SDÜ FEN EDEBİYAT FAKÜLTESİ
FEN DERGİSİ (E-DERGİ). 2008, 3(1), 107-112

ITERATIVE INVERSE ALGORITHM FOR PERTURBED MATRIX

Kemal AYDIN, Gülnur ÇELIK KIZILKAN

Selçuk University, Art and Science Faculty, Department of Mathematics,
Konya, Turkey e- mail: kaydin@selcuk.edu.tr, gckizilkan@selcuk.edu.tr
Received: 01 January 2008, Accepted: 24 April 2008

Abstract: In this study, we have given an iterative inverse method (IIM) to compute the inverse of a perturbed matrix $A+D$ and an iterative inverse algorithm (IIA) based on IIM. IIM is also used to control the regularity of any square matrix A and to compute its inverse taking $A = I + D$, where I is unit matrix. We have also given numerical example using IIA.

Key words: Inverse of perturbed matrix, Shermann-Morrison formula, Iterative inverse algorithm.

Mathematics Subject Classifications (2000): 15A09, 65F05

PERTÜRBE MATRİSLER İÇİN İTERATİF TERS ALGORİTMASI

Özet: Bu çalışmada, $A+D$ matrisinin tersini hesaplamak için iteratif ters metodu (IIM) ve IIM üzerine kurulan iteratif ters algoritması (IIA) verildi. IIM, I birim matris olmak üzere, $A = I + D$ alarak A matrisinin regülerliğini kontrol etmek ve tersini hesaplamak için de kullanılır. Ayrıca, IIA kullanılarak nümerik örnekler verildi.

Anahtar kelimeler: Pertürbe matrisin tersi, Shermann- Morrison formülü, İteratif ters algoritması.

1. INTRODUCTION

Studying on solution of the systems of linear algebraic equation $AX = f$ is a classical problem which is important in both linear algebra and applied mathematics. Since the solution of given system is $X = A^{-1} f$, where A is $n\times n$-regular matrix, X and f are n-vectors, the solution of perturbed system as $(A+D)Y = f$ is also

$$Y = (A + D)^{-1} f \tag{1.1}$$

where D is $n\times n$ -perturbation matrix such as matrix $(A+D)$ to be regular. Therefore, it has to be noted that calculating inverse of the perturbed matrix $A+D$ is important.

For U, V-$n\times k$ matrices and $D = UV^T$, the inverse of matrix $(A+D)$ can be computed by Shermann-Morrison-Woodbury formula

$$(A+ UV^T)^{-1}= A^{-1} – A^{-1}U(I+ V^TA^{-1}U)^{-1}V^TA^{-1},$$

where matrix $(I+ V^TA^{-1}U)$ is regular (GOLUB & VAN LOAN 1991).

It is well known that matrix C^{-1} is given to be $C^{-1}= A^{-1} - C^{-1}(C-A)A^{-1}$ which shows how the inverse changes as the matrix changes(GOLUB & VAN LOAN 1991, BULGAK & BULGAK 2001). This identity can be rewritten as

$$(A+D)^{-1}= A^{-1} - A^{-1}(A^{-1}+ D^{-1})^{-1}A^{-1} \tag{1.2}$$

by taking $C = A + D$ where D-invertible matrix. A variant of Sherman Morrison formula was given in (HOUSEHOLDER 1953, BULGAK 2003) as follows,

$$(A+aE_{pq})^{-1} = A^{-1} - \frac{a}{1+a\,G_{qp}}\left(G_{ip}\right)(G_{qj}), \tag{1.3}$$

where $1+aG_{qp} \neq 0$, a is a scalar, (G_{ip}) - p^{th} column vector of matrix A^{-1}, (G_{qj}) - q^{th} row vector of matrix A^{-1} and $E_{pq}= (e_{ij})$; $e_{ij}=\begin{cases}1 & i=p, j=q \\ 0 & otherwise\end{cases}$.

A modification of (1.3) is given for the symmetric matrix A as follows,

$$(A+aE_{pq}+aE_{qp})^{-1} = S - \frac{a}{1+a\,S_{pq}}\left(S_{iq}\right)(S_{pj}), \tag{1.4}$$

where $S = (A+aE_{pq})^{-1}$, $1+aS_{pq} \neq 0$ and $S_{pq} = G_{pq} - \frac{a}{1+aG_{qp}} G_{pp}G_{qq}$ (AYDIN 2004). Therefore, for a non-symmetric matrix, the formula (1.4) can be written simply as

$$(A+aE_{pq}+bE_{mn})^{-1} = S - \frac{b}{1+bS_{nm}}\left(S_{im}\right)(S_{nj}), \tag{1.5}$$

where $S = (A+aE_{pq})^{-1}$, $1+bS_{nm} \neq 0$, b - scalar and $S_{nm} = G_{nm} - \frac{a}{1+aG_{qp}} G_{np}G_{qm}$. We note that the formulas (1.4) and (1.5) are only depend on the elements of matrix A^{-1}.

In (CHANG 2006), the inverse of matrix $(A+D)$ is given taking $A^{-1} = B$ as follows,

$$(A+D)^{-1}= B - B(I+ DB)^{-1}DB = B - BD(I+BD)^{-1}B. \tag{1.6}$$

Thus, author of (CHANG 2006) has said that the formula (1.2) is not feasible for computing the matrix $(A+D)^{-1}$ because both matrices D and $(A^{-1}+D^{-1})$ are regular matrices and therefore, the regularity requirement of matrix D is removed by courtesy of the formula (1.6). In addition, the matrices D and B have been partitioned in (CHANG 2006) to be

$$D=\begin{pmatrix}\overline{\underline{D}} & 0 \\ 0 & 0\end{pmatrix},\ B=\begin{pmatrix}\overline{\underline{B}} & B_2 \\ B_1 & B_3\end{pmatrix} \text{ and } \overline{B}=\begin{pmatrix}\overline{\underline{B}} \\ B_1\end{pmatrix},\ \underline{B}=\begin{pmatrix}\overline{\underline{B}} & B_2\end{pmatrix},$$

where $\overline{\underline{D}}$ is formed through by the selected rows and columns scattering non-zero elements within D, the element positions of $\overline{\underline{B}}$ are the transport elements positions of $\overline{\underline{D}}$. With respect to the partitions of matrices D and B, the formula (1.6) has been rewritten as

$$(A+D)^{-1}= B - \overline{B}\,(\mathrm{I}+ \overline{\underline{D}}\,\overline{\underline{B}}\,)^{-1}\overline{\underline{D}}\,\underline{B} = B - \overline{B}\,\overline{\underline{D}}\,(I+ \overline{\underline{B}}\,\overline{\underline{D}}\,)^{-1}\underline{B}\,. \tag{1.7}$$

In this equality, it has been used the inverse of matrix $(I+\underline{\overline{D}}\,\underline{\overline{B}})$ (or $(I+\underline{\overline{B}}\,\underline{\overline{D}})$) for computing the matrix $(A+D)^{-1}$. Computing inverse process needs any procedure to control the regularity of and compute the inverse of matrix $(I+\underline{\overline{D}}\,\underline{\overline{B}})$ instead of the requirement regularity of matrix D. So in fact, we should emphasize that the formula (1.7) (or (1.6)) is not feasible as (1.2), too. Even though the regularity of matrix $(I+\underline{\overline{D}}\,\underline{\overline{B}})$ is guaranteed, order of matrix $(I+\underline{\overline{D}}\,\underline{\overline{B}})$ for $D = \underline{\overline{D}}$ is same as order of matrix A. In this case, computation of the matrix $(I+\underline{\overline{D}}\,\underline{\overline{B}})^{-1}$ costs as much computation of the matrix $(A+D)^{-1}$. In addition to these, the formula (1.7) is equal to the formula (1.3) for $\underline{\overline{D}} = (d)$ where d is a scalar.

We have given an iterative inverse method (IIM) which computes the inverse of a perturbed matrix $A+D$ in Section 2 and an iterative inverse algorithm (IIA) based on this method in Section 3 and also numerical example using this algorithm in Section 4.

2. ITERATIVE INVERSE METHOD (IIM)

Let matrix A is perturbed by matrix D. Any matrix $D = (d_{ij})$ can be written in form of

$$D = \sum_{i=1}^{N}\sum_{j=1}^{N} d_{ij} E_{ij} \ .$$

If $(A+D)$ is regular and matrix A^{-1} is known, there are two situations to compute the inverse of matrix $(A+D)$.

i. For $k, r = 1(1)N$, $l=(k-1)N+r$ and $A_0 = A$,

$$A_l^{-1} = (A_{l-1}+d_{kr}E_{kr})^{-1}$$

$$= A_{l-1}^{-1} - \frac{d_{kr}}{1+d_{kr}\left(A_{l-1}^{-1}\right)_{rk}} \left(\left(A_{l-1}^{-1}\right)_{ik}\right)\left(\left(A_{l-1}^{-1}\right)_{rj}\right), \tag{2.1}$$

where $A_l = A + \sum_{i=1}^{k}\sum_{j=1}^{r} d_{ij}E_{ij}$ and $1+d_{kr}\left(A_{l-1}^{-1}\right)_{rk} \neq 0$. The formula (2.1) has been obtained by applying (1.5), successively. Notice that $A_l^{-1} = A_{N^2}^{-1} = A+D$ for $k,r = N$.

ii. In case of $1+d_{mn}\left(A_s^{-1}\right)_{nm} = 0$ for some elements d_{mn} of D, the inverse of the matrix $(A_s+d_{pq}E_{pq})$, which composed of the element d_{pq} of D such that $1+d_{pq}\left(A_s^{-1}\right)_{qp} \neq 0$, is computed by

$$(A_s+d_{pq}E_{pq})^{-1} = A_s^{-1} - \frac{d_{pq}}{1+d_{pq}\left(A_s^{-1}\right)_{qp}} \left(\left(A_s^{-1}\right)_{ip}\right)\left(\left(A_s^{-1}\right)_{qj}\right),$$

where the matrix A_s is the last matrix which has been computed using the formula (2.1). It has been continued the same procedure until the matrix $A+D$ is obtained. If there is not an element d_{pq} of D such that $1+d_{pq}\left(A_s^{-1}\right)_{qp} \neq 0$, then the matrix $A+D$ is singular.

Note. IIM also can *directly* be used to control the regularity of any square matrix A and compute the inverse matrix if the matrix A is regular.

3. ITERATIVE INVERSE ALGORITHM (IIA)

Input. $A = (a_{ij})$ –regular matrix, $A^{-1} = G = (G_{ij})$ – inverse matrix, $D = (d_{ij})$ perturbation matrix.

Step 1. Compose the set $D(0)$ which consists of non-zero elements of D. Let m is number of elements of set $D(0)$.

Step 2. $k = 1(1)m$;

2.1. Find an element of $D(k\text{-}1)$ such that $1 + d_{ij} G_{ji}^{(k-1)} \neq 0$, where $G^{(0)} = G = A^{-1}$; let $d_{ij} = d_{pq}$. If there is not an element d_{pq} of D such that $1 + d_{pq}\left(A_s^{-1}\right)_{qp} \neq 0$, go to Output 2.

2.2. Compute $A(k) = A(k\text{-}1) + d_{pq} E_{pq}$; get $A(0) = A$, $E_{pq} = (e_{ij})$.

2.3. Compute $G^{(k)} = G^{(k-1)} - \dfrac{d_{pq}}{1 + d_{pq} G_{qp}^{(k-1)}} \left(G_{ip}^{(k-1)}\right) \left(G_{qj}^{(k-1)}\right)$.

2.4. Constitute $D(k) = D(k\text{-}1) - \{ d_{pq} \}$; $D(m) = \varnothing$.

Output 1. $(A+D)^{-1} = G^{(m)}$.

Output 2. The matrix $(A+D)$ is singular.

4. ILLUSTRATIVE EXAMPLE

In present section, let us give an example which shows how the inverse of perturbed matrix and inverse of any given matrix A can be computed by applying IIA.

Input. Let $I = I^{-1} = G$ and $D = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{pmatrix}$; $A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} = I + D.$.

Step 1. $D(0) = \{d_{11} = 1,\ d_{13} = -1,\ d_{31} = 1,\ d_{33} = -2\}$ and $m = 4$.
Step 2. $k = 1(1)4$;

2.**1**.1. For ***k*=1,** $d_{11} \in D(0)$, $1 + d_{11} G_{11}^{(0)} = 1+1\times 1 = 2 \neq 0 \Rightarrow d_{pq} = d_{11}$.

2.1.2. $A(1) = A(0) + d_{11}E_{11} = I + E_{11} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

2.1.3. $G^{(1)} = G^{(0)} - \dfrac{d_{11}}{1 + d_{11} G_{11}^{(0)}} \left(G_{i1}^{(0)}\right)\left(G_{1j}^{(0)}\right) = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

2.1.4. $D(1) = D(0) - \{ d_{11} \} = \{ d_{13} = -1, d_{31} = 1, d_{33} = -2 \}$.

2.**2**.1. For ***k*=2,** $d_{13} \in D(1)$, $1 + d_{13} G_{31}^{(1)} = 1\text{-}1\times 0 = 1 \neq 0 \Rightarrow d_{pq} = d_{13}$.

2.2.2. $A(2) = A(1) + d_{13}E_{13} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

2.2.3. $G^{(2)} = G^{(1)} - \dfrac{d_{13}}{1+d_{13}G^{(1)}_{31}}\left(G^{(1)}_{i1}\right)\left(G^{(1)}_{3j}\right) = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

2.2.4. $D(2) = D(1) - \{ d_{13} \} = \{ d_{31} = 1, d_{33} = -2\}$.

2.**3**.1. For ***k*=3,** $d_{31} \in D(2)$, $1 + d_{31}G^{(2)}_{13} = 1 + 1\times(1/2) = 3/2 \neq 0 \Rightarrow d_{pq} = d_{31}$.

2.3.2. $A(3) = A(2) + d_{31}E_{31} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

2.3.3. $G^{(3)} = G^{(2)} - \dfrac{d_{31}}{1+d_{31}G^{(2)}_{13}}\left(G^{(2)}_{i3}\right)\left(G^{(2)}_{1j}\right) = \begin{pmatrix} 1/3 & 0 & 1/3 \\ 0 & 1 & 0 \\ -1/3 & 0 & 2/3 \end{pmatrix}$.

2.3.4. $D(3) = D(2) - \{ d_{31} \} = \{ d_{33} = -2\}$.

2.**4**.1. For ***k*=4,** $d_{33} \in D(3)$, $1 + d_{33}G^{(3)}_{33} = 1 - 2\times(2/3) = -1/3 \neq 0 \Rightarrow d_{pq} = d_{33}$.

2.4.2. $A(4) = A(3) + d_{33}E_{33} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$.

2.4.3. $G^{(4)} = G^{(3)} - \dfrac{d_{33}}{1+d_{33}G^{(3)}_{33}}\left(G^{(3)}_{i3}\right)\left(G^{(3)}_{3j}\right) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{pmatrix}$.

2.4.4. $D(m) = D(4) = D(3) - \{ d_{33} \} = \varnothing$.

Output 1. $A^{-1} = (I+D)^{-1} = G^{(4)} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{pmatrix}$ is obtained.

5. CONCLUSION

The formulas (1.7) (or (1.6)) and (1.2) can not compute *directly* the inverse of a perturbed matrix. These formulas need some usual matrix inversion method. IIM (IIA) computes *directly* the inverse of a perturbed matrix without using any matrix inversion method. IIM (IIA) also can be used *directly* to compute the inverse matrix and to control the regularity of any square matrix A taking $A = I + D$, where I is unit matrix. IIM (IIA) can be applied manually or using computer programming. There is an argument that IIA based on IIM.

REFERENCES

AYDIN K, 2004. Algorithm for checking a practical regularity of symmetric interval matrices, *Sibirskii Zhurnal Industrial'noi Matematiki,* 7(3),15-20 (in Russian).

BULGAK A, 2003. Algorithm for checking a practical regularity of interval matrices, *Sibirskii Zhurnal Industrial'noi Matematiki,* 6(1), 17-23 (in Russian).

BULGAK A, BULGAK H, 2001. Linear Algebra, *Selçuk Univ. Research Center of Applied Mathematics*, Konya (in Turkish).

CHANG F.C., 2006. Inversion of a perturbed matrix, *Applied Mathematics Letters*, 19, 169-173

GOLUB GH, VAN LOAN CF, 1991. *Matrix Computation*, 2nd Edition, The Johns Hopkins University Press, Baltimore.

HOUSEHOLDER AS, 1953. *Principles of Numerical Analysis*, McGraw-Hill Book Company, Inc., New York, Toronto, London.