# Numerical approximation with Newton polynomial for the solution of a tumor growth model including fractional differential operators

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**Abstract**

In this study, a mathematical model about tumor growth is handled and this model is modified with new differential and integral operators. Numerical method with Newton polynomial which is introduced by Atangana and Seda is used for numerical solution of this model. Also numerical simulations are presented to show the accuracy and the effectiveness of the method.

**Keywords:** Numerical method, fractional derivative and integral, tumor growth model.

### Kesirli diferansiyel operatörler içeren bir tümör büyüme modelinin çözümü için Newton polinomlu nümerik yaklaşım

**Öz**

Bu çalışmada, tümör büyüme ile ilgili bir matematiksel model ele alınır ve bu modeli yeni diferansiyel ve integral operatörlerle modifiye edilir. Bu modelin nümerik çözümü için Atangana ve Seda tarafından tanıtılan Newton polinomuna sahip bir nümerik metot kullanılır. Ayrıca metodun etkinliğini ve doğruluğunu göstermek için nümerik simülasyonlar sunulur.

**Anahtar Kelimeler:** Nümerik metot, kesirli türev ve integral, tümör büyüme modeli.

## 1. Introduction

Mathematical modeling makes important contributions in understanding and analyzing biological phenomena. Therefore, both the modeling and biological processes have become the focus of many researchers (Kim et al. 2007; Barillot et al. 2013). Such models are often constructed with classical derivative. Using the new mathematical concepts called as fractional differential operators, these mathematical models can be modified with these new concepts. Thus, we can predict and understand events in nature

thanks to these concepts which are Caputo, Caputo-Fabrizio and Atangana-Baleanu fractional derivatives (Caputo et. al., 2015; Atangana et al., 2016).

However, we need to find exact solutions these mathematical models to predict and interpret these biological processes. But since these models are mostly nonlinear, it is quite difficult to obtain exact solutions of these models. That's why we need to find numerical solutions of such models. One can find many numerical methods that are based on some polynomials, for instance Lagrange

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polynomial, Newton polynomial, Legendre polynomials, Bernstein polynomial and so on (Owolabi et al., 2020; Mekkaoui et al., 2017). Subaşı vd. (2017; 2019) used Galerkin method for solving differential equations in their studies.

Toufik and Atangana (2017) constructed an efficient numerical method which based on Lagrange polynomial. Recently, Atangana and Araz (2021) presented a numerical method with Newton polynomial which is quite efficient, practical and accurate. Also they presented the error analysis for the suggested method. Alkahtani (2020) applied the newly introduced method with Newton polynomial to fractional nonlinear differential equations.

When the literature is examined, one can find many cancer models that are handled different perspectives of cancer. Watanabe et al. (2016) introduced a mathematical model of tumor growth and its response to single irradiation in their studies. In this study, we modify this model with new mathematical tools. We present new numerical method to find numerical solution of such models. Also we provide numerical simulation about solution of this model.

Let us give definition of these new fractional derivatives and integrals (Caputo et. al., 2015; Atangana et al., 2016).
The Caputo derivative is

$$_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t f^{(n)}(\tau)(t-\tau)^{\alpha-n+1} d\tau .(1)$$

The Caputo-Fabrizio derivative is

$$_0^{CF} D_t^\alpha f(t) = \frac{M(\alpha)}{1-\alpha} \int_0^t f'(\tau) exp\left[-\alpha \frac{(t-\tau)}{1-\alpha}\right] d\tau .(2)$$

The Atangana-Baleanu derivative is

$$_0^{ABC} D_t^\alpha f(t) = \frac{B(\alpha)}{1-\alpha} \int_0^t f'(\tau) E_\alpha \left[-\alpha \frac{(t-\tau)^\alpha}{1-\alpha}\right] d\tau .(3)$$

The associate fractional integrals are given as:

$$_a^C I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau)(t-\tau)^{\alpha-1} d\tau . \qquad (4)$$

$$_0^{CF} I_t^\alpha f(t) = \frac{1-\alpha}{M(\alpha)} f(t) + \frac{\alpha}{M(\alpha)} \int_0^t f(\tau) d\tau \qquad (5)$$

where $M(\alpha)$ is normalization function and $M(0) = 1$, $M(1) = 1$.

$$_0^{AB} I_t^\alpha f(t) = \frac{1-\alpha}{B(\alpha)} f(t)$$

$$+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1} d\tau . \qquad (6)$$

where $B(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$ is normalization function and $B(0) = 1, B(1) = 1$.

The organization of this paper is as follows. In section 2, we present the new model including fractional operators. Also for such model with these operators, we establish numerical scheme which contains two step Newton polynomial. In section 3, we depict numerical simulations for the solution of this model. In section 4, we discuss the obtained results about the considered model.

## 2. Material and method
Watanabe et al. (2016) introduced a tumor growth model for the time evolution of the tumor volume before and after a radio surgical procedure.
In this study, we will deal with the model which is taken into account the growth of tumor after a single instantaneous irradiation of dose $D$ given to the tumor at time $t$.

We deal with for the tumor growth model which is given by

$$\begin{cases} \dfrac{dx}{dt} = z(t)k(D)x(t) - l(D)x(t) \\[2mm] \dfrac{dy}{dt} = l(D)x(t) - a_d y(t) \\[2mm] \dfrac{dz}{dt} = -vz(0)z(t). \end{cases} \quad (7)$$

where the derivative is the classical derivative.

For equation, we have the following functions and parameters for the considered model.

$D$: Radiation dose

$x(t)$: The volume of proliferation tumor

$y(t)$: The volume composed of non-dividing cells

$z(t)$: The tumor growth rate

$z(0)$: The initial tumor growth rate

$a_d$: The cell clearance rate

$v$: The vascular growth retardation factor

$k(D)$: The cell proliferation probability

$l(D)$: The non-dividing state with a rate

We will modify this model using the newly introduced fractional differential operators which are Caputo, Caputo-Fabrizio and Atangana-Baleanu derivative. We start with Caputo-Fabrizio case

$$\begin{cases} {}^{CF}_0 D_t^\alpha x(t) = z(t)k(D)x(t) - l(D)x(t) \\ {}^{CF}_0 D_t^\alpha y(t) = l(D)x(t) - a_d y(t) \\ {}^{CF}_0 D_t^\alpha z(t) = -vz(0)z(t). \end{cases} \quad (8)$$

For brevity, we write

$$\begin{cases} {}^{CF}_0 D_t^\alpha x(t) = X(x,y,z,t) \\ {}^{CF}_0 D_t^\alpha y(t) = Y(x,y,z,t) \\ {}^{CF}_0 D_t^\alpha z(t) = Z(x,y,z,t). \end{cases} \quad (9)$$

Applying Caputo-Fabrizio fractional integral, we have

$$x(t) = x(0) + \frac{1-\alpha}{M(a)} X(x,y,z,t) \quad (10)$$
$$+ \frac{\alpha}{M(a)} \int_0^t X(x,y,z,\tau)d\tau$$

$$y(t) = y(0) + \frac{1-\alpha}{M(a)} Y(x,y,z,t)$$
$$+ \frac{\alpha}{M(a)} \int_0^t Y(x,y,z,\tau)d\tau$$

$$z(t) = z(0) + \frac{1-\alpha}{M(a)} Z(x,y,z,t)$$
$$+ \frac{\alpha}{M(a)} \int_0^t Z(x,y,z,\tau)d\tau$$

At the point $t = t_{k+1}$

$$x(t_{k+1}) = x(0) + \frac{1-\alpha}{M(a)} X(x^k,y^k,z^k,t_k) \quad (11)$$
$$+ \frac{\alpha}{M(a)} \int_0^{t_{k+1}} X(x,y,z,\tau)d\tau$$

$$y(t_{k+1}) = y(0) + \frac{1-\alpha}{M(a)} Y(x^k,y^k,z^k,t_k)$$
$$+ \frac{\alpha}{M(a)} \int_0^{t_{k+1}} Y(x,y,z,\tau)d\tau$$

$$z(t_{k+1}) = z(0) + \frac{1-\alpha}{M(a)} Z(x^k,y^k,z^k,t_k)$$
$$+ \frac{\alpha}{M(a)} \int_0^{t_{k+1}} Z(x,y,z,\tau)d\tau$$

and at the point $t = t_k$, we have

$$x(t_k) = x(0) + \frac{1-\alpha}{M(a)} X(x^{k-1},y^{k-1},z^{k-1},t_{k-1}) \quad (12)$$
$$+ \frac{\alpha}{M(a)} \int_0^{t_k} X(x,y,z,\tau)d\tau$$

$$y(t_k) = y(0) + \frac{1-\alpha}{M(a)} Y(x^{k-1},y^{k-1},z^{k-1},t_{k-1})$$
$$+ \frac{\alpha}{M(a)} \int_0^{t_k} Y(x,y,z,\tau)d\tau$$

$$z(t_k) = z(0) + \frac{1-\alpha}{M(a)} Z(x^{k-1},y^{k-1},z^{k-1},t_{k-1})$$
$$+ \frac{\alpha}{M(a)} \int_0^{t_k} Z(x,y,z,\tau)d\tau.$$

Thus we have

$$x(t_{k+1}) = x(t_k) + \frac{1-\alpha}{M(a)} [X(x^k,y^k,z^k,t_k)$$
$$- X(x^{k-1},y^{k-1},z^{k-1},t_{k-1})]$$
$$+ \frac{\alpha}{M(a)} \int_{t_k}^{t_{k+1}} X(x,y,z,\tau)d\tau \quad (13)$$

$$y(t_{k+1}) = y(t_k) + \frac{1-\alpha}{M(a)}[Y(x^k, y^k, z^k, t_k)$$
$$- Y(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1})]$$
$$+ \frac{\alpha}{M(a)} \int_{t_k}^{t_{k+1}} Y(x, y, z, \tau) d\tau$$

$$z(t_{k+1}) = z(t_k)$$
$$+ \frac{1-\alpha}{M(a)}[Z(x^k, y^k, z^k, t_k)$$
$$- Z(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1})]$$
$$+ \frac{\alpha}{M(a)} \int_{t_k}^{t_{k+1}} Z(x, y, z, \tau) d\tau$$

The function $X(x, y, z, \tau)$ taken place on the right hand side of first equation can be approximated as

$$X(x, y, z, \tau) = X(x^{k-2}, y^{k-2}, z^{k-2}, t_{k-2}) \quad (14)$$
$$+ \left[ \frac{X(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1})}{\Delta t} \right.$$
$$\left. - \frac{X(x^{k-2}, y^{k-2}, z^{k-2}, t_{k-2})}{\Delta t} \right] \times (\tau - t_{k-2})$$
$$+ \left[ \frac{X(x^k, y^k, z^k, t_k)}{2(\Delta t)^2} \right.$$
$$- \frac{2X(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1})}{2(\Delta t)^2}$$
$$\left. + \frac{X(x^{k-2}, y^{k-2}, z^{k-2}, t_{k-2})}{2(\Delta t)^2} \right]$$
$$\times (\tau - t_{k-2})(\tau - t_{k-1})$$

and we can do same routine for the functions $Y(x, y, z, \tau)$ and $Z(x, y, z, \tau)$. Then we can revise the above equation as follows

$$x(t_{k+1}) = x(0) + \frac{1-\alpha}{M(a)} \left[ \begin{array}{c} X(x^k, y^k, z^k, t_k) \\ -X(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1}) \end{array} \right]$$
$$+ \frac{\alpha}{M(a)} \int_{t_k}^{t_{k+1}} X(x^{k-2}, y^{k-2}, z^{k-2}, t_{k-2}) d\tau \quad (15)$$
$$+ \frac{\alpha}{M(a)} \int_{t_k}^{t_{k+1}} \left[ \frac{X(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1})}{\Delta t} \right.$$
$$\left. - \frac{X(x^{k-2}, y^{k-2}, z^{k-2}, t_{k-2})}{\Delta t} \right]$$
$$\times (\tau - t_{k-2}) d\tau$$
$$+ \frac{\alpha}{M(a)} \int_{t_k}^{t_{k+1}} \left[ \frac{X(x^k, y^k, z^k, t_k)}{2(\Delta t)^2} \right.$$
$$- \frac{2X(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1})}{2(\Delta t)^2}$$
$$\left. + \frac{X(x^{k-2}, y^{k-2}, z^{k-2}, t_{k-2})}{2(\Delta t)^2} \right]$$
$$\times (\tau - t_{k-2})(\tau - t_{k-1}) d\tau$$

$$y(t_{k+1})$$
$$= y(0) + \frac{1-\alpha}{M(a)} \left[ \begin{array}{c} Y(x^k, y^k, z^k, t_k) \\ -Y(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1}) \end{array} \right]$$
$$+ \frac{\alpha}{M(a)} \int_{t_k}^{t_{k+1}} Y(x^{k-2}, y^{k-2}, z^{k-2}, t_{k-2}) d\tau$$

$$+ \frac{\alpha}{M(a)} \int_{t_k}^{t_{k+1}} \left[ \frac{Y(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1})}{\Delta t} \right.$$
$$\left. - \frac{Y(x^{k-2}, y^{k-2}, z^{k-2}, t_{k-2})}{\Delta t} \right]$$
$$\times (\tau - t_{k-2}) d\tau$$
$$+ \frac{\alpha}{M(a)} \int_{t_k}^{t_{k+1}} \left[ \frac{Y(x^k, y^k, z^k, t_k)}{2(\Delta t)^2} \right.$$
$$- \frac{2Y(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1})}{2(\Delta t)^2}$$
$$\left. + \frac{Y(x^{k-2}, y^{k-2}, z^{k-2}, t_{k-2})}{2(\Delta t)^2} \right]$$
$$\times (\tau - t_{k-2})(\tau - t_{k-1}) d\tau$$

$$z(t_{k+1})$$
$$= z(0) + \frac{1-\alpha}{M(a)} \left[ \begin{array}{c} Z(x^k, y^k, z^k, t_k) \\ -Z(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1}) \end{array} \right]$$
$$+ \frac{\alpha}{M(a)} \int_{t_k}^{t_{k+1}} Z(x^{k-2}, y^{k-2}, z^{k-2}, t_{k-2})(t_{k+1} - \tau)^{a-1} d\tau$$
$$+ \frac{\alpha}{M(a)} \int_{t_k}^{t_{k+1}} \left[ \frac{Z(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1})}{\Delta t} \right.$$
$$\left. - \frac{Z(x^{k-2}, y^{k-2}, z^{k-2}, t_{k-2})}{\Delta t} \right]$$
$$\times (\tau - t_{k-2}) d\tau$$
$$+ \frac{\alpha}{M(a)} \int_{t_k}^{t_{k+1}} \left[ \frac{Z(x^k, y^k, z^k, t_k)}{2(\Delta t)^2} \right.$$
$$- \frac{2Z(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1})}{2(\Delta t)^2}$$
$$\left. + \frac{Z(x^{k-2}, y^{k-2}, z^{k-2}, t_{k-2})}{2(\Delta t)^2} \right]$$
$$\times (\tau - t_{k-2})(\tau - t_{k-1}) d\tau$$

For the integrals on the right hand side of the above equations, we have the following

$$\int_{t_k}^{t_{k+1}} (\tau - t_{k-2}) d\tau = \frac{5}{2}(\Delta t)^2$$
$$\int_{t_k}^{t_{k+1}} (\tau - t_{k-2})(\tau - t_{k-1}) d\tau = \frac{23}{6}(\Delta t)^3 \quad (16)$$

Replacing these results into the above system,

$$x(t_{k+1}) = x(0) + \frac{1-\alpha}{M(a)}[X(x^k, y^k, z^k, t_k)$$
$$- X(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1})]$$
$$+ \frac{\alpha}{M(a)} X(x^{k-2}, y^{k-2}, z^{k-2}, t_{k-2}) \Delta t \quad (17)$$
$$+ \frac{\alpha}{M(a)}[X(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1})$$
$$- X(x^{k-2}, y^{k-2}, z^{k-2}, t_{k-2})] \frac{5}{2} \Delta t$$

$$+ \frac{\alpha}{M(a)} [X(x^k, y^k, z^k, t_k)$$
$$- 2X(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1})$$
$$+ X(x^{k-2}, y^{k-2}, z^{k-2}, t_{k-2})] \frac{23}{12} \Delta t$$

$$y(t_{k+1}) = y(0) + \frac{1-\alpha}{M(a)} [Y(x^k, y^k, z^k, t_k)$$
$$- Y(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1})]$$
$$+ \frac{\alpha}{M(a)} Y(x^{k-2}, y^{k-2}, z^{k-2}, t_{k-2}) \Delta t$$
$$+ \frac{\alpha}{M(a)} [Y(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1})$$
$$- Y(x^{k-2}, y^{k-2}, z^{k-2}, t_{k-2})] \frac{5}{2} \Delta t$$

$$+ \frac{\alpha}{M(a)} [Y(x^k, y^k, z^k, t_k)$$
$$- 2Y(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1})$$
$$+ Y(x^{k-2}, y^{k-2}, z^{k-2}, t_{k-2})] \frac{23}{12} \Delta t$$

$$z(t_{k+1}) = z(0) + \frac{1-\alpha}{M(a)} [Z(x^k, y^k, z^k, t_k)$$
$$- Z(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1})]$$
$$+ \frac{\alpha}{M(a)} Z(x^{k-2}, y^{k-2}, z^{k-2}, t_{k-2}) \Delta t$$
$$+ \frac{\alpha}{M(a)} [Z(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1})$$
$$- Z(x^{k-2}, y^{k-2}, z^{k-2}, t_{k-2})] \frac{5}{2} \Delta t$$
$$+ \frac{\alpha}{M(a)} [Z(x^k, y^k, z^k, t_k)$$
$$- 2Z(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1})$$
$$+ Z(x^{k-2}, y^{k-2}, z^{k-2}, t_{k-2})] \frac{23}{12} \Delta t$$

Thus we can have the following numerical scheme the considered model with Caputo-Fabrizio

$$x(t_{k+1}) = x(t_k) + \frac{1-\alpha}{M(a)} [X(x^k, y^k, z^k, t_k) - X(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1})] \quad (18)$$

$$+ \frac{\alpha}{M(a)} \begin{bmatrix} \frac{23}{12} X(x^k, y^k, z^k, t_k) \Delta t \\ -\frac{4}{3} X(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1}) \Delta t \\ +\frac{5}{2} X(x^{k-2}, y^{k-2}, z^{k-2}, t_{k-2}) \Delta t \end{bmatrix}$$

$$y(t_{k+1}) = y(t_k) + \frac{1-\alpha}{M(a)} [Y(x^k, y^k, z^k, t_k)$$
$$- Y(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1})]$$

$$+ \frac{\alpha}{M(a)} \begin{bmatrix} \frac{23}{12} Y(x^k, y^k, z^k, t_k) \Delta t \\ -\frac{4}{3} Y(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1}) \Delta t \\ +\frac{5}{2} Y(x^{k-2}, y^{k-2}, z^{k-2}, t_{k-2}) \Delta t \end{bmatrix}$$

$$z(t_{k+1}) = z(t_k) + \frac{1-\alpha}{M(a)} [Z(x^k, y^k, z^k, t_k)$$
$$- Z(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1})]$$

$$+ \frac{\alpha}{M(a)} \begin{bmatrix} \frac{23}{12} Z(x^k, y^k, z^k, t_k) \Delta t \\ -\frac{4}{3} Z(x^{k-1}, y^{k-1}, z^{k-1}, t_{k-1}) \Delta t \\ +\frac{5}{2} Z(x^{k-2}, y^{k-2}, z^{k-2}, t_{k-2}) \Delta t \end{bmatrix}$$

Now, we handle the following model having Atangana-Baleanu derivative

$$\begin{cases} {}^{ABC}_0 D_t^\alpha x(t) = X(x, y, z, t) \\ {}^{ABC}_0 D_t^\alpha y(t) = Y(x, y, z, t) \\ {}^{ABC}_0 D_t^\alpha z(t) = Z(x, y, z, t). \end{cases} \quad (19)$$

Integrating each equation, we obtain the following

$$x(t) = x(0) + \frac{1-\alpha}{B(a)} X(x, y, z, t)$$
$$+ \frac{\alpha}{B(a)\Gamma(a)} \int_0^t X(x, y, z, \tau)(t-\tau)^{a-1} d\tau \quad (20)$$

$$y(t) = y(0) + \frac{1-\alpha}{B(a)} Y(x, y, z, t)$$
$$+ \frac{\alpha}{B(a)\Gamma(a)} \int_0^t Y(x, y, z, \tau)(t - \tau)^{a-1} d\tau$$

$$z(t) = z(0) + \frac{1-\alpha}{B(a)} Z(x, y, z, t)$$
$$+ \frac{\alpha}{B(a)\Gamma(a)} \int_0^t Z(x, y, z, \tau)(t - \tau)^{a-1} d\tau$$

At the point $t = t_{k+1}$, we have

$$x(t_{k+1}) = x(0) + \frac{1-\alpha}{B(a)} X(x^k, y^k, z^k, t_k)$$
$$+ \frac{\alpha}{B(a)\Gamma(a)} \sum_{s=2}^k \int_{t_s}^{t_{s+1}} X(x, y, z, \tau) \times (t_{k+1} - \tau)^{a-1} d\tau \quad (21)$$

$$y(t_{k+1}) = y(0) + \frac{1-\alpha}{B(a)} Y(x^k, y^k, z^k, t_k)$$
$$+ \frac{\alpha}{B(a)\Gamma(a)} \sum_{s=2}^k \int_{t_s}^{t_{s+1}} Y(x, y, z, \tau)(t_{k+1} - \tau)^{a-1} d\tau$$

$$z(t_{k+1})$$
$$= z(0) + \frac{1-\alpha}{B(a)} Z(x^k, y^k, z^k, t_k)$$

$$+ \frac{\alpha}{B(a)\Gamma(a)} \sum_{s=2}^{k} \int_{t_s}^{t_{s+1}} Z(x, y, z, \tau)(t_{k+1} - \tau)^{a-1} d\tau$$

After putting Newton polynomial into the above equations, we rearrange the above system as follows

$$x(t_{k+1}) = x(0) + \frac{1-\alpha}{B(a)} X(x^k, y^k, z^k, t_k)$$

$$+ \frac{\alpha}{B(a)\Gamma(a)} \sum_{s=2}^{k} \int_{t_s}^{t_{s+1}} X(x^{s-2}, y^{s-2}, z^{s-2}, t_{s-2}) \quad (22)$$
$$\times (t_{k+1} - \tau)^{a-1} d\tau$$

$$+ \frac{\alpha}{B(a)\Gamma(a)} \sum_{s=2}^{k} \int_{t_s}^{t_{s+1}} + \left[ \frac{X(x^{s-1}, y^{s-1}, z^{s-1}, t_{s-1})}{\Delta t} \right.$$
$$\left. - \frac{X(x^{s-2}, y^{s-2}, z^{s-2}, t_{s-2})}{\Delta t} \right]$$
$$\times (\tau - t_{s-2})(t_{k+1} - \tau)^{a-1} d\tau$$

$$+ \frac{\alpha}{B(a)\Gamma(a)} \sum_{s=2}^{k} \int_{t_s}^{t_{s+1}} \left[ \begin{array}{c} \frac{X(x^s, y^s, z^s, t_s)}{2(\Delta t)^2} \\ - \frac{2X(x^{s-1}, y^{s-1}, z^{s-1}, t_{s-1})}{2(\Delta t)^2} \\ + \frac{X(x^{k-2}, y^{k-2}, z^{k-2}, t_{k-2})}{2(\Delta t)^2} \end{array} \right]$$
$$\times (\tau - t_{s-2})(\tau - t_{s-1})(t_{k+1} - \tau)^{a-1} d\tau$$

$$y(t_{k+1})$$
$$= y(0) + \frac{1-\alpha}{B(a)} Y(x^k, y^k, z^k, t_k)$$

$$+ \frac{\alpha}{B(a)\Gamma(a)} \sum_{s=2}^{k} \int_{t_s}^{t_{s+1}} Y(x^{s-2}, y^{s-2}, z^{s-2}, t_{s-2})$$
$$\times (t_{k+1} - \tau)^{a-1} d\tau$$

$$+ \frac{\alpha}{B(a)\Gamma(a)} \sum_{s=2}^{k} \int_{t_s}^{t_{s+1}} + \left[ \frac{Y(x^{s-1}, y^{s-1}, z^{s-1}, t_{s-1})}{\Delta t} \right.$$
$$\left. - \frac{Y(x^{s-2}, y^{s-2}, z^{s-2}, t_{s-2})}{\Delta t} \right]$$
$$\times (\tau - t_{s-2})(t_{k+1} - \tau)^{a-1} d\tau$$

$$+ \frac{\alpha}{B(a)\Gamma(a)} \sum_{s=2}^{k} \int_{t_s}^{t_{s+1}} \left[ \begin{array}{c} \frac{Y(x^s, y^s, z^s, t_s)}{2(\Delta t)^2} \\ - \frac{2Y(x^{s-1}, y^{s-1}, z^{s-1}, t_{s-1})}{2(\Delta t)^2} \\ + \frac{Y(x^{s-2}, y^{s-2}, z^{s-2}, t_{s-2})}{2(\Delta t)^2} \end{array} \right]$$
$$\times (\tau - t_{s-2})(\tau - t_{s-1})(t_{k+1} - \tau)^{a-1} d\tau$$

$$z(t_{k+1})$$
$$= z(0) + \frac{1-\alpha}{B(a)} Z(x^k, y^k, z^k, t_k)$$

$$+ \frac{\alpha}{B(a)\Gamma(a)} \sum_{s=2}^{k} \int_{t_s}^{t_{s+1}} Z(x^{s-2}, y^{s-2}, z^{s-2}, t_{s-2})$$
$$\times (t_{k+1} - \tau)^{a-1} d\tau$$

$$+ \frac{\alpha}{B(a)\Gamma(a)} \sum_{s=2}^{k} \int_{t_s}^{t_{s+1}} + \left[ \frac{Z(x^{s-1}, y^{s-1}, z^{s-1}, t_{s-1})}{\Delta t} \right.$$
$$\left. - \frac{Z(x^{s-2}, y^{s-2}, z^{s-2}, t_{s-2})}{\Delta t} \right]$$
$$\times (\tau - t_{s-2})(t_{k+1} - \tau)^{a-1} d\tau$$

$$+ \frac{\alpha}{B(a)\Gamma(a)} \sum_{s=2}^{k} \int_{t_s}^{t_{s+1}} \left[ \begin{array}{c} \frac{Z(x^s, y^s, z^s, t_s)}{2(\Delta t)^2} \\ - \frac{2Z(x^{s-1}, y^{s-1}, z^{s-1}, t_{s-1})}{2(\Delta t)^2} \\ + \frac{Z(x^{s-2}, y^{s-2}, z^{s-2}, t_{s-2})}{2(\Delta t)^2} \end{array} \right]$$
$$\times (\tau - t_{s-2})(\tau - t_{s-1})(t_{k+1} - \tau)^{a-1} d\tau$$

We have the following calculations for the above integrals

$$\int_{t_s}^{t_{s+1}} (t_{k+1} - \tau)^{a-1} d\tau = \frac{(\Delta t)^\alpha}{\alpha} [(k - s + 1)^\alpha - (k-s)^\alpha] \quad (23)$$

$$\int_{t_s}^{t_{s+1}} (\tau - t_{s-2})(t_{k+1} - \tau)^{a-1} d\tau$$
$$= \frac{(\Delta t)^\alpha}{\alpha(\alpha+1)} \left[ \begin{array}{c} (k-s+1)^\alpha(k-s+3+2\alpha) \\ -(k-s)^\alpha(k-s+3+3\alpha) \end{array} \right]$$

$$\int_{t_s}^{t_{s+1}} (\tau - t_{s-2})(\tau - t_{s-1})(t_{k+1} - \tau)^{a-1} d\tau$$
$$= \frac{(\Delta t)^\alpha}{\alpha(\alpha+1)(\alpha+2)}$$
$$\times \left[ \begin{array}{c} (k-s+1)^\alpha \left( \begin{array}{c} 2(k-s)^2 + (3\alpha+10)(k-s) \\ +2\alpha^2 + 9\alpha + 12 \end{array} \right) \\ -(k-s)^\alpha \left( \begin{array}{c} 2(k-s)^2 + (5\alpha+10)(k-s) \\ +6\alpha^2 + 18\alpha + 12 \end{array} \right) \end{array} \right]$$

Putting these calculations into above system, the following numerical scheme is written

$$x(t_{k+1}) = x(0) + \frac{1-\alpha}{B(a)} X(x^k, y^k, z^k, t_k) +$$

$$\frac{\alpha(\Delta t)^\alpha}{B(a)\Gamma(a+1)} \sum_{s=2}^{k} X(x^{s-2}, y^{s-2}, z^{s-2}, t_{s-2})[(k - s + 1)^\alpha - (k-s)^\alpha] \quad (24)$$

$$+ \frac{\alpha(\Delta t)^\alpha}{B(a)\Gamma(a+2)} \sum_{s=2}^{k} [X(x^{s-1}, y^{s-1}, z^{s-1}, t_{s-1}) - X(x^{s-2}, y^{s-2}, z^{s-2}, t_{s-2})]$$
$$\times \left[ \begin{array}{c} (k-s+1)^\alpha(k-s+3+2\alpha) \\ -(k-s)^\alpha(k-s+3+3\alpha) \end{array} \right]$$

$$+\frac{\alpha(\Delta t)^{\alpha}}{2B(a)\Gamma(a+3)}\sum_{s=2}^{k}[X(x^{s},y^{s},z^{s},t_{s})$$
$$-2X(x^{s-1},y^{s-1},z^{s-1},t_{s-1})$$
$$+X(x^{s-2},y^{s-2},z^{s-2},t_{s-2})]$$

$$\times\left[\begin{array}{l}(k-s+1)^{\alpha}\binom{2(k-s)^{2}+(3\alpha+10)(k-s)}{+2\alpha^{2}+9\alpha+12}\\-(k-s)^{\alpha}\binom{2(k-s)^{2}+(5\alpha+10)(k-s)}{+6\alpha^{2}+18\alpha+12}\end{array}\right]$$

$$y(t_{k+1})$$
$$=y(0)+\frac{1-\alpha}{B(a)}Y(x^{k},y^{k},z^{k},t_{k})$$

$$+\frac{\alpha(\Delta t)^{\alpha}}{B(a)\Gamma(a+1)}\sum_{s=2}^{k}Y(x^{s-2},y^{s-2},z^{s-2},t_{s-2})[(k-s+1)^{\alpha}-(k-s)^{\alpha}]$$

$$+\frac{\alpha(\Delta t)^{\alpha}}{B(a)\Gamma(a+2)}\sum_{s=2}^{k}[Y(x^{s-1},y^{s-1},z^{s-1},t_{s-1})$$
$$-Y(x^{s-2},y^{s-2},z^{s-2},t_{s-2})]$$
$$\times\left[\begin{array}{l}(k-s+1)^{\alpha}(k-s+3+2\alpha)\\-(k-s)^{\alpha}(k-s+3+3\alpha)\end{array}\right]$$

$$+\frac{\alpha(\Delta t)^{\alpha}}{2B(a)\Gamma(a+3)}\sum_{s=2}^{k}[Y(x^{s},y^{s},z^{s},t_{s})$$
$$-2Y(x^{s-1},y^{s-1},z^{s-1},t_{s-1})$$
$$+Y(x^{s-2},y^{s-2},z^{s-2},t_{s-2})]$$
$$\times\left[\begin{array}{l}(k-s+1)^{\alpha}\binom{2(k-s)^{2}+(3\alpha+10)(k-s)}{+2\alpha^{2}+9\alpha+12}\\-(k-s)^{\alpha}\binom{2(k-s)^{2}+(5\alpha+10)(k-s)}{+6\alpha^{2}+18\alpha+12}\end{array}\right]$$

$$z(t_{k+1})$$
$$=z(0)+\frac{1-\alpha}{B(a)}Z(x^{k},y^{k},z^{k},t_{k})$$

$$+\frac{\alpha(\Delta t)^{\alpha}}{B(a)\Gamma(a+1)}\sum_{s=2}^{k}Z(x^{s-2},y^{s-2},z^{s-2},t_{s-2})[(k-s+1)^{\alpha}-(k-s)^{\alpha}]$$

$$+\frac{\alpha(\Delta t)^{\alpha}}{B(a)\Gamma(a+2)}\sum_{s=2}^{k}[Z(x^{s-1},y^{s-1},z^{s-1},t_{s-1})$$
$$-Z(x^{s-2},y^{s-2},z^{s-2},t_{s-2})]$$
$$\times\left[\begin{array}{l}(k-s+1)^{\alpha}(k-s+3+2\alpha)\\-(k-s)^{\alpha}(k-s+3+3\alpha)\end{array}\right]$$

$$+\frac{\alpha(\Delta t)^{\alpha}}{2B(a)\Gamma(a+3)}\sum_{s=2}^{k}[Z(x^{s},y^{s},z^{s},t_{s})$$
$$-2Z(x^{s-1},y^{s-1},z^{s-1},t_{s-1})$$
$$+Z(x^{s-2},y^{s-2},z^{s-2},t_{s-2})]$$

$$\times\left[\begin{array}{l}(k-s+1)^{\alpha}\binom{2(k-s)^{2}+(3\alpha+10)(k-s)}{+2\alpha^{2}+9\alpha+12}\\-(k-s)^{\alpha}\binom{2(k-s)^{2}+(5\alpha+10)(k-s)}{+6\alpha^{2}+18\alpha+12}\end{array}\right]$$

Now we consider our model with Caputo derivative

$$\begin{cases}{}_{0}^{C}D_{t}^{\alpha}x(t)=X(x,y,z,t)\\{}_{0}^{C}D_{t}^{\alpha}y(t)=Y(x,y,z,t)\\{}_{0}^{C}D_{t}^{\alpha}z(t)=Z(x,y,z,t).\end{cases}\quad(25)$$

We convert each equation as

$$x(t)=x(0)+\frac{1}{\Gamma(a)}\int_{0}^{t}X(x,y,z,\tau)(t-\tau)^{a-1}d\tau$$
$$y(t)=y(0)+\frac{1}{\Gamma(a)}\int_{0}^{t}Y(x,y,z,\tau)(t-\tau)^{a-1}d\tau\quad(26)$$
$$z(t)=z(0)+\frac{1}{\Gamma(a)}\int_{0}^{t}Z(x,y,z,\tau)(t-\tau)^{a-1}d\tau$$

At the point $t=t_{k+1}$, we have

$$x(t_{k+1})=x(0)$$
$$+\frac{1}{\Gamma(a)}\sum_{s=2}^{k}\int_{t_{s}}^{t_{s+1}}X(x,y,z,\tau)(t_{k+1}-\tau)^{a-1}d\tau$$
$$(27)$$

$$y(t_{k+1})=y(0)+\frac{1}{\Gamma(a)}\sum_{s=2}^{k}\int_{t_{s}}^{t_{s+1}}Y(x,y,z,\tau)(t_{k+1}-\tau)^{a-1}d\tau$$

$$z(t_{k+1})=z(0)+\frac{1}{\Gamma(a)}\sum_{s=2}^{k}\int_{t_{s}}^{t_{s+1}}Z(x,y,z,\tau)(t_{k+1}-\tau)^{a-1}d\tau$$

Substituting Newton polynomial into the above equations, we reorder the above system as follows

$$x(t_{k+1})=x(0)+$$
$$\frac{1}{\Gamma(a)}\sum_{s=2}^{k}\int_{t_{s}}^{t_{s+1}}X(x^{s-2},y^{s-2},z^{s-2},t_{s-2})\times$$
$$(t_{k+1}-\tau)^{a-1}d\tau\qquad(28)$$

$$+\frac{1}{\Gamma(a)}\sum_{s=2}^{k}\int_{t_{s}}^{t_{s+1}}+\left[\frac{X(x^{s-1},y^{s-1},z^{s-1},t_{s-1})}{\Delta t}\right.$$
$$\left.-\frac{X(x^{s-2},y^{s-2},z^{s-2},t_{s-2})}{\Delta t}\right]$$
$$\times(\tau-t_{s-2})(t_{k+1}-\tau)^{a-1}d\tau$$

$$+\frac{1}{\Gamma(a)}\sum_{s=2}^{k}\int_{t_s}^{t_{s+1}}\left[\begin{array}{c}\dfrac{X(x^s,y^s,z^s,t_s)}{2(\Delta t)^2}\\[2mm]-\dfrac{2X(x^{s-1},y^{s-1},z^{s-1},t_{s-1})}{2(\Delta t)^2}\\[2mm]+\dfrac{X(x^{s-2},y^{s-2},z^{s-2},t_{s-2})}{2(\Delta t)^2}\end{array}\right]$$
$$\times(\tau-t_{s-2})(\tau-t_{s-1})(t_{k+1}-\tau)^{a-1}d\tau$$

$$y(t_{k+1})=y(0)$$
$$+\frac{1}{\Gamma(a)}\sum_{s=2}^{k}\int_{t_s}^{t_{s+1}}Y(x^{s-2},y^{s-2},z^{s-2},t_{s-2})(t_{k+1}-\tau)^{a-1}d\tau$$
$$+\frac{1}{\Gamma(a)}\sum_{s=2}^{k}\int_{t_s}^{t_{s+1}}\left[+\left[\frac{Y(x^{s-1},y^{s-1},z^{s-1},t_{s-1})}{\Delta t}\right.\right.$$
$$\left.\left.-\frac{Y(x^{s-2},y^{s-2},z^{s-2},t_{s-2})}{\Delta t}\right]\right.$$
$$\times(\tau-t_{s-2})(t_{k+1}-\tau)^{a-1}d\tau$$
$$+\frac{1}{\Gamma(a)}\sum_{s=2}^{k}\int_{t_s}^{t_{s+1}}\left[\begin{array}{c}\dfrac{Y(x^s,y^s,z^s,t_s)}{2(\Delta t)^2}\\[2mm]-\dfrac{2Y(x^{s-1},y^{s-1},z^{s-1},t_{s-1})}{2(\Delta t)^2}\\[2mm]+\dfrac{Y(x^{s-2},y^{s-2},z^{s-2},t_{s-2})}{2(\Delta t)^2}\end{array}\right]$$
$$\times(\tau-t_{s-2})(\tau-t_{s-1})(t_{k+1}-\tau)^{a-1}d\tau$$

$$z(t_{k+1})=z(0)$$
$$+\frac{1}{\Gamma(a)}\sum_{s=2}^{k}\int_{t_s}^{t_{s+1}}Z(x^{s-2},y^{s-2},z^{s-2},t_{s-2})(t_{k+1}-\tau)^{a-1}d\tau$$
$$+\frac{1}{\Gamma(a)}\sum_{s=2}^{k}\int_{t_s}^{t_{s+1}}\left[+\left[\frac{Z(x^{s-1},y^{s-1},z^{s-1},t_{s-1})}{\Delta t}\right.\right.$$
$$\left.\left.-\frac{Z(x^{s-2},y^{s-2},z^{s-2},t_{s-2})}{\Delta t}\right]\right.$$
$$\times(\tau-t_{s-2})(t_{k+1}-\tau)^{a-1}d\tau$$
$$+\frac{1}{\Gamma(a)}\sum_{s=2}^{k}\int_{t_s}^{t_{s+1}}\left[\begin{array}{c}\dfrac{Z(x^s,y^s,z^s,t_s)}{2(\Delta t)^2}\\[2mm]-\dfrac{2Z(x^{s-1},y^{s-1},z^{s-1},t_{s-1})}{2(\Delta t)^2}\\[2mm]+\dfrac{Z(x^{s-2},y^{s-2},z^{s-2},t_{s-2})}{2(\Delta t)^2}\end{array}\right]$$
$$\times(\tau-t_{s-2})(\tau-t_{s-1})(t_{k+1}-\tau)^{a-1}d\tau$$

We have the following

$$\int_{t_s}^{t_{s+1}}(t_{k+1}-\tau)^{a-1}d\tau=\frac{(\Delta t)^\alpha}{\alpha}\left[\begin{array}{c}(k-s+1)^\alpha\\-(k-s)^\alpha\end{array}\right]\qquad(29)$$

$$\int_{t_s}^{t_{s+1}}(\tau-t_{s-2})(t_{k+1}-\tau)^{a-1}d\tau$$
$$=\frac{(\Delta t)^\alpha}{\alpha(\alpha+1)}\left[\begin{array}{c}(k-s+1)^\alpha(k-s+3+2\alpha)\\-(k-s)^\alpha(k-s+3+3\alpha)\end{array}\right]$$

$$\int_{t_s}^{t_{s+1}}(\tau-t_{s-2})(\tau-t_{s-1})(t_{k+1}-\tau)^{a-1}d\tau$$
$$=\frac{(\Delta t)^\alpha}{\alpha(\alpha+1)(\alpha+2)}$$
$$\times\left[\begin{array}{c}(k-s+1)^\alpha\left(\begin{array}{c}2(k-s)^2+(3\alpha+10)(k-s)\\+2\alpha^2+9\alpha+12\end{array}\right)\\-(k-s)^\alpha\left(\begin{array}{c}2(k-s)^2+(5\alpha+10)(k-s)\\+6\alpha^2+18\alpha+12\end{array}\right)\end{array}\right]$$

Replacing these calculations into above system, we obtain the following numerical scheme

$$x(t_{k+1})=x(0)$$
$$+\frac{(\Delta t)^\alpha}{\Gamma(a+1)}\sum_{s=2}^{k}X(x^{s-2},y^{s-2},z^{s-2},t_{s-2})[(k-s+1)^\alpha-(k-s)^\alpha]$$
$$+\frac{(\Delta t)^\alpha}{\Gamma(a+2)}\sum_{s=2}^{k}[X(x^{s-1},y^{s-1},z^{s-1},t_{s-1})-X(x^{s-2},y^{s-2},z^{s-2},t_{s-2})]$$
$$\times\left[\begin{array}{c}(k-s+1)^\alpha(k-s+3+2\alpha)\\-(k-s)^\alpha(k-s+3+3\alpha)\end{array}\right]\qquad(30)$$
$$+\frac{(\Delta t)^\alpha}{2\Gamma(a+3)}\sum_{s=2}^{k}[X(x^s,y^s,z^s,t_s)-2X(x^{s-1},y^{s-1},z^{s-1},t_{s-1})+X(x^{s-2},y^{s-2},z^{s-2},t_{s-2})]$$
$$\times\left[\begin{array}{c}(k-s+1)^\alpha\left(\begin{array}{c}2(k-s)^2+(3\alpha+10)(k-s)\\+2\alpha^2+9\alpha+12\end{array}\right)\\-(k-s)^\alpha\left(\begin{array}{c}2(k-s)^2+(5\alpha+10)(k-s)\\+6\alpha^2+18\alpha+12\end{array}\right)\end{array}\right]$$

$$y(t_{k+1})=y(0)$$
$$+\frac{(\Delta t)^\alpha}{\Gamma(a+1)}\sum_{s=2}^{k}Y(x^{s-2},y^{s-2},z^{s-2},t_{s-2})[(k-s+1)^\alpha-(k-s)^\alpha]$$
$$+\frac{(\Delta t)^\alpha}{\Gamma(a+2)}\sum_{s=2}^{k}[Y(x^{s-1},y^{s-1},z^{s-1},t_{s-1})-Y(x^{s-2},y^{s-2},z^{s-2},t_{s-2})]$$
$$\times\left[\begin{array}{c}(k-s+1)^\alpha(k-s+3+2\alpha)\\-(k-s)^\alpha(k-s+3+3\alpha)\end{array}\right]$$

$$+\frac{(\Delta t)^\alpha}{2\Gamma(a+3)}\sum_{s=2}^{k}[Y(x^s,y^s,z^s,t_s)$$

$$-2Y(x^{s-1},y^{s-1},z^{s-1},t_{s-1})$$
$$+Y(x^{s-2},y^{s-2},z^{s-2},t_{s-2})]$$

$$\times\begin{bmatrix}(k-s+1)^\alpha\begin{pmatrix}2(k-s)^2+(3\alpha+10)(k-s)\\+2\alpha^2+9\alpha+12\end{pmatrix}\\-(k-s)^\alpha\begin{pmatrix}2(k-s)^2+(5\alpha+10)(k-s)\\+6\alpha^2+18\alpha+12\end{pmatrix}\end{bmatrix}$$

$$z(t_{k+1})$$
$$=z(0)$$
$$+\frac{(\Delta t)^\alpha}{\Gamma(a+1)}\sum_{s=2}^{k}Z(x^{s-2},y^{s-2},z^{s-2},t_{s-2})[(k-s+1)^\alpha-(k-s)^\alpha]$$

$$+\frac{(\Delta t)^\alpha}{\Gamma(a+2)}\sum_{s=2}^{k}[Z(x^{s-1},y^{s-1},z^{s-1},t_{s-1})$$
$$-Z(x^{s-2},y^{s-2},z^{s-2},t_{s-2})]$$
$$\times\begin{bmatrix}(k-s+1)^\alpha(k-s+3+2\alpha)\\-(k-s)^\alpha(k-s+3+3\alpha)\end{bmatrix}$$

$$+\frac{(\Delta t)^\alpha}{2\Gamma(a+3)}\sum_{s=2}^{k}[Z(x^s,y^s,z^s,t_s)$$
$$-2Z(x^{s-1},y^{s-1},z^{s-1},t_{s-1})$$
$$+Z(x^{s-2},y^{s-2},z^{s-2},t_{s-2})]$$

$$\times\begin{bmatrix}(k-s+1)^\alpha\begin{pmatrix}2(k-s)^2+(3\alpha+10)(k-s)\\+2\alpha^2+9\alpha+12\end{pmatrix}\\-(k-s)^\alpha\begin{pmatrix}2(k-s)^2+(5\alpha+10)(k-s)\\+6\alpha^2+18\alpha+12\end{pmatrix}\end{bmatrix}$$

## 3. Numerical simulation

In this section, we consider the following model with Atangana-Baleanu derivative

$$\begin{cases}{}^{AB}_{0}D_t^\alpha x(t)=z(t)k(D)x(t)-l(D)x(t)\\{}^{AB}_{0}D_t^\alpha y(t)=l(D)x(t)-a_d y(t)\\{}^{AB}_{0}D_t^\alpha z(t)=-vz(0)z(t).\end{cases}\quad(31)$$

Numerical simulations are performed for different alphas in Figure 1, 2, 3 and 4 with the parameters $k(D)=1, l(D)=0.7, a_d=0.6, v=0.8$. Also initial conditions are

$$x(0)=10, z(0)=10, z(0)=2.$$



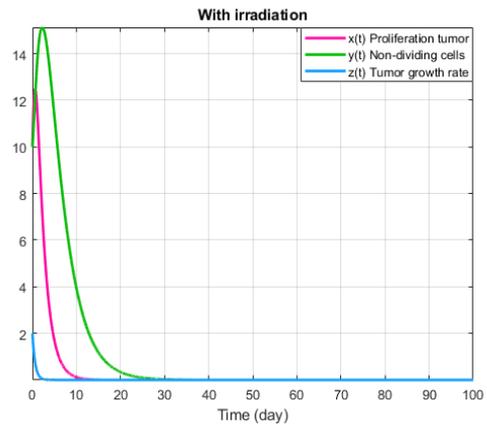**Figure 1.** Numerical simulation for tumor growth model for $\alpha=1$.
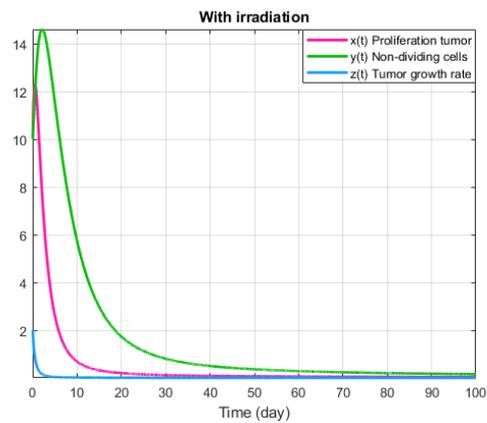


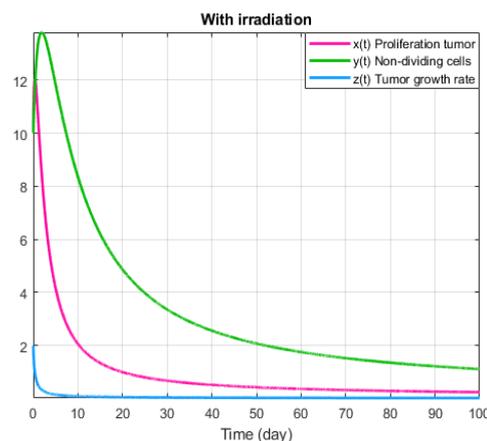**Figure 2.** Numerical simulation for tumor growth model for $\alpha=0.9$.



**Figure 3.** Numerical simulation for tumor growth model for $\alpha=0.72$.
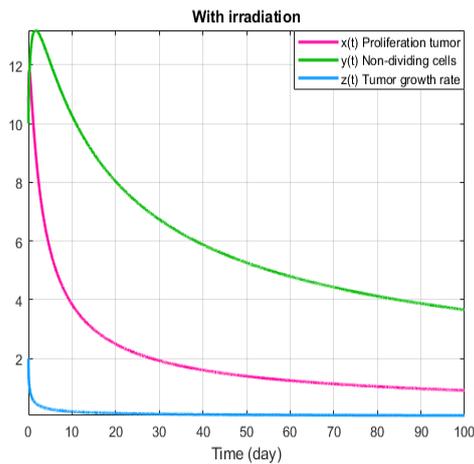
**Figure 4.** Numerical simulation for tumor growth model for $\alpha = 0.54$.

## 4. Conclusion

In this study, we deal with a tumor growth model to discuss the effect of irradiation to tumor growth rate. We reconsider such model with the fractional differential operators and we offer the numerical scheme for the new model having Caputo, Caputo-Fabrizio and Atangana-Baleanu derivative. We present the numerical simulations for the solution of the system and we can say that the considered method is efficient and the results are accurate. Also one can see that tumor growth decreases under irradiation when these simulations examined. All calculations are performed using MATLAB.

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