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Multi-Parametric Families of Solutions of Order N to the Boussinesq and KP Equations and the Degenerate Rational Case

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Abstract

From elementary exponential functions which depend on several parameters, we construct multi-parametric solutions to the Boussinesq equation. When we perform a passage to the limit when one of these parameters goes to 0, we get rational solutions as a quotient of a polynomial of degree $N(N+1)-2$ in x and t, by a polynomial of degree $N(N+1)$ in x and t for each positive integer N depending on $3N$ real parameters. We restrict ourself to give the explicit expressions of these rational solutions for $N=1$ until $N=3$ to shortened the paper.
We easily deduce the corresponding explicit rational solutions to the Kadomtsev Petviashvili equation for the same orders from 1 to 3.

1. Introduction

The Boussinesq equation in the following normalization is considered

$$u_{tt} - u_{xx} + (u^2)_{xx} + \frac{1}{3}u_{xxxx} = 0. \tag{1.1}$$

The subscripts x and t denote as usual partial derivatives.
This equation (1.1) is an equation solvable [3, 4] by inverse scattering. It was introduced for the first time by Boussinesq [1, 2] in 1871. This equation appears in a wide range of physical problems dealing with propagation of nonlinear waves; for example, in one-dimensional nonlinear lattice-waves [5], vibrations in a nonlinear string [6], ion sound waves in a plasma [7],...
The first solutions were constructed by Hirota [8] in 1977 with Bäcklund transformations. Non singular rational solutions were constructed by Ablowitz and Satsuma by using the Hirota bilinear method [9] in 1978. Freemann and Nimmo [10] gave in 1983 wronskians representations of the solutions. Other approaches were used; in particular, an algebro-geometrical method was given by Matveev et al. [11] in 1987; Darboux transformations [12] was used by Matveev; the $\overline{\partial}$ dressing method [13] was considered by Bogdanov.
Clarkson obtained solutions in terms of particular polynomials in a series of papers [14, 15] and recently, in 2017 gives new solutions [16] as second derivatives of polynomials.

Solutions to the Boussinesq equation and the Kadomtsev Petviashvili equation are considered in this paper. We give solutions from elementary exponential functions depending on several parameters. Then we construct rational solution in performing a passage to the limit when one of these parameters goes to 0. We obtain rational solutions as a quotient of a polynomial of degree $N(N+1)-2$ in x and t by a polynomial of degree $N(N+1)$ in x and t, depending on $2N$ parameters. We give explicit solutions in the simplest cases where $N = 1, 2, 3$. We deduce and give explicit expressions of rational solutions to the Kadomtsev-Petviashvili (KP) equation for the cases of orders from 1 to 3.

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2. Solutions to the Boussinesq equation

2.1. Solutions to the Boussinesq equation in terms of elementary exponentials

We consider the Boussinesq equation

$$u_{tt}-u_{xx}+(u^2)_{xx}+\frac{1}{3}u_{xxxx}=0.$$

We define the following notations.
We consider e, a_j, c_j, d_j, $1\leq j\leq N$, arbitrary real numbers, and α_j, β_j the numbers defined by

$$\alpha_j=\frac{3}{2}a_je+\frac{1}{2}\sqrt{1-3a_j^2e^2} \tag{2.1}$$

and

$$\beta_j=-\frac{3}{2}a_je+\frac{1}{2}\sqrt{1-3a_j^2e^2}. \tag{2.2}$$

We consider the following elementary functions

$$f_{ij}(x,t)=\alpha_j^{i-1}\exp(\alpha_jx-\alpha_j^2t+c_je^{2N-1})-\beta_j^{i-1}\exp(\beta_jx-\beta_j^2t+d_je^{2N-1}), \tag{2.3}$$

for $1\leq i\leq N$.
Then, we have the following statement:

Theorem 2.1. *The function v defined by*

$$v(x,t)=2\partial_x^2\ln(\det(f_{ij})_{(i,j)\in[1,N]}) \tag{2.4}$$

is a solution to the Boussinesq equation (1.1) with e, a_j, c_j and d_j, $1\leq j\leq N$ arbitrarily real parameters.

Proof. The corresponding Lax pair to the Boussinesq equation (1.1) is

$$\begin{cases}\phi_{xxx}+\frac{3}{2}u\phi_x-\frac{3}{4}\phi_x+u\phi=\lambda\phi,\\ \phi_t=-\phi_{xx}-u\phi.\end{cases} \tag{2.5}$$

The compatibility condition of the preceding system can be written as [12]

$$\begin{cases}w_x=\frac{3}{4}u_{xx}-\frac{3}{4}u_t,\\ w_t=\frac{1}{4}u_{xxx}+\frac{3}{4}(u^2)_x-\frac{3}{4}u_x+\frac{3}{4}u_{xt}.\end{cases} \tag{2.6}$$

The Boussinesq equation is obtained by excluding w from the above equations.
This system is covariant by the Darboux transformation. If $\phi_1,\ldots,\phi_N$ are solutions of the system (2.6), then $\phi[N]$ defined by $\phi[N]=\frac{W(\phi_1,\ldots,\phi_N,\phi)}{W(\phi_1,\ldots,\phi_N)}$ is another solution of this system (2.6) where u is replaced by $u[N]=u+2(\ln W(\phi_1,\ldots,\phi_N)_{xx}$ [12].
We choose $u=0$. Then the functions $\phi_j=f_{1j}$ verify the following system

$$\begin{cases}\phi_{xxx}-\frac{3}{4}\phi_x=\lambda\phi,\\ \phi_t=-\phi_{xx}.\end{cases} \tag{2.7}$$

Then the solution of (1.1) can be written as $v(x,t)=2(\ln W(\phi_1,\ldots,\phi_N)_{xx}$which is nothing else that (2.4) $v(x,t)=2\partial_x^2\ln(\det(f_{ij})_{(i,j)\in[1,N]})$. □

2.2. Rational solutions to the Boussinesq equation

To obtain rational solutions to the Boussinesq equation, we are going to perform a limit when the parameter e tends to 0.

2.2.1. Rational solutions as a limit case

We get the following result :

Theorem 2.2. *The function v defined by*

$$v(x,t)=\lim_{e\to 0}2\partial_x^2\ln(\det(f_{ij})_{(i,j)\in[1,N]}) \tag{2.8}$$

is a rational solution to the Boussinesq equation (1.1) depending on $3N$ parameters a_j, c_j and d_j, $1\leq j\leq N$; the numerator is a polynomial of degree $N(N+1)-2$ in x and t, the denominator a polynomial of degree $N(N+1)$ in x and t.

Proof. It is a consequence of the previous result. □

2.2.2. Degenerate rational solutions

A more precise result can be formulated in the following way.
We consider e, a_j, c_j, d_j, $1 \leq j \leq N$, arbitrary real numbers, and γ_j, δ_j the numbers defined by

$$\gamma_j = \frac{3}{2}\left(\sum_{k=1}^{N} a_k (je)^{2k-1}\right) + \frac{1}{2}\sqrt{1-3\left(\sum_{k=1}^{N} a_k (je)^{2k-1}\right)^2},$$
$$\delta_j = -\frac{3}{2}\left(\sum_{k=1}^{N} a_k (je)^{2k-1}\right) + \frac{1}{2}\sqrt{1-3\left(\sum_{k=1}^{N} a_k (je)^{2k-1}\right)^2} \tag{2.9}$$

We consider the following elementary functions

$$g_{ij}(x,t,e) = \gamma_j^{i-1}\exp\left(\gamma_j x - \gamma_j^2 t + \sum_{k=1}^{N} c_k (je)^{2k-1}\right) - \delta_j^{i-1}\exp\left(\delta_j x - \delta_j^2 t + \sum_{k=1}^{N} d_k (je)^{2k-1})\right), \tag{2.10}$$

$$\varphi_{ij}(x,t) = \frac{\partial^j g_{i1}(x,t,0)}{\partial e^j}, \text{ for } 1 \leq i \leq N, \quad 1 \leq j \leq N. \tag{2.11}$$

Then get the following result :

Theorem 2.3. *The function v defined by*

$$v(x,t) = 2\partial_x^2 \ln(\det(\varphi_{ij})_{(i,j)\in[1,N]} \tag{2.12}$$

is a rational solution to the Boussinesq equation (1.1) depending on $3N$ parameters a_j, c_j and d_j, $1 \leq j \leq N$; the numerator is a polynomial of degree $N(N+1)-2$ in x and t, the denominator a polynomial of degree $N(N+1)$ in x and t.

Proof. In the coefficients α_j and β_j defined in (2.1,2.2), we replace a_j by $\sum_{k=1}^{N} a_k (je)^{2k-1}$, and in the functions f_{ij} defined in (2.3), c_j by $\sum_{k=1}^{N} c_k (je)^{2k-1}$ and d_j by $\sum_{k=1}^{N} d_k (je)^{2k-1}$; this gives functions g_{ij} defined by (2.10). Then, it is sufficient to combine the columns of the determinant obtained from this defined by (2.8) by replacing f_{ij} by g_{ij} and to take a passage to the limit when e tends to 0. So we get the solution v given by (2.12). ☐

So we obtain an infinite hierarchy of rational solutions to the Boussinesq equation depending on the integer N.
In the following we give some examples of rational solutions.
These results are consequences of the previous result (2.12).
But, it is also to possible to prove it directly in replacing the expressions of each of the solutions given in the corresponding equation and check that the relation is verified.

2.3. First order rational solutions

We have the following result at order $N = 1$:

Theorem 2.4. *The function v defined by*

$$v(x,t) = \frac{-18 a_1{}^2}{(-3a_1 x - c_1 + 3ta_1 + d_1)^2}, \tag{2.13}$$

is a solution to the Boussinesq equation (1.1) with a_1, c_1, d_1 arbitrarily real parameters.

Remark 2.5. *If $a_1 = 0$, then the solution is the trivial solution* 0.

Remark 2.6. *The solution (2.13) can be simplified and be rewritten as a solution depending on one parameter C_1.*

$$v(x,t) = \frac{-18}{(-3x + 3t + C_1)^2}. \tag{2.14}$$

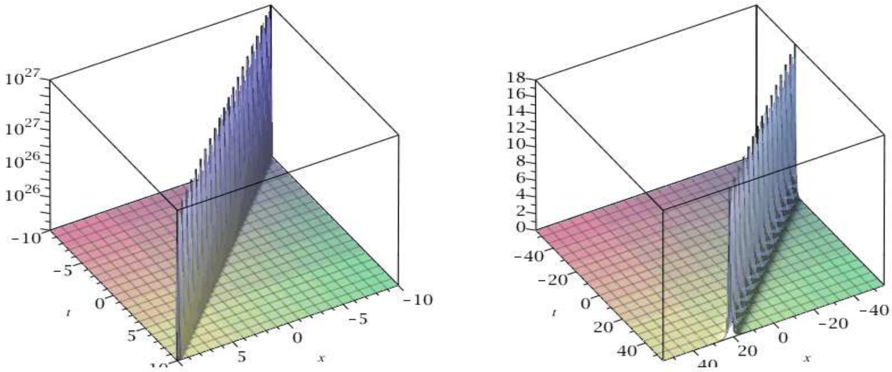

Figure 2.1: Solution of order 1 to (1.1), on the left $a_1 = 10^{13}$, $c_1 = 1$, $c_1 = 0$; on the right $a_1 = 1$, $c_1 = 10^2$, $d_1 = 0$.

Remark 2.7. *The case where $a_1 = 1$, $c_1 = 0$, $d_1 = 10^2$ gives the same figure as the case $a_1 = 1$, $c_1 = 10^2$, $d_1 = 0$. The roles played by the parameters c and d being the same, we only give the figures with parameters d equal to 0.*

2.4. Second order rational solutions

Theorem 2.8. *The function v defined by*

$$v(x,t) = -2\,\frac{n(x,t)}{d(x,t)^2}, \tag{2.15}$$

with

$n(x,t) = 9a_1a_2(27a_1{}^5a_2 + 243a_2{}^5a_1 - 162a_2{}^3a_1{}^3)x^4 + 9a_1a_2(-972a_2{}^5ta_1 - 324a_2{}^5a_1 + 216a_2{}^3a_1{}^3 + 648a_2{}^3a_1{}^3t - 36a_1{}^5a_2 - 108a_1{}^5ta_2)x^3$
$+9a_1a_2(972a_2{}^5ta_1 - 648a_2{}^3a_1{}^3t - 108a_2{}^3a_1{}^3 - 972a_2{}^3a_1{}^3t^2 + 162a_1{}^5t^2a_2 + 162a_2{}^5a_1 + 18a_1{}^5a_2 + 108a_1{}^5ta_2 + 1458a_2{}^5t^2a_1)x^2 + 9a_1a_2(-108a_1{}^5t^3a_2 +$
$216a_2{}^2c_2a_1 + 72d_2a_1{}^3 - 432a_2{}^3a_1{}^3t + 648a_2{}^3a_1{}^3t^3 - 24a_1{}^2d_1a_2 + 648a_2{}^3a_1{}^3t^2 + 648a_2{}^5ta_1 - 72a_1{}^3c_2 + 72d_1a_2{}^3 + 24a_1{}^2c_1a_2 - 972a_2{}^5a_1t^3 - 72a_2{}^3c_1 -$
$972a_2{}^5t^2a_1 + 72a_1{}^5ta_2 - 216a_2{}^2d_2a_1 - 108a_1{}^5t^2a_2)x + 9a_1a_2(24a_2{}^3c_1 + 24a_1{}^3c_2 + 324a_2{}^5a_1t^3 + 540a_2{}^3a_1{}^3t^2 + 216a_2{}^3a_1{}^3t - 216a_2{}^3a_1{}^3t^3 + 8a_1{}^2d_1a_2 -$
$90a_1{}^5t^2a_2 - 36a_1{}^5ta_2 + 36a_1{}^5t^3a_2 + 243t^4a_2{}^5a_1 - 162t^4a_2{}^3a_1{}^3 + 27a_1{}^5t^4a_2 + 72a_2{}^2d_2a_1 - 810a_2{}^5t^2a_1 - 324a_2{}^5ta_1 - 72ta_2{}^3d_1 - 72a_1{}^3td_2 + 216ta_2{}^2d_2a_1 +$
$24a_1{}^2td_1a_2 - 216ta_2{}^2c_2a_1 - 24a_1{}^2tc_1a_2 - 24d_2a_1{}^3 - 24d_1a_2{}^3 - 8a_1{}^2c_1a_2 + 72ta_2{}^3c_1 + 72a_1{}^3tc_2 - 72a_2{}^2c_2a_1)$,

and

$d(x,t) = (-9a_1{}^3a_2 + 27a_1a_2{}^3)x^3 + (27ta_1{}^3a_2 - 81ta_1a_2{}^3 + 9a_1{}^3a_2 - 27a_1a_2{}^3)x^2 + (-27t^2a_1{}^3a_2 + 81t^2a_1a_2{}^3 - 18ta_1{}^3a_2 + 54ta_1a_2{}^3)x + 9t^3a_1{}^3a_2 -$
$27t^3a_1a_2{}^3 + 9t^2a_1{}^3a_2 - 27t^2a_1a_2{}^3 + 18ta_1{}^3a_2 - 54ta_1a_2{}^3 - 12a_1c_2 + 12a_1d_2 + 4a_2c_1 - 4a_2d_1$,
is a rational solution to the Boussinesq equation (1.1), quotient of two polynomials with numerator of order 4 in x and t, denominator of degree 6 in x and t.

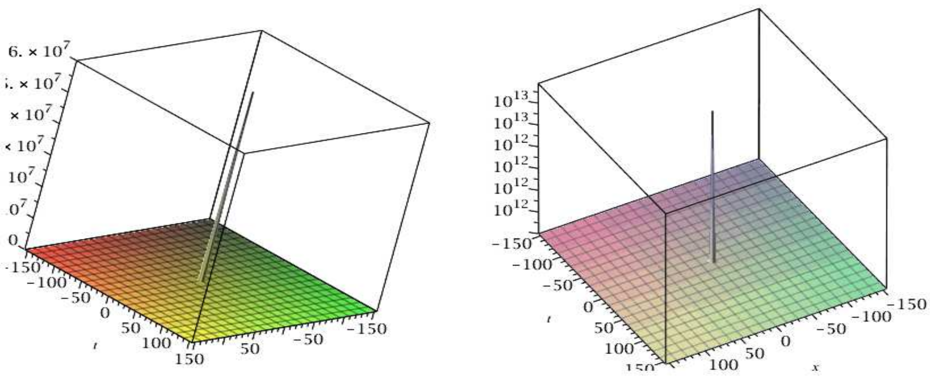

Figure 2.2: Solution of order 2 to (1.1); on the left, $a_1 = 10^7$, $a_2 = 1$, $c_1 = 1$, $c_2 = 1$, $d_1 = 0$, $d_2 = 0$; on the right, $a_1 = 1$, $a_2 = 10^7$, $c_1 = 1$, $c_2 = 1$, $d_1 = 0$, $d_2 = 0$.

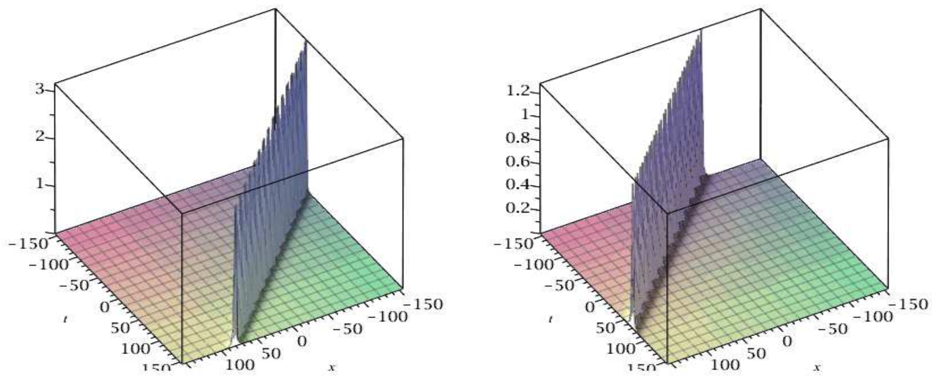

Figure 2.3: Solution of order 2 to (1.1); on the left, $a_1 = 1$, $a_2 = 1$, $c_1 = 10^7$, $c_2 = 0$, $d_1 = 1$, $d_2 = 1$; on the right, $a_1 = 1$, $a_2 = 1$, $c_1 = 0$, $c_2 = 10^7$, $d_1 = 0$, $d_2 = 0$.

2.5. Rational solutions of order three

We get the following rational solutions given by :

Theorem 2.9. *The function v defined by*

$$v(x,t) = -2\,\frac{n(x,t)}{d(x,t)^2}, \tag{2.16}$$

is a rational solution to the Boussinesq equation (1.1), quotient of two polynomials with numerator of order 10 *in x and t, denominator of degree* 12 *in x and t.*

Because of the length of the solution, we give it only in the appendix.

Remark 2.10. *If $c_1 = c_2 = c_3 = d_1 = d_2 = d_3 = 0$, then the determinant in the formula (2.12) can be simplified by* $\frac{(177147}{80}a_1a_2a_3^3(-80a_2^4+360a_3^2a_2^2+a_1^4-30a_3^2a_1^2)$ *and the solution to the Boussinseq equation depends no longer on any parameter.*
If one of the parameters a_1, a_2 or a_3 is equal to 0 *then the solution of the Boussinesq equation is the trivial solution (equal to* 0*).*

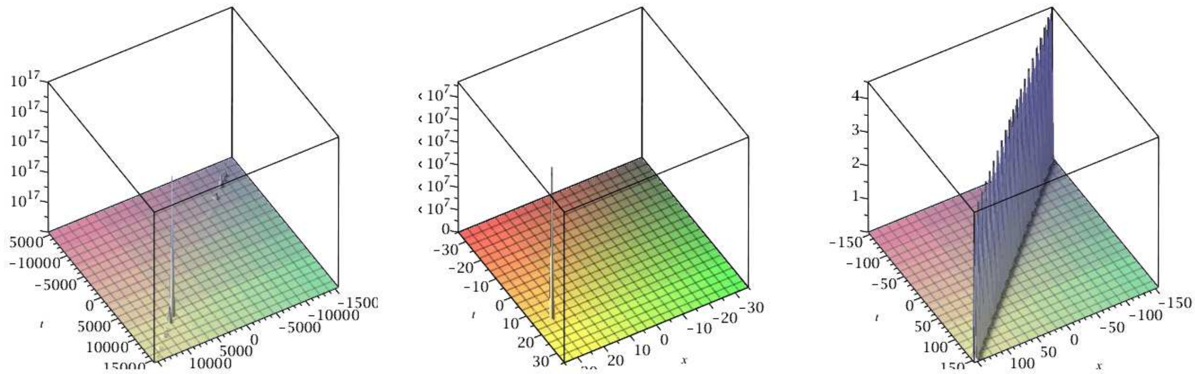

Figure 2.4: Solution of order 3 to (1.1); on the left, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $c_1 = 0$, $c_2 = 0$,$c_3 = 10^7$, $d_1 = 0$, $d_2 = 0$, $d_3 = 0$; in the center, $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $c_1 = 0$, $c_2 = 10^7$,$c_3 = 1$, $d_1 = 0$, $d_2 = 0$, $d_3 = 0$; on the right, $a_1 = 1$, $a_2 = 1$, $a_3 = 10^7$, $c_1 = 1$, $c_2 = 1$,$c_3 = 1$, $d_1 = 0$, $d_2 = 0$, $d_3 = 0$.

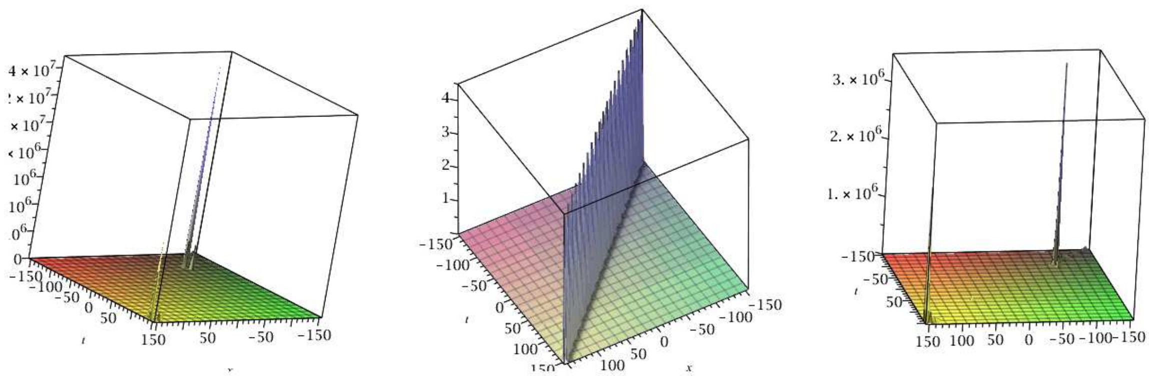

Figure 2.5: Solution of order 3 to (1.1)on the left, $a_1 = 1$, $a_2 = 10^7$, $a_3 = 1$, $c_1 = 1$, $c_2 = 1$,$c_3 = 10^7$, $d_1 = 0$, $d_2 = 0$, $d_3 = 0$; in the center, $a_1 = 10^7$, $a_2 = 1$, $a_3 = 1$, $c_1 = 1$, $c_2 = 10^7$,$c_3 = 1$, $d_1 = 0$, $d_2 = 0$, $d_3 = 0$; on the right, $a_1 = 1$, $a_2 = 1$, $a_3 = 10^7$, $c_1 = 10^5$, $c_2 = 1$,$c_3 = 1$, $d_1 = 0$, $d_2 = 0$, $d_3 = 0$.

3. Solutions to the Kadomtsev Petviashvili equation

We consider the Kadomtsev Petviashvili equation (KP) which can be written in the form

$$(4u_T - 6uu_X + u_{XXX})_X - 3u_{YY} = 0, \tag{3.1}$$

where subscripts X, Y and T denote as usual partial derivatives.
From the previous study, we can deduce easily solutions to the KP equation. It is sufficient for this, to use the following transformations $x = \imath X + \frac{3\imath T}{4}$, $t = \imath Y$ from the solutions to the Boussinesq equation to obtain solutions to the KP equation.

3.1. Solutions to the KP equation

3.2. First order rational solutions

We have the following result at order $N = 1$:

Theorem 3.1. *The function v defined by*

$$v(X.Y,T) = \frac{-288\,{a_1}^2}{(12\,ia_1X + 9\,ia_1T + 4\,c_1 - 12\,iYa_1 - 4\,d_1)^2}, \tag{3.2}$$

is a solution to the KP equation (3.1).

Remark 3.2. *The solution* (3.2) *can be simplified and be rewritten as depending on one parameter* $v(X.Y,T) = \frac{-288}{(12\,iX + 9\,iT + 4C_1 - 12\,iY)^2}$

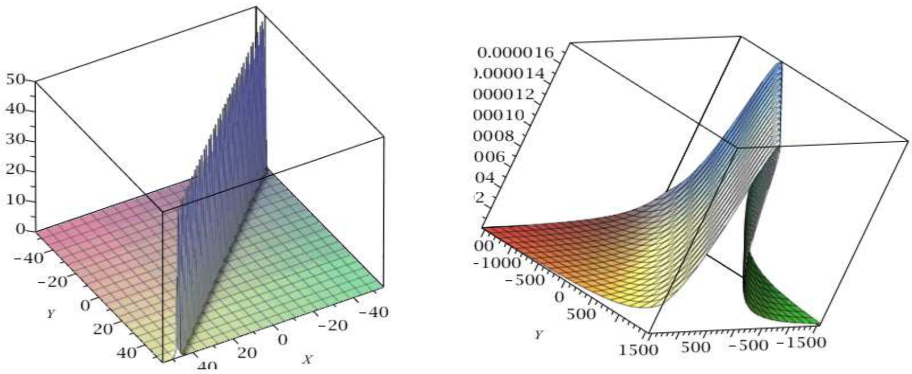

Figure 3.1: Solution of order 1 to (3.1), on the left $T = 10$, $a_1 = 10^6$, $c_1 = 1$, $d_1 = 1$; on the right $T = 10$, $a_1 = 1$, $c_1 = 10^3$, $d_1 = 1$.

Remark 3.3. *The case where* $T = 10$, $a_1 = 1$, $c_1 = 1$, $d_1 = 10^3$ *gives the same figure as the case* $T = 10$, $a_1 = 1$, $c_1 = 10^3$, $d_1 = 1$.

3.3. Second order rational solutions

We obtain the following solutions :

Theorem 3.4. *The function v defined by*

$$v(X.Y,T) = -2\,\frac{n(X.Y,T)}{d(X.Y,T)^2}, \tag{3.3}$$

with

$$n(X.Y,T) = 144\,a_1a_2(-41472\,{a_2}^3{a_1}^3 + 62208\,{a_2}^5a_1 + 6912\,{a_1}^5a_2)X^4 + 144\,a_1a_2(-124416\,{a_2}^3{a_1}^3T - 27648\,{a_1}^5Ya_2 + 165888\,{a_2}^3{a_1}^3Y + 9216\,i{a_1}^5a_2$$
$$-55296\,i{a_2}^3{a_1}^3 + 186624\,{a_2}^5a_1T + 82944\,i{a_2}^5a_1 + 20736\,{a_1}^5a_2T - 248832\,{a_2}^5Ya_1)X^3 + 144\,a_1a_2(-248832\,{a_2}^3{a_1}^3Y^2 - 27648\,i{a_1}^5Ya_2 + 23328\,{a_1}^5a_2T^2 +$$
$$373248\,{a_2}^3{a_1}^3YT + 165888\,i{a_2}^3{a_1}^3Y - 124416\,i{a_2}^3{a_1}^3T + 41472\,{a_1}^5Y^2a_2 - 62208\,{a_1}^5Ya_2T - 559872\,{a_2}^5Ya_1T - 139968\,{a_2}^3{a_1}^3T^2 - 41472\,{a_2}^5a_1 + 27648\,{a_2}^3{a_1}^3 +$$
$$209952\,{a_2}^5a_1T^2 + 186624\,i{a_2}^5a_1T + 373248\,{a_2}^5Y^2a_1 - 248832\,i{a_2}^5Ya_1 + 20736\,i{a_1}^5a_2T - 4608\,{a_1}^5a_2)X^2 + 144\,a_1a_2(-419904\,{a_2}^5Ya_1T^2 + 279936\,{a_2}^3{a_1}^3YT^2 -$$
$$46656\,{a_1}^5Ya_2T^2 + 62208\,{a_1}^5Y^2a_2T + 559872\,{a_2}^5Y^2a_1T - 373248\,{a_2}^3{a_1}^3Y^2T - 373248\,i{a_2}^5Ya_1T + 248832\,i{a_2}^3{a_1}^3YT - 41472\,i{a_1}^5Ya_2T - 18432\,i{a_2}^3c_1 +$$
$$41472\,{a_2}^3{a_1}^3T - 6912\,{a_1}^5a_2T - 62208\,{a_2}^5a_1T + 18432\,id_2{a_1}^3 - 18432\,i{a_1}^3c_2 + 18432\,id_1{a_2}^3 + 139968\,i{a_2}^5a_1T^2 - 93312\,i{a_2}^3{a_1}^3T^2 + 15552\,i{a_1}^5a_2T^2 +$$
$$27648\,i{a_1}^5Y^2a_2 + 248832\,i{a_2}^5Y^2a_1 - 165888\,i{a_2}^3{a_1}^3Y^2 - 55296\,i{a_2}^2d_2a_1 - 6144\,i{a_1}^2d_1a_2 + 55296\,i{a_2}^2c_2a_1 + 6144\,i{a_1}^2c_1a_2 - 69984\,{a_2}^3{a_1}^3T^3 + 11664\,{a_1}^5a_2T^3 +$$
$$104976\,{a_2}^5a_1T^3 - 248832\,{a_2}^5a_1Y^3 + 110592\,{a_2}^3{a_1}^3Y + 165888\,{a_2}^3{a_1}^3Y^3 - 18432\,{a_1}^5Ya_2 - 27648\,{a_1}^5Y^3a_2 - 165888\,{a_2}^5Ya_1)X + 144\,a_1a_2(6144\,{a_2}^3c_1 +$$
$$6144\,{a_1}^3c_2 + 41472\,i{a_2}^2c_2a_1T + 4608\,i{a_1}^2c_1a_2T - 104976\,{a_2}^5Ya_1T^3 + 69984\,{a_2}^3{a_1}^3YT^3 - 11664\,{a_1}^5Ya_2T^3 + 55296\,iY{a_2}^2d_2a_1 + 6144\,i{a_1}^2Yd_1a_2$$
$$-55296\,iY{a_2}^2c_2a_1 - 6144\,i{a_1}^2Yc_1a_2 - 139968\,i{a_2}^5Ya_1T^2 + 93312\,i{a_2}^3{a_1}^3YT^2 - 15552\,i{a_1}^5Ya_2T^2 + 20736\,i{a_1}^5Y^2a_2T + 186624\,i{a_2}^5Y^2a_1T - 124416\,i{a_2}^3{a_1}^3Y^2T -$$
$$41472\,i{a_2}^2d_2a_1T - 4608\,i{a_1}^2d_1a_2T - 13122\,{a_2}^3{a_1}^3T^4 + 19683\,{a_2}^5a_1T^4 + 2187\,{a_1}^5a_2T^4 + 13824\,id_2{a_1}^3T - 13824\,i{a_1}^3c_2T + 13824\,id_1{a_2}^3T - 23328\,i{a_2}^3{a_1}^3T^3 +$$
$$3888\,i{a_1}^5a_2T^3 - 82944\,i{a_2}^5a_1Y^3 + 55296\,i{a_2}^3{a_1}^3Y + 55296\,i{a_2}^3{a_1}^3Y^3 - 9216\,i{a_1}^5Ya_2 - 9216\,i{a_1}^5Y^3a_2 - 82944\,i{a_2}^5Ya_1 - 18432\,iY{a_2}^3d_1 - 18432\,i{a_1}^3Yd_2 +$$
$$18432\,iY{a_2}^3c_1 + 18432\,i{a_1}^3Yc_2 + 34992\,i{a_2}^5a_1T^3 - 13824\,i{a_2}^3c_1T - 124416\,{a_2}^5Ya_1T + 82944\,{a_2}^3{a_1}^3YT - 186624\,{a_2}^5a_1Y^3T + 124416\,{a_2}^3{a_1}^3Y^3T - 13824\,{a_1}^5Ya_2T$$

$-20736a_1{}^5Y^3a_2T+209952a_2{}^5Y^2a_1T^2-139968a_2{}^3a_1{}^3Y^2T^2+23328a_1{}^5Y^2a_2T^2+62208Y^4a_2{}^5a_1-41472Y^4a_2{}^3a_1{}^3+6912a_1{}^5Y^4a_2+2048a_1{}^2d_1a_2+18432a_2{}^2d_2a_1-6144d_2a_1{}^3-6144d_1a_2{}^3+15552a_2{}^3a_1{}^3T^2-2592a_1{}^5a_2T^2-23328a_2{}^5a_1T^2-138240a_2{}^3a_1{}^3Y^2+207360a_2{}^5Y^2a_1+23040a_1{}^5Y^2a_2-2048a_1{}^2c_1a_2-18432a_2{}^2c_2a_1)$,

and

$d(X.Y,T)=-1728ia_1a_2{}^3X^3+576ia_1{}^3a_2X^3+1728a_1a_2{}^3X^2-576a_1{}^3a_2X^2-3888ia_1a_2{}^3X^2T+5184iYa_1a_2{}^3X^2-1728iYa_1{}^3a_2X^2+1296ia_1{}^3a_2TX^2+7776iYa_1a_2{}^3TX+972ia_1{}^3a_2T^2X-2916ia_1a_2{}^3XT^2-5184iY^2a_1a_2{}^3X+1728iY^2a_1{}^3a_2X-2592iYa_1{}^3a_2XT-864a_1{}^3a_2TX-3456Ya_1a_2{}^3X+2592a_1a_2{}^3TX+1152Ya_1{}^3a_2X-768a_1c_2-256a_2d_1-3888iY^2a_1a_2{}^3T+2916iYa_1a_2{}^3T^2+1728iY^3a_1a_2{}^3-576iY^3a_1{}^3a_2-3456iYa_1a_2{}^3+1152iYa_1{}^3a_2-972iYa_1{}^3a_2T^2-729ia_1a_2{}^3T^3+256a_2c_1+243ia_1{}^3a_2T^3+768a_1d_2+1296iY^2a_1{}^3a_2T+972a_1a_2{}^3T^2-2592Ya_1a_2{}^3T-324a_1{}^3a_2T^2+864Ya_1{}^3a_2T+1728Y^2a_1a_2{}^3-576Y^2a_1{}^3a_2$,

is a rational solution to the KP equation (3.1), quotient of two polynomials with numerator of degree 4 *in x, y and t and denominator of degree* 6 *in x, y and t.*

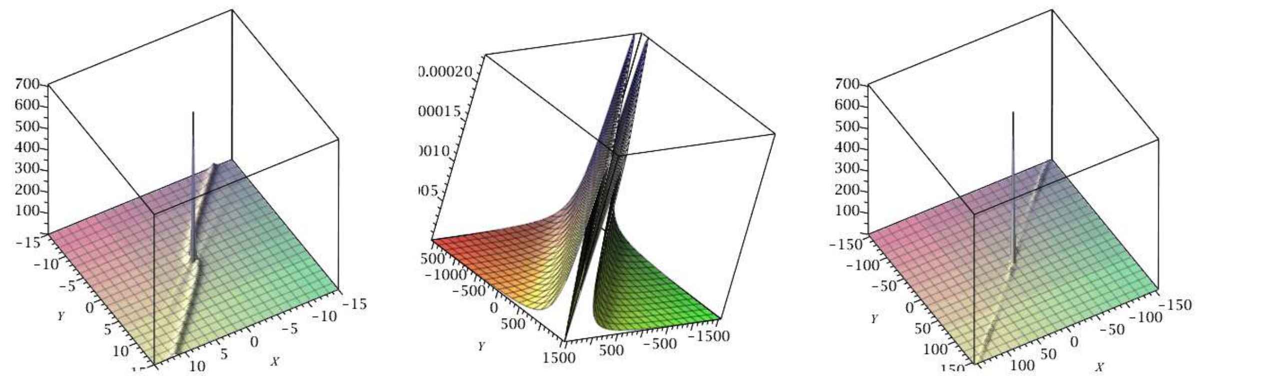

Figure 3.2: Solution of order 2 to (3.1); on the left $T=0,1$, $a_1=1$, $a_2=1$, $c_1=0$, $c_2=0$, $d_1=0$, $d_2=0$; in the center $T=0,1$, $a_1=1$, $a_2=1$, $c_1=0$, $c_2=10^8$, $d_1=0$, $d_2=0$; on the right $T=0,1$, $a_1=1$, $a_1=10^9$, $c_1=1$, $c_2=1$, $d_1=0$, $d_2=0$.

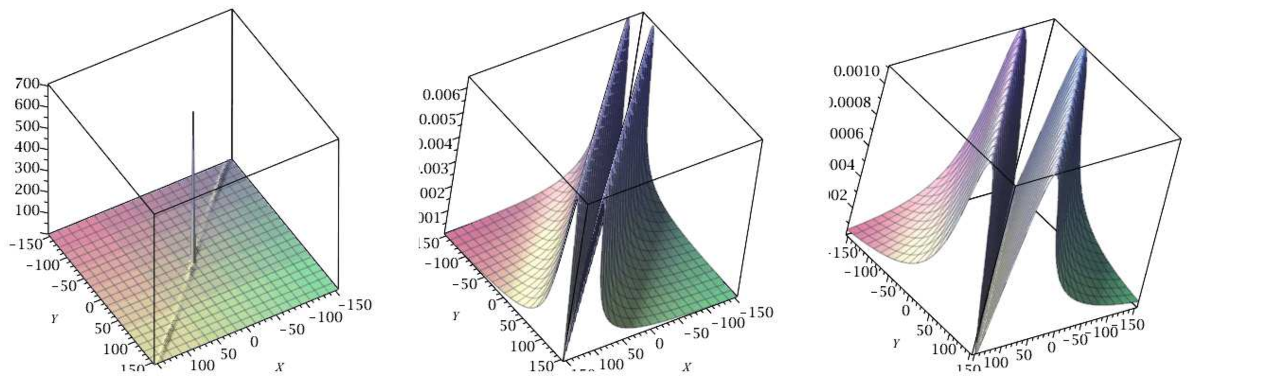

Figure 3.3: Solution of order 2 to (3.1); on the left $T=0,1$, $a_1=10^6$, $a_2=1$, $c_1=1$, $c_2=1$, $d_1=0$, $d_2=0$; in the center $T=0,1$, $a_1=1$, $a_2=1$, $c_1=10^6$, $c_2=0$, $d_1=0$, $d_2=0$; on the right $T=10$, $a_1=1$, $a_2=1$, $c_1=1$, $c_2=10^7$, $d_1=0$, $d_2=0$.

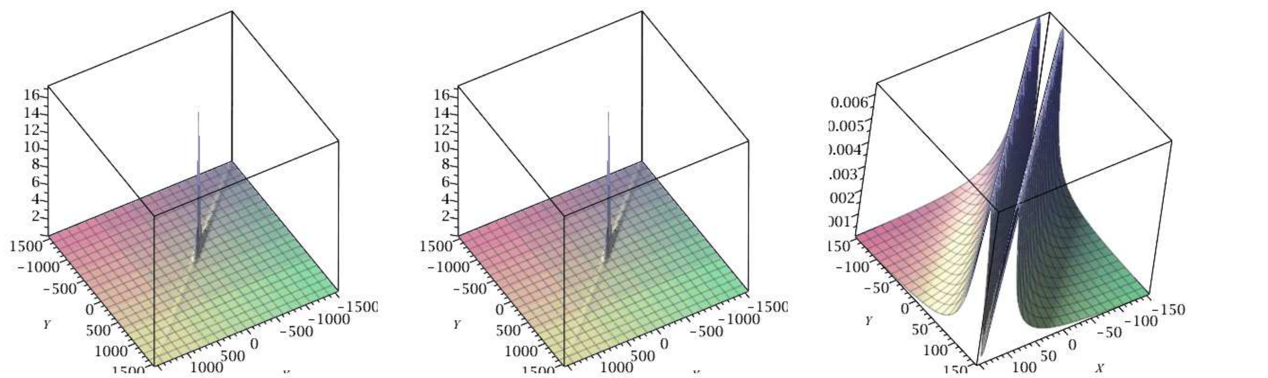

Figure 3.4: Solution of order 2 to (3.1); on the left $T=10$, $a_1=1$, $a_2=10^9$, $c_1=1$, $c_2=1$, $d_1=0$, $d_2=0$; in the center $T=10$, $a_1=10^{10}$, $a_2=1$, $c_1=1$, $c_2=1$, $d_1=0$, $d_2=0$; on the right $T=10$, $a_1=1$, $a_2=1$, $c_1=1$, $c_2=10^6$, $d_1=0$, $d_2=0$.

3.4. Rational solutions of order 3

We get the non singular rational solutions given by :

Theorem 3.5. *The function v defined by*

$$v(X.Y,T) = -2\,\frac{n(X.Y,T)}{d(X.Y,T)^2}, \tag{3.4}$$

is a rational solution to the KP equation (3.1), quotient of two polynomials with numerator of degree 10 *in X, Y, T and denominator of degree* 12 *in X, Y and T.*

Because of the length of the solution, we only give it in the appendix.

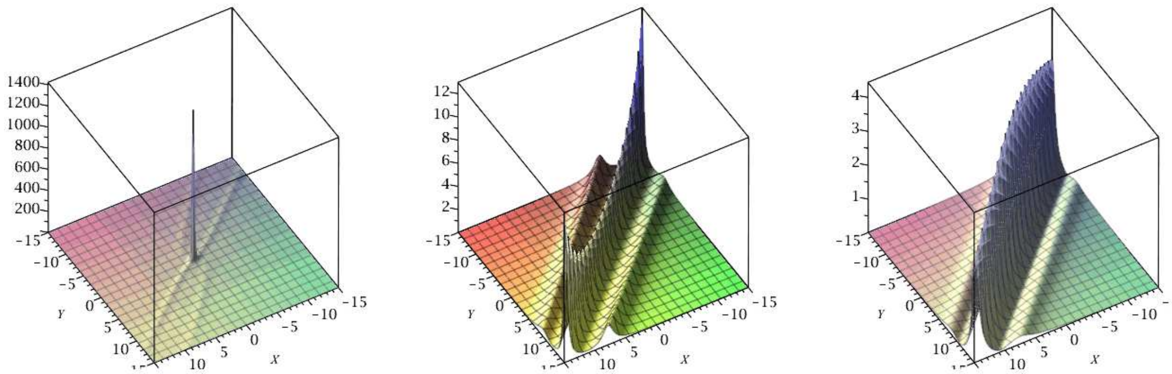

Figure 3.5: Solution of order 3 to (3.1); on the left $T=0,1$, $a_1=1$, $a_2=1$, $a_3=1$, $c_1=0$, $c_2=0$ $c_3=0$, $d_1=0$, $d_2=0$, $d_3=0$; in the center $T=0,1$, $a_1=1$, $a_2=1$, $a_3=1$, $c_1=1$, $c_2=0$, $c_3=10^6$, $d_1=0$, $d_2=0$, $d_3=0$; on the right $T=0,1$, $a_1=1$, $a_2=1$, $a_3=1$, $c_1=0$, $c_2=10^6$, $c_3=1$, $d_1=0$, $d_2=0$, $d_3=0$.

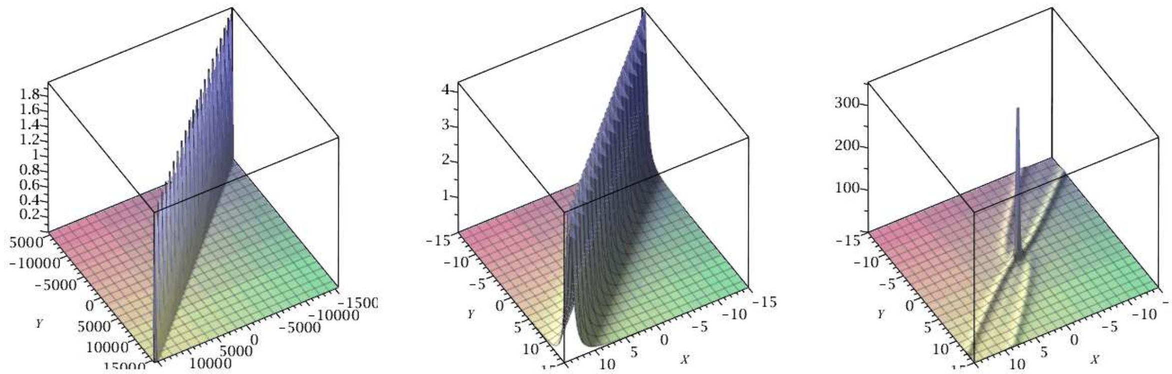

Figure 3.6: Solution of order 3 to (3.1); on the left $T=0,1$, $a_1=1$, $a_2=1$, $a_3=10^{24}$, $c_1=1$, $c_2=1$ $c_3=1$, $d_1=0$, $d_2=0$, $d_3=0$; in the center $T=0,1$, $a_1=1$, $a_2=10^4$, $a_3=1$, $c_1=1$, $c_2=1$, $c_3=1$, $d_1=0$, $d_2=0$, $d_3=0$; on the right $T=0,1$, $a_1=10$, $a_2=1$, $a_3=1$, $c_1=1$, $c_2=10$, $c_3=1$, $d_1=0$, $d_2=0$, $d_3=0$.

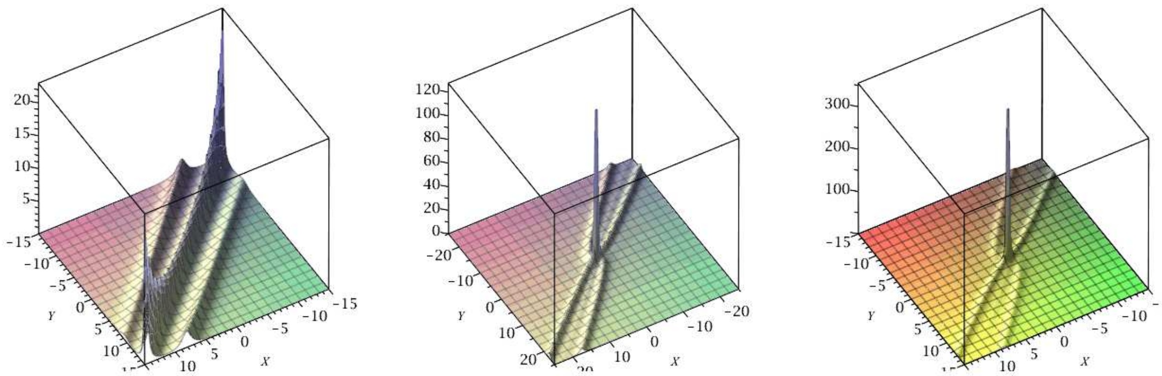

Figure 3.7: Solution of order 3 to (3.1); on the left $T=0,1$, $a_1=1$, $a_2=1$, $a_3=1$, $c_1=10^6$, $c_2=1$ $c_3=1$, $d_1=0$, $d_2=0$, $d_3=0$; in the center $T=1$, $a_1=1$, $a_2=1$, $a_3=1$, $c_1=0$, $c_2=0$, $c_3=0$, $d_1=0$, $d_2=0$, $d_3=0$; on the right $T=1$, $a_1=10^6$, $a_2=1$, $a_3=1$, $c_1=1$, $c_2=1$, $c_3=1$, $d_1=0$, $d_2=0$, $d_3=0$.

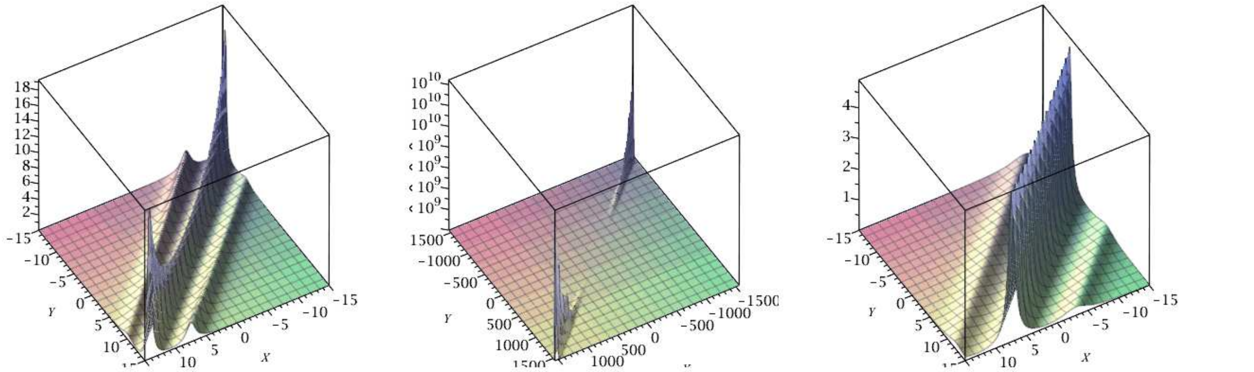

Figure 3.8: Solution of order 3 to (3.1); on the left $T=1$, $a_1=1$, $a_2=1$, $a_3=1$, $c_1=10^6$, $c_2=1$ $c_3=1$, $d_1=0$, $d_2=0$, $d_3=0$; in the center $T=10$, $a_1=10^6$, $a_2=1$, $a_3=1$, $c_1=1$, $c_2=1$, $c_3=1$, $d_1=0$, $d_2=0$, $d_3=0$; on the right $T=10$, $a_1=1$, $a_2=1$, $a_3=1$, $c_1=1$, $c_2=1$, $c_3=10^7$, $d_1=0$, $d_2=0$, $d_3=0$.

4. Conclusion

We have given three types of representations of solutions to the Boussinesq equation. First, solutions in terms of elementary exponential functions have been constructed. In particular, performing a passage to the limit when one parameter goes to 0 we get rational solutions to the Boussinesq equation. We give an other representation in terms of determinants without the presence of a limit. So we obtain an infinite hierarchy of multiparametric families of rational solutions to the Boussinesq equation as a quotient of a polynomial of degree $N(N+1)-2$ in x, t by a polynomial of degree $N(N+1)$ in x, t depending on $3N$ real parameters.
As a byproduct, we get easily similar rational solutions to the Kadomtsev Petviashvili equation as the quotient of determinants of polynomials, where the numerator is a polynomial of degree $N(N+1)-2$ in X, Y, T and the denominator is a polynomial of degree $N(N+1)$ in X, Y, T.
In particular, we construct explicit rational solutions to the Boussinesq equation of order 1, 2, 3.
Unlike other equations such as NLS, there are no specific structures that emerge as a function of the parameters.

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