing α by $\alpha-1$ in Theorem 2.4, and then by using (1.11), we attain:

$$P_{0+}^{(\eta,\lambda)}\left[t^{\frac{\beta}{k}-1}E_{k,\alpha,\beta}^{\tau}(wt^{\frac{\alpha}{k}})\right](x)=\frac{x^{\eta+\frac{\beta}{k}+1}\Gamma(1-\frac{\eta}{(\lambda-1)})}{(-a(1-\lambda)^{\frac{\beta}{k}}k^{\frac{\alpha+1}{k}-1}}E_{k,\alpha,\beta+k(n\alpha+k-\frac{\eta}{\lambda-1})}^{\tau}\left(w(\frac{x}{-a(\lambda-1)})^{\frac{\alpha}{k}}\right). \tag{3.15}$$

(16) If we let $\mu=\nu=\sigma=\rho=\zeta=p=q=k=1$ and replacing α by $\alpha-1$ in Theorem 2.4, and then by using (1.12), we obtain:

$$P_{0+}^{(\eta,\lambda)}\left[t^{\beta-1}E_{,\alpha,\beta}^{\tau,q}(wt^{\alpha})\right](x)=\frac{x^{\eta+\beta+1}\Gamma(1-\frac{\eta}{(\lambda-1)})}{(-a(1-\lambda)^{\frac{\beta}{k}}}E_{\alpha,\beta+1(n\alpha+1-\frac{\eta}{\lambda-1})}^{\tau,q}\left(w(\frac{x}{-a(\lambda-1)})^{\alpha}\right). \tag{3.16}$$

(17) If we let $\mu=\nu=\sigma=\rho=\zeta=\tau=p=q=k=1,\alpha=0$ and replacing α by $\alpha-1$ in Theorem 2.4, and then by using (1.13), we find:

$$P_{0+}^{(\eta,\lambda)}\left[t^{\beta-1}E_{\beta}(w)\right](x)=\frac{x^{\eta+\beta+1}\Gamma(1-\frac{\eta}{(\lambda-1)})}{(-a(1-\lambda)^{\beta}}E_{\beta+1(1-\frac{\eta}{\lambda-1})}\left(w(\frac{x}{-a(\lambda-1)})^{\alpha}\right). \tag{3.17}$$

(18) If we let α by $\alpha-1$ in Theorem 2.7, and then by using (**??**), we get:

$$\int_0^1 t^{\frac{\upsilon}{k}-1}(1-t)^{\frac{\xi}{k}-1}[at+b(1-t)]^{\frac{-\upsilon-\xi}{k}}E_{k,\alpha,\beta,\nu,\sigma,\zeta,p}^{\mu,\rho,\tau,q}\left[\frac{-2abt(1-t)}{(at+b(1-t))^2}\right]^{\frac{1}{k}}dt$$

$$=\frac{\Gamma_k(\zeta)\Gamma_k(\mu)}{\Gamma_k(\tau)\Gamma_k(\mu)a^{\tau}b^{\lambda}}\sum_{s=0}^{\infty}\frac{\Gamma_k(\mu+s\rho)\Gamma_k(\gamma+sq)(2)^{\frac{s}{k}}a^{\frac{s}{k}}b^{\frac{s}{k}}}{\Gamma_k(s\beta+\alpha+1)\Gamma_k(\zeta+ps)\Gamma_k(\nu+s\sigma)\Gamma}\frac{\Gamma_k(\upsilon+s)\Gamma_k(\lambda+s)}{\Gamma_k(\upsilon+\lambda+2s)} \tag{3.18}$$

(19) If we let $\mu=\nu=\sigma=\rho=k=1$ and replacing α by $\alpha-1$ in Theorem 2.7, and then by using (1.7), we get:

$$\int_0^1 t^{\upsilon-1}(1-t)^{\xi-1}[at+b(1-t)]^{-\upsilon-\xi}E_{\alpha,\beta,p}^{\tau,\zeta,q}\left[\frac{-2abt(1-t)}{(at+b(1-t))^2}\right]dt={}_4\Psi_3\left[\begin{array}{c}(\tau,q),(\upsilon,1),(\lambda,1),(1,1);\\ \\ (\alpha,\beta),(\zeta,p),(\upsilon+\lambda,2),;\end{array}\quad -2ab\right]. \tag{3.19}$$

(20) If we let $\mu=\nu=\sigma=\rho=\zeta=p=1$ and replacing α by $\alpha-1$ in Theorem 2.7, and then by using (1.8), we get:

$$\int_0^1 t^{\frac{\upsilon}{k}-1}(1-t)^{\frac{\xi}{k}-1}[at+b(1-t)]^{\frac{-\upsilon-\xi}{k}}E_{k,\alpha,\beta}^{\tau,q}\left[\frac{-2abt(1-t)}{(at+b(1-t))^2}\right]^{\frac{1}{k}}dt$$

$$= \frac{1}{\Gamma_k(\tau)a^{\upsilon}b^{\lambda}} \sum_{s=0}^{\infty} \frac{\Gamma_k(1+s)\Gamma_k(\tau+sq)(2)^{\frac{s}{k}}a^{\frac{s}{k}}b^{\frac{s}{k}}}{\Gamma_k(s\beta+\alpha+1)\Gamma_k(1+s)\Gamma_k(1+s)\Gamma} \frac{\Gamma_k(\upsilon+s)\Gamma_k(\lambda+s)}{\Gamma_k(\upsilon+\lambda+2s)}. \quad (3.20)$$

(21) If we let $\mu = \nu = \sigma = \rho = \zeta = p = k = 1$ and replacing α by $\alpha - 1$ in Theorem 2.7,and then by using (1.9), we attain:

$$\int_0^1 t^{\upsilon-1}(1-t)^{\xi-1}[at+b(1-t)]^{-\upsilon-\xi} E_{\alpha,\beta}^{\tau,q}\left[\frac{2abt(1-t)}{(at+b(1-t))^2}\right] dt = {}_3\Psi_2 \left[\begin{array}{cc} (\tau,q),(\upsilon,1),(\lambda,1); & \\ & -2ab \\ (\alpha,\beta),(\upsilon+\lambda,2),; & \end{array}\right]. \quad (3.21)$$

(22) If we let $\mu = \nu = \sigma = \rho = q = p = k = 1$ and replacing α by $\alpha - 1$ in Theorem 2.7,and then by using (1.10), we attain:

$$\int_0^1 t^{\upsilon-1}(1-t)^{\xi-1}[at+b(1-t)]^{-\upsilon-\xi} E_{\alpha,\beta}^{\tau,\zeta}\left[\frac{2abt(1-t)}{(at+b(1-t))^2}\right] dt = {}_3\Psi_3 \left[\begin{array}{cc} (\tau,1),(\upsilon,1),(\lambda,1); & \\ & -2ab \\ (\alpha,\beta),(\upsilon+\lambda,2),(\zeta,1),; & \end{array}\right]. \quad (3.22)$$

(23) If we let $\mu = \nu = \sigma = \rho = \zeta = p = q = 1$ and replacing α by $\alpha - 1$ in Theorem 2.7, and then by using (1.11), we attain:

$$\int_0^1 t^{\frac{\upsilon}{k}-1}(1-t)^{\frac{\xi}{k}-1}[at+b(1-t)]^{\frac{-\upsilon-\xi}{k}} E_{k,\alpha,\beta}^{\tau}\left[\frac{-2abt(1-t)}{(at+b(1-t))^2}\right]^{\frac{1}{k}} dt$$

$$= \frac{1}{\Gamma_k(\tau)a^{\upsilon}b^{\lambda}} \sum_{s=0}^{\infty} \frac{\Gamma_k(1+s)\Gamma_k(\upsilon+s)(2)^{\frac{s}{k}}a^{\frac{s}{k}}b^{\frac{s}{k}}}{\Gamma_k(s\beta+\alpha+1)\Gamma_k(1+s)\Gamma_k(1+s)\Gamma} \frac{\Gamma_k(\upsilon+s)\Gamma_k(\lambda+s)}{\Gamma_k(\upsilon+\lambda+2s)}. \quad (3.23)$$

(24) If we let $\mu = \nu = \sigma = \rho = \zeta = p = q = k = 1$ and replacing α by $\alpha - 1$ in Theorem 2.7, and then by using (1.12), we obtain:

$$\int_0^1 t^{\upsilon-1}(1-t)^{\xi-1}[at+b(1-t)]^{-\upsilon-\xi} E_{\alpha,\beta}^{\tau}\left[\frac{-2abt(1-t)}{(at+b(1-t))^2}\right] dt = {}_3\Psi_2 \left[\begin{array}{cc} (\tau,1),(\upsilon,1),(\lambda,1); & \\ & -2ab \\ (\alpha,\beta),(\upsilon+\lambda,2),; & \end{array}\right]. \quad (3.24)$$

(25) If we let $\mu = \nu = \sigma = \rho = \zeta = \tau = p = q = k = 1$ and replacing α by $\alpha - 1$ in Theorem 2.7, and then by using (1.13) we obtain

$$\int_0^1 t^{\upsilon-1}(1-t)^{\xi-1}[at+b(1-t)]^{-\upsilon-\xi} E_{\alpha,\beta}\left[\frac{-2abt(1-t)}{(at+b(1-t))^2}\right] dt = {}_3\Psi_2 \left[\begin{array}{cc} (1,1),(\upsilon,1),(\lambda,1); & \\ & -2ab \\ (\alpha,\beta),(\upsilon+\lambda,2),; & \end{array}\right]. \quad (3.25)$$

(26) If we let $\mu = \nu = \sigma = \rho = \zeta = \tau = p = q = k = 1, \alpha = 0$ and replacing α by $\alpha - 1$ in Theorem 2.7, and then by using (1.14) we obtain:

$$\int_0^1 t^{\upsilon-1}(1-t)^{\xi-1}[at+b(1-t)]^{-\upsilon-\xi} E_{\beta}\left[\frac{-2abt(1-t)}{(at+b(1-t))^2}\right] dt = {}_3\Psi_2 \left[\begin{array}{cc} (1,1),(\upsilon,1),(\lambda,1); & \\ & -2ab \\ (0,\beta),(\upsilon+\lambda,2),; & \end{array}\right]. \quad (3.26)$$

4. **Conclusion**

In the present article, we derive a new generalization of k-Beseel Maitland function and obtain the fractional calculus formula for the same. We also define and study a new fractional integral operators, which contain the extended Bessel Maitland function. If $k = 0$, then all the results of extended Bessel Maitland function will lead to the well-known results of Bessel Maitland function (see [9]).

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References

[1] Agarwal, P; Purohit, S. D . The unified pathway fractional integral formulae, Fract. Calc. Appl., **4**(9)(2013), 1-8.

[2] Bairwa, R. K; Sharma, S. C. Certain properties and integral transforms of the k-Generalized Mittag-leffler type function, J. Int. Acad. Physical Sciences, **19**(4)(2015), 277-294.

[3] Chaudhry, M. A; Qadir, A; Rafiq, M; Zubair, S. M. Extension of Euler's Beta function, J. Comput. Appl. Math., **78**(1)(1997),19-32.

[4] Chand, M., Prajapati, J.C., Bonyah, E. Fractional integral and solution of fractional kinetic equation involving k-MittagLeffler function. Trans. A. Razmadze Math. Inst. 171(2017), 144166.

[5] Choi, J; Agrawal, P. Certain unified integrals associated with special functions, Boundary Value Problems, Vol. 2013(2013),(95).

[6] Dorrego, G. A; Cerutti, R. A. The k-Mittag-Leffer function, Int. J. Contemp. Math. Sci., **7** (2012), 705-716.

[7] Ghayasuddin, M; Khan, W. A; Araci, S. A new extension of Bessel Maitland function and its properties, Matematicki, Vesnik, Mathe. bechnk, **70**(4)(2018), 292-302.

[8] Khan, M. A., Ahmed, S. On some properties of the generalized Mittag-Leffler function, Springer Plus, 2:(2013),337.

[9] Khan, A. W; Khan, A. I; Ahmad, M. On Certain integral transforms involving generalized Bessel-Baitland function, J. Appl. Pure Math., **2** (1-2)(2020), 63-78.

[10] Luke, Y. L. The Special functions and their approximations,vol.1, New York, Academic Press 1969.

[11] MacRobert, T. M. Beta functions formulae and integral involving E-function Math. Annalen, **142**(1961),450-452.

[12] Marichev, Ö. I. Handbook of integral transform and higher transcendental function. Theory Algorithm Tables, Ellis Horwood, Chichester [John Wiley and sons], New York 1983.

[13] Mathai, A. M. A pathway to matrix-variate gamma and normal densities, Linear Algebra Appl., **396**(2005),317-328.

[14] Mathai, A. M; Haubold, H. J. Pathway model super statistics, trellis statistics and generalized measure of entropy, Phys. A. **375**(2007), 110-122.

[15] Mathai, A. M; Haubold, H. J. On generalized distributions and pathways phys. LCH. A. **372**(2008), 2019-2113.

[16] Mittag-Leffler, G. M, and Sur la. Nouvelle function $E_\alpha(x)$, C. R .Acad. Sci Paris, **137**(1903), 554-558.

[17] Nair, S. S. Pathway fractional integral operator, Fract. Calc. Appl, Anal, **12**(3)(2009), 237-252.

[18] Nisar, K. S; Mondal, S. R. Pathway fractional integral operators involving k-Struve function, arXIV: 1611;(2016) 09157[math. C. A].

[19] Nisar, K. S; Mondal, S. R; Agraval, P. Pathway fractional integral operator associated with Struve function of first kind, Advanced Studies Contemporary Math. **26**(2016),63-70.

[20] Nisar, K. S; Eata, A. F; Dhatallah, M; Choi, J. fractional calculus of generalized k-Mittag-Leffler function and its application, Adv. Diff. Equations, **1**(2016), 304.

[21] Rainville, E. D. Special functions,The Macmillan Company, New York 1960.

[22] Prabhakar, T. R. A singular integral equation with a generalized Mittag-Leffler function in the kernal, Yokohama Math. J., **19**(1971), 7-15.

[23] Salim, T. O; Faraj, W. A generalization of Mittag-Leffler function and integral operator associated with fractional calculus, Appl. Math. Comput., **3**(5)(2012),1-13.

[24] Salim, T. O. Some properties relating to the generalized Mittag-Leffler function, Adv. Appl. Math. Anal., **4**(2009), 21-80.

[25] Singh. M; Khan. M; Khan A. H. On some properties of a generalization of Bessel Maitland function, Int. J. Math. Trends Tech., **14**(1)(2016), 46-54.

[26] Shukla, A.K; Prajapati, J. C. On a generalization of Mittag-Leffler function and its properties, J. Math. Anal. Appl., **336**(2007), 797-811.

[27] Watson, G. N. A treatise on the theory of bessel functions, Cambridge University Press 1962.

[28] Wiman, A. Uber den fundamental satz in der theory der funktionen, Acta Math., **29**(1905), 191-201.

[29] Wright, E. M. The asymptotic expansion of the generalized hypergeometric function, J. Lond. Math. Soc, **10**(1935), 286-293.

[30] Wright, E. M. The asymptotic expansion of integral functions defined by Taylor series, Philos. Trans. R. Soc. Lond. A, **238**(1940), 435-451.

[31] Wright, E. M. The asymptotic expansion of the generalized hypergeometric function II, Proc. Lond. Math. Soc, **46**(1940), 389.

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