Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 50 (3) (2021), 659 – 667
DOI : 10.15672/hujms.754006

Research Article

The representations of the g-Drazin inverse in a Banach algebra

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Abstract

The aim of this paper is to establish an explicit representation of the generalized Drazin inverse $(a+b)^d$ under the condition

$$ab^2 = 0, ba^2 = 0, a^\pi b^\pi (ba)^2 = 0.$$

Furthermore, we apply our results to give some representation of generalized Drazin inverse for a 2×2 block operator matrix. These extend the results on Drazin inverse of Bu, Feng and Bai [Appl. Math. Comput. **218**, 10226-10237, 2012] and Dopazo and Martinez-Serano [Linear Algebra Appl. **432**, 1896-1904, 2010].

Mathematics Subject Classification (2020). 15A09, 47L10, 32A65

Keywords. g-Drazin inverse, additive property, perturbation, Banach algebra

1. Introduction

Let $\mathcal{A}$ be a complex Banach algebra. An element $a \in \mathcal{A}$ has g-Drazin inverse, i.e., generalized Drazin inverse, if there exists $b \in \mathcal{A}$ such that

$$b = bab, ab = ba, a - a^2 b \in A^{qnil}.$$

Here, $\mathcal{A}^{qnil} = \{a \in \mathcal{A} \mid 1 + ax \in \mathcal{A}$ is invertible for every $x \in comm(a)\}$. We note that $a \in \mathcal{A}^{qnil} \Leftrightarrow \lim\limits_{n\to\infty} \parallel a^n \parallel^{\frac{1}{n}} = 0$. Such b, if it exists, is unique, and is called the g-Drazin inverse of a, and denote it by a^d. The g-Drazin inverse in a Banach algebra has various applications in singular differential equations, Markov chains and iterative methods (see [3, 4, 11]). New additive results for the g-Drazin inverse in a Banach algebra are presented.

In [2, Theorem 3.1], Bu, Feng and Bai gave some formulas of the Drazin inverse of the sum of two complex matrices under the condition $PQ^2 = 0, QP^2 = 0$. In Section 2, we extend this result and establish an explicit representation of the generalized Drazin inverse $(a+b)^d$ under the condition

$$ab^2 = 0, ba^2 = 0, a^\pi b^\pi (ba)^2 = 0,$$

where $a^\pi = 1 - aa^d$ is the spectral idempotent of $a \in \mathcal{A}$.

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Received: 17.06.2020; Accepted: 08.10.2020

In Section 3, we consider the g-Drazin inverse of a 2×2 operator matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.1}$$

where $A, B, C, D \in \mathcal{L}(X)$. Here, M is a bounded linear operator on $X \oplus X$. This problem has been expensively studied by many authors (see [1,2,6,7,9]. We then apply our results to establish new conditions under which M has g-Drazin inverse. This also generalize [7, Theorem 2.2] from the Drazin inverse of complex matrix to the g-Drazin inverse in a Banach algebra under a weaker condition.

Throughout the paper, $\mathcal{A}$ is a complex Banach algebra, X is a Banach space. We use $\mathcal{A}^d$ to stands for the set of all g-Drazin invertible $a \in \mathcal{A}$.

Let $x \in \mathcal{A}$ and $p^2 = p \in \mathcal{A}$. Then we have Pierce matrix decomposition $x = pxp + px(1-p) + (1-p)xp + (1-p)x(1-p)$. Set $a = pxp, b = px(1-p), c = (1-p)xp, d = (1-p)x(1-p)$. We use the following matrix version to express the Pierce matrix decomposition of x about the idempotent p:

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}_p$$

2. Additive results

In this section we establish some additive properties of g-Drazin inverse in Banach algebras. We begin with

Lemma 2.1. *Let $\mathcal{A}$ be a Banach algebra, $a, b \in \mathcal{A}^d$. Let*

$$x = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}_p \text{ or } \begin{pmatrix} b & c \\ 0 & a \end{pmatrix}_p.$$

Then

$$x^d = \begin{pmatrix} a^d & 0 \\ z & b^d \end{pmatrix}_p \text{ or } \begin{pmatrix} b^d & z \\ 0 & a^d \end{pmatrix}_p,$$

where

$$z = (b^d)^2 \Big(\sum_{i=0}^{\infty} (b^d)^i c a^i \Big) a^\pi + b^\pi \Big(\sum_{i=0}^{\infty} b^i c (a^d)^i \Big) (a^d)^2 - b^d c a^d.$$

Proof. See [5, Theorem 2.3]. □

Lemma 2.2. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}^{qnil}$. If*

$$ab^2 = 0, ba^2 = 0, (ba)^2 = 0,$$

then $a + b \in \mathcal{A}^{qnil}$.

Proof. Set

$$M = \begin{pmatrix} a^3 + a^2b + aba & a^3b + abab \\ a^2 + ab + ba + b^2 & a^2b + bab + b^3 \end{pmatrix}.$$

Then

$$\begin{aligned} M &= \begin{pmatrix} a^2b + aba & a^3b + abab \\ 0 & a^2b + bab \end{pmatrix} + \begin{pmatrix} a^3 & 0 \\ a^2 + ab + ba + b^2 & b^3 \end{pmatrix} \\ &:= G + F. \end{aligned}$$

We see that $G^2 = 0, GFG = 0$ and $GF^2 = 0$. Moreover, we have

$$\begin{aligned} F &= \begin{pmatrix} a^3 & 0 \\ a^2 + ba & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ b^2 + ab & b^3 \end{pmatrix} \\ &:= H + K. \end{aligned}$$

Since H, K are quasinilpotent and $HK = 0$, we see that F is quasinilpotent. Therefore M is quasinilpotent. Obviously, $M = (\begin{pmatrix} a \\ 1 \end{pmatrix}(1,b))^3$. It is obvious that $(1,b)\begin{pmatrix} a \\ 1 \end{pmatrix}$ is quasinilpotent. This completes the proof. $\square$

Lemma 2.3. *Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}^d, b \in \mathcal{A}^{qnil}$. If*

$$ab^2 = 0, ba^2 = 0, a^{\pi}(ba)^2 = 0,$$

then $a + b \in \mathcal{A}^d$ and

$$(a+b)^d = a^d + \sum_{n=0}^{\infty}(a^d)^{n+2}b(a+b)^n a^{\pi}.$$

Proof. Let $p = aa^d$. Then we have the Pierce decomposition relatively to the idempotent p:

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p, b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}_p.$$

Moreover,

$$a^d = \begin{pmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}_p \text{ and } a^{\pi} = \begin{pmatrix} 0 & 0 \\ 0 & 1 - aa^d \end{pmatrix}_p.$$

Since $ba^2 = 0$, we see that $baa^d = (ba^2)a^d = 0$, we see that $b_1 = b_3 = 0$.

We easily see that $a_2 = a - a^2a^d \in ((1-p)\mathcal{A}(1-p))^{qnil}$. Since $b(1 - aa^d) = b \in \mathcal{A}^{qnil}$, it follows by [8, Theorem 2.1] that $b_4 = (1 - aa^d)b(1 - aa^d) \in \mathcal{A}^{qnil}$. One easily checks that

$$a_2b_4^2 = 0, b_4a_2^2 = 0, (b_4a_2)^2 = 0.$$

In light of Lemma 2.2, $a_2 + b_4 \in ((1-p)\mathcal{A}(1-p))^{qnil}$. Thus $(a_2 + b_4)^d = 0$, and so

$$a + b = \begin{pmatrix} a_1 & b_2 \\ 0 & a_2 + b_4 \end{pmatrix}_p,$$

It follows by Lemma 2.1 that

$$(a+b)^d = \begin{pmatrix} a_1 & b_2 \\ 0 & a_2 + b_4 \end{pmatrix}^d = \begin{pmatrix} a^d & z \\ 0 & 0 \end{pmatrix}_p,$$

where $z = \sum\limits_{n=0}^{\infty}(a^d)^{n+2}b_2(a_2 + b_4)^n$. Therefore

$$(a+b)^d = a^d + \sum_{n=0}^{\infty}(a^d)^{n+2}b(a+b)^n a^{\pi},$$

as asserted. $\square$

We are now ready to prove the following.

Theorem 2.4. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b \in \mathcal{A}^d$. If*

$$ab^2 = 0, ba^2 = 0, a^{\pi}b^{\pi}(ba)^2 = 0,$$

then $a + b \in \mathcal{A}^d$. In this case,

$$(a+b)^d = a^d + b^d + \sum_{n=0}^{\infty}(a^d)^{n+2}b(a+b)^n + \sum_{n=0}^{\infty}(b^d)^{n+2}a(a+b)^n.$$

Proof. Let $q = bb^d$. Then we have

$$b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}_q, a = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}_q.$$

Moreover,

$$b^d = \begin{pmatrix} b_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}_q \text{ and } b^\pi = \begin{pmatrix} 0 & 0 \\ 0 & 1-bb^d \end{pmatrix}_q.$$

Since $ab^2 = 0$, we see that $ab^d = 0$; hence, $a_1b_1^{-1} = 0$ and $a_3b_1^{-1} = 0$. It follows that $a_1 = a_3 = 0$. Thus

$$a+b = \begin{pmatrix} b_1 & a_2 \\ 0 & a_4+b_2 \end{pmatrix}_p.$$

We easily see that $b_2 = b - b^2b^d \in ((1-p)\mathcal{A}(1-p))^{qnil}$. Since $a(1-bb^d) = a \in \mathcal{A}^d$, by using Cline's formula, we have $a_4 = (1-bb^d)a(1-bb^d) \in \mathcal{A}^d$.

Since $ab^2 = 0$, we see that $a_4b_2^2 = (1-bb^d)a(1-bb^d)b^2 = 0$. Also we have

$$b_2a_4^2 = (1-bb^d)ba(1-bb^d)a(1-bb^d) = (1-bb^d)ba^2(1-bb^d) = 0.$$

As $a^\pi b^\pi (ba)^2 = 0$, we have

$$\begin{pmatrix} a_1 & a_2 \\ 0 & a_4 \end{pmatrix}_q^\pi \begin{pmatrix} 0 & 0 \\ 0 & 1-bb^d \end{pmatrix}_q \left(\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}_q \begin{pmatrix} a_1 & a_2 \\ 0 & a_4 \end{pmatrix}_q\right)^2 = 0,$$

and so $a_4^\pi(b_2a_4)^2 = 0$. In light of Lemma 2.3, we get

$$\begin{array}{rcl} (a_4+b_2)^d & = & a_4^d + \sum\limits_{n=0}^{\infty}(a_4^d)^{n+2}b_2(a_4+b_2)^n a_4^\pi \\ & = & a^d + \sum\limits_{n=0}^{\infty}(a^d)^{n+2}b(a+b)^n. \end{array}$$

In view of Lemma 2.1, we have

$$(a+b)^d = \begin{pmatrix} b_1 & a_2 \\ 0 & a_4+b_2 \end{pmatrix}^d = \begin{pmatrix} b_1^{-1} & z \\ 0 & (a_4+b_2)^d \end{pmatrix}_p,$$

where

$$z = \sum_{n=0}^{\infty}(b^d)^{n+2}a_2(a_4+b_2)^n(a_4+b_2)^\pi - b^da_2(a_4+b_2)^d.$$

Since $b^da^2 = (b^d)^2(ba^2) = 0$ and $b^da = 0$, we have $b^da_2(a_4+b_2)^d = 0$ and

$$\begin{array}{rl} & (b^d)^{n+2}a_2(a_4+b_2)^n(a_4+b_2)^\pi \\ = & (b^d)^{n+2}a(a+b)^n(a^\pi - \sum\limits_{n=0}^{\infty}(a^d)^{n+1}b(a+b)^n) \\ = & (b^d)^{n+2}a(a+b)^n. \end{array}$$

Hence,

$$z = \sum_{n=0}^{\infty}(b^d)^{n+2}a(a+b)^n.$$

Therefore

$$(a+b)^d = a^d + b^d + \sum_{n=0}^{\infty}(a^d)^{n+2}b(a+b)^n + \sum_{n=0}^{\infty}(b^d)^{n+2}a(a+b)^n.$$

as asserted. □

Example 2.5. Let

$$a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in M_2(\mathbb{C}).$$

Then

$$ab^2 = 0, ba^2 = 0, a^\pi b^\pi (ba)^2 = 0.$$

It is obvious by computing that

$$ab^2 = 0, ba^2 = 0, a^\pi b^\pi (ba)^2 = 0.$$

Also

$$a^d = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, b^d = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and by the formula of Theorem 2.4 we have,

$$(a+b)^d = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

3. Block operator matrices

In this section, we apply our results to establish new conditions under which a 2×2 operator matrix over Banach spaces has g-Drazin inverse. Let $\mathcal{A} = \mathcal{L}(x)$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_2(\mathcal{A})$. We now derive

Theorem 3.1. *Let A and D have g-Drazin inverses. If $ABD = 0, CBD = 0, BCA = 0, DCA = 0, BCBC = 0$ and $D^\pi CBC = 0$, then $M \in M_2(\mathcal{A})^d$. In this case*

$$M^d = \begin{pmatrix} A^d & B(D^d)^2 \\ C(A^d)^2 & D^d \end{pmatrix} + \sum_{n=0}^{\infty} \begin{pmatrix} B(D^d)^{n+3}C & (A^d)^{n+2}B \\ (D^d)^{n+2}C & C(A^d)^{n+3}B \end{pmatrix} M^n.$$

Proof. Let $M = P + Q$, where $P = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}$, and $Q = \begin{pmatrix} 0 & B \\ 0 & D \end{pmatrix}$. Then P, Q have g-Drazin inverses. Moreover, we have

$$P^d = \begin{pmatrix} A^d & 0 \\ C(A^d)^2 & 0 \end{pmatrix}, Q^d = \begin{pmatrix} 0 & B(D^d)^2 \\ 0 & D^d \end{pmatrix}.$$

Then

$$P^\pi = \begin{pmatrix} A^\pi & 0 \\ -CA^d & I \end{pmatrix}, Q^\pi = \begin{pmatrix} I & -BD^d \\ 0 & D^\pi \end{pmatrix}.$$

Thus, we have

$$\begin{aligned} PQ^2 &= \begin{pmatrix} 0 & ABD \\ 0 & CBD \end{pmatrix} = 0; \\ QP^2 &= \begin{pmatrix} BCA & 0 \\ DCA & 0 \end{pmatrix} = 0; \\ (QP)^2 &= \begin{pmatrix} BCBC & 0 \\ DCBC & 0 \end{pmatrix}; \\ P^\pi Q^\pi &= \begin{pmatrix} A^\pi & -A^\pi BD^d \\ -CA^d & CA^d BD^d + D^\pi \end{pmatrix}. \end{aligned}$$

It is obvious by computing that $P^\pi Q^\pi (QP)^2 = 0$. In light of Theorem 2.4, M has g-Drazin inverse. Moreover, we have

$$\begin{aligned} M^d &= P^d + Q^d + \sum_{n=0}^{\infty} (P^d)^{n+2} Q M^n + \sum_{n=0}^{\infty} (Q^d)^{n+2} P M^n \\ &= \begin{pmatrix} A^d & B(D^d)^2 \\ C(A^d)^2 & D^d \end{pmatrix} + \sum_{n=0}^{\infty} \begin{pmatrix} 0 & (A^d)^{n+2}B \\ 0 & C(A^d)^{n+3}B \end{pmatrix} M^n \\ &+ \sum_{n=0}^{\infty} \begin{pmatrix} B(D^d)^{n+3}C & 0 \\ (D^d)^{n+2}C & 0 \end{pmatrix} M^n \end{aligned}$$

This completes the proof. □

Corollary 3.2. *Let A and D have g-Drazin inverses. If $ABD = 0, CBD = 0, BCA = 0, DCA = 0$ and $CBC = 0$, then $M \in M_2(\mathcal{A})^d$. In this case,*

$$M^d = \begin{pmatrix} A^d & B(D^d)^2 \\ C(A^d)^2 & D^d \end{pmatrix} + \sum_{n=0}^{\infty} \begin{pmatrix} B(D^d)^{n+3}C & (A^d)^{n+2}B \\ (D^d)^{n+2}C & C(A^d)^{n+3}B \end{pmatrix} M^n.$$

Proof. This is obvious by Theorem 3.1. □

Theorem 3.3. *Let A and D have g-Drazin inverses. If $DCA = 0, BCA = 0, CBD = 0, ABD = 0, CBCB = 0$ and $A^{\pi}BCB = 0$, then $M \in M_2(\mathcal{A})^d$. In this case.*

$$M^d = \begin{pmatrix} A^d & B(D^d)^2 \\ C(A^d)^2 & D^d \end{pmatrix} + \sum_{n=0}^{\infty} \begin{pmatrix} B(D^d)^{n+3}C & (A^d)^{n+2}B \\ (D^d)^{n+2}C & C(A^d)^{n+3}B \end{pmatrix} M^n.$$

Proof. By virtue of Theorem 3.1, the matrix $\begin{pmatrix} D & C \\ B & A \end{pmatrix}$ has g-Drazin inverse. Moreover, we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^d = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} D & C \\ B & A \end{pmatrix}^d \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Therefore we obtain the result. □

Corollary 3.4. *Let A and D have g-Drazin inverses. If $DCA = 0, BCA = 0, CBD = 0, ABD = 0$ and $BCB = 0$, then $M \in M_2(\mathcal{A})^d$. In this case.*

$$M^d = \begin{pmatrix} A^d & B(D^d)^2 \\ C(A^d)^2 & D^d \end{pmatrix} + \sum_{n=0}^{\infty} \begin{pmatrix} B(D^d)^{n+3}C & (A^d)^{n+2}B \\ (D^d)^{n+2}C & C(A^d)^{n+3}B \end{pmatrix} M^n.$$

Proof. This is obvious by Theorem 3.3. □

Lemma 3.5. *Let P and $Q \in \mathcal{A}$ have g-Drazin inverses. If $PQ^2 = 0, PQP = 0$, then $P + Q$ has g-Drazin inverse and*

$$\begin{aligned}(P+Q)^d = \;& Q^{\pi} \sum_{i=0}^{\infty} Q^i (P^d)^{i+1} + \sum_{i=0}^{\infty} (Q^d)^{i+1} P^i P^{\pi} + Q^{\pi} \sum_{i=0}^{\infty} Q^i (P^d)^{i+2} Q \\ & + \sum_{i=0}^{\infty} (Q^d)^{i+3} P^{i+1} P^{\pi} Q - Q^d P^d Q - (Q^d)^2 P P^d Q.\end{aligned}$$

Proof. This is proved as in [10, Theorem 2.1]. □

In [7, Theorem 2.2], Dopazo and Martinez-Serrano investigated Drazin inverse of a 2×2 block complex matrix under the condition $BC = 0, BDC = 0$ and $BD^2 = 0$. We now generalize it to the g-Drazin inverse with a weaker condition.

Theorem 3.6. *Let A and D have g-Drazin inverses. If $BCA = 0, CBCB = 0, A^{\pi}BCB = 0, BDC = 0$ and $BD^2 = 0$, then $M \in M_2(\mathcal{A})^d$. In this case,*

$$\begin{aligned}M^d = \;& \sum_{i=0}^{\infty} \begin{pmatrix} 0 & 0 \\ 0 & D^{\pi} D^i \end{pmatrix} (P^d)^{i+1} \begin{pmatrix} I & \sum_{n=0}^{\infty} (A^d)^{n+2} B D_n D \\ 0 & I + \sum_{n=0}^{\infty} C (A^d)^{n+3} B D_n D \end{pmatrix} \\ & + \begin{pmatrix} \Gamma & \Delta \\ \Lambda & \Xi \end{pmatrix},\end{aligned}$$

where

$$P^d = \begin{pmatrix} A^d + \sum_{n=0}^{\infty} (A^d)^{n+2} B C_n & \sum_{n=0}^{\infty} (A^d)^{n+2} B D_n \\ C(A^d)^2 + \sum_{n=0}^{\infty} C (A^d)^{n+3} B C_n & \sum_{n=0}^{\infty} C (A^d)^{n+3} B D_n \end{pmatrix},$$

$$A_1 = A, B_1 = B, C_1 = C, D_1 = 0; C_0 = 0 \text{ and } D_0 = 1$$
$$A_{n+1} = AA_n + BC_n, B_{n+1} = AB_n + BD_n, C_{n+1} = CA_n, D_{n+1} = CB_n,$$

and

$$\begin{aligned}
\Gamma &= A^{\pi} - \sum_{n=0}^{\infty}(A^d)^{n+1}BC_n,\\
\Delta &= -\sum_{n=0}^{\infty}(A^d)^{n+1}BD_n,\\
\Lambda &= \sum_{i=0}^{\infty}\big[(D^d)^{i+1}C_i\big(A^{\pi} - \sum_{n=0}^{\infty}(A^d)^{n+1}BC_n\\
&\quad +(D^d)^{i+1}D_i\big(-CA^d - \sum_{n=0}^{\infty}C(A^d)^{n+2}BC_n\big)\big]\\
&\quad +DD^{\pi}\big[-CA^d - \sum_{n=0}^{\infty}C(A^d)^{n+2}BC_n\big],
\end{aligned}$$

$$\begin{aligned}
\Xi &= \sum_{i=0}^{\infty}\big[(D^d)^{i+1}C_i\big(-\sum_{n=0}^{\infty}(A^d)^{n+1}BD_n\\
&\quad +(D^d)^{i+1}D_i\big(I_n - \sum_{n=0}^{\infty}C(A^d)^{n+2}BD_n\big)\big]\\
&\quad +\sum_{i=0}^{\infty}\big[(D^d)^{i+3}C_{i+1}\big(-\sum_{n=0}^{\infty}(A^d)^{n+1}BD_n\\
&\quad +(D^d)^{i+3}C_{i+1}\big(I - \sum_{n=0}^{\infty}C(A^d)^{n+2}BD_n\big)D\big] + (D^d)^2C\\
&\quad \big[-\sum_{n=0}^{\infty}(A^d)^{n+1}BD_n - D^d\big[I - \sum_{n=0}^{\infty}C(A^d)^{n+2}BD_n\big]D\\
&\quad +DD^{\pi}\big[I - \sum_{n=0}^{\infty}C(A^d)^{n+2}BD_n\big] \qquad (*)
\end{aligned}.$$

Proof. Obviously, we have $M = P + Q$, where

$$P = \begin{pmatrix} A & B\\ C & 0\end{pmatrix}, Q = \begin{pmatrix} 0 & 0\\ 0 & D\end{pmatrix}.$$

Clearly, we see that Q has g-Drazin inverse. Since $BCA = 0, CBCB = 0$ and $A^{\pi}BCB = 0$, it follows by Theorem 3.3 that P has g-Drazin inverse and

$$P^d = \begin{pmatrix} A^d & 0\\ C(A^d)^2 & 0\end{pmatrix} + \sum_{n=0}^{\infty}\begin{pmatrix} 0 & (A^d)^{n+2}B\\ 0 & C(A^d)^{n+3}B\end{pmatrix}P^n.$$

We directly compute that

$$\begin{aligned}
PQP &= \begin{pmatrix} BDC & 0\\ 0 & 0\end{pmatrix} = 0;\\
PQ^2 &= \begin{pmatrix} 0 & BD^2\\ 0 & 0\end{pmatrix} = 0.
\end{aligned}$$

Write $P^n = \begin{pmatrix} A_n & B_n\\ C_n & D_n\end{pmatrix}$. Then $A_1 = A, B_1 = B, C_1 = C, D_1 = 0$ and

$$A_{n+1} = AA_n + BC_n, B_{n+1} = AB_n + BD_n, C_{n+1} = CA_n, D_{n+1} = CB_n.$$

Then

$$P^d = \begin{pmatrix} A^d + \sum_{n=0}^{\infty}(A^d)^{n+2}BC_n & \sum_{n=0}^{\infty}(A^d)^{n+2}BD_n\\ C(A^d)^2 + \sum_{n=0}^{\infty}C(A^d)^{n+3}BC_n & \sum_{n=0}^{\infty}C(A^d)^{n+3}BD_n\end{pmatrix},$$

and so $P^\pi = (P_{ij})$, where

$$\begin{array}{rcl} P_{11} & = & A^\pi - \sum\limits_{n=0}^{\infty}(A^d)^{n+1}BC_n \\ P_{12} & = & -\sum\limits_{n=0}^{\infty}(A^d)^{n+1}BD_n, \\ P_{21} & = & -CA^d - \sum\limits_{n=0}^{\infty}C(A^d)^{n+2}BC_n, \\ P_{22} & = & I - \sum\limits_{n=0}^{\infty}C(A^d)^{n+2}BD_n. \end{array}$$

According to Lemma 3.5, we have

$$\begin{array}{rcl} M & = & (P+Q)^d \\ & = & Q^\pi \sum\limits_{i=0}^{\infty} Q^i(P^d)^{i+1} + \sum\limits_{i=0}^{\infty}(Q^d)^{i+1}P^iP^\pi + Q^\pi \sum\limits_{i=0}^{\infty} Q^i(P^d)^{i+2}Q \\ & & + \sum\limits_{i=0}^{\infty}(Q^d)^{i+3}P^{i+1}P^\pi Q - Q^dP^dQ - (Q^d)^2PP^dQ. \\ & = & \sum\limits_{i=1}^{\infty}\begin{pmatrix} 0 & 0 \\ 0 & D^\pi D^i \end{pmatrix}(P^d)^{i+1}\begin{pmatrix} I & \sum\limits_{n=0}^{\infty}(A^d)^{n+2}BD_nD \\ 0 & I+\sum\limits_{n=0}^{\infty}C(A^d)^{n+3}BD_nD \end{pmatrix} \\ & & + \sum\limits_{i=0}^{\infty}(Q^d)^{i+1}P^iP^\pi + \sum\limits_{i=0}^{\infty}(Q^d)^{i+3}P^{i+1}P^\pi Q - Q^dP^dQ \\ & & -(Q^d)^2PP^dQ + Q^\pi P^d \\ & = & \sum\limits_{i=0}^{\infty}\begin{pmatrix} 0 & 0 \\ 0 & D^\pi D^i \end{pmatrix}(P^d)^{i+1}\begin{pmatrix} I & \sum\limits_{n=0}^{\infty}(A^d)^{n+2}BD_nD \\ 0 & I+\sum\limits_{n=0}^{\infty}C(A^d)^{n+3}BD_nD \end{pmatrix} \\ & & + \begin{pmatrix} \Gamma & \Delta \\ \Lambda & \Xi \end{pmatrix}, \end{array}$$

where Γ, Δ, Λ and Ξ are given as in $(*)$ by direct computation. □

Acknowledgment. The author is so grateful to the referees for their careful reading and valuable comments.

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