On the Join Operation of Graphs Obtained by Monogenic Semigroups

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ABSTRACT. For each commutative ring R we associate a simple graph $\Gamma(R)$. This relationship presents a link between algebra and graph theory. Our main scope in this study is to extend this study over the special algebraic graphs to join graph operations. In this paper, we will give some graph parameters for the join of monogenic semigroup graphs.

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1. Introduction

In [6], Beck put forward the opinion of the zero divisor graph. This concept has been studied in various forms to the present day. Similarly, this concept was defined in semigroups and many studies were done [3–5, 8, 9].

In a recently study Das et.al [7], the graph $\Gamma(S_M)$ defined. They considered the monogenic semigroup (with zero) $S_M = \{0, x, x^2, ..., x^p\}$. In here, $V(\Gamma(S_M)) = \{x, x^2, ..., x^p\}$ and the vertices x^i and x^j are adjacent with the rule $x^i.x^j = 0$ if and only if $i + j \geq p + 1$, $(1 \leq i, j \leq p)$. The other words $x^i \sim x^j$ iff $i + j \geq p + 1$, $(1 \leq i, j \leq p)$.

In previous studies, The Cartesian product, the lexicographic product, the strong product, the disjunctive product and the corona product of monogenic semigroup graphs were defined and some properties were given in [1,2,7,13,14].

In this paper, by considering join operation of monogenic semigroup graphs we will give some results for some graph parameters such as the diameter, radius, girth etc.

2. Main Results

The join of two graphs G_1 and G_2 is denoted by $G_1 + G_2$. It has the following vertex-set and edge set:

$$V(G_1 + G_2) = V(G_1) \cup V(G_2) \quad and \quad E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{ij : i \in V(G_1) \quad and \quad j \in V(G_2)\}.$$

In here we replace G_1 by $\Gamma(S_M^1)$ and G_2 by $\Gamma(S_M^2)$, where $S_M^1 = \{x, x^2, ..., x^p\}$ with 0 and $S_M^2 = \{y, y^2, ..., y^s\}$ with 0. We have rules for monogenic semigroup graphs as follows:

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$\Gamma(S_M^1) + \Gamma(S_M^1)$ has vertex set $V(\Gamma(S_M^1) + \Gamma(S_M^2)) = V(\Gamma(S_M^1)) \cup V(\Gamma(S_M^2))$ and let us take any two vertices of $\Gamma(S_M^1) + \Gamma(S_M^2)$ are adjacent if and only if

$$
\begin{cases}
x^i x^j = 0 \Leftrightarrow i + j \geq p + 1, & \text{where} \quad x^i, x^j \in V(\Gamma(S_M^1)) \\
\qquad\qquad\qquad or \\
y^i y^j = 0 \Leftrightarrow i + j \geq s + 1, & \text{where} \quad y^i, y^j \in V(\Gamma(S_M^2)) \\
\qquad\qquad\qquad or \\
x^i y^j = 0 \Leftrightarrow i + j \geq 2, & \text{where} \quad x^i \in V(\Gamma(S_M^1)) \quad and \quad y^j \in V(\Gamma(S_M^2))
\end{cases}
$$

In the following parts of the study, we will use the Γ_1 and Γ_2 notation instead of $\Gamma(S_M^1)$ and $\Gamma(S_M^2)$ for convenience, respectively. Now, we give some properties of the graph $\Gamma_1 + \Gamma_2$.

For two vertices i and j of $V(G)$ we define their distance $d_G(i, j)$ as the length of a shortest path connecting i and j in G. The diameter of G is defined as $diam(G) = max\{d(i, j) : i, j \in V(G)\}$ [10].

Theorem 2.1. *Let $S_M^1 = \{0, x, x^2, ..., x^p\}$ and $S_M^2 = \{0, y, y^2, ..., y^s\}$ be two monogenic semigroups. Then we have*

$$diam(\Gamma_1 + \Gamma_2) = 2.$$

Proof. We know that $x^i \sim x^p$ for all $1 \leq i \leq p - 1$ and $y^j \sim y^s$ for all $1 \leq j \leq s - 1$. Also from definition of join operation we have $x^i \sim y^j$ for all $1 \leq i \leq p$ and $1 \leq j \leq s$. Then we get $diam(\Gamma_1 + \Gamma_2) = 2$. □

If $i \in V(G)$ is a vertex of G, its eccentricity $e(i)$ is defined as $e(i) = max\{d(i, j) : j \in V(G)\}$. The radius of G is defined as $radius(G) = rad(G) = min\{e(i) : i \in V(G)\}$ [10].

Theorem 2.2. *Let $S_M^1 = \{0, x, x^2, ..., x^p\}$ and $S_M^2 = \{0, y, y^2, ..., y^s\}$ be two monogenic semigroups. Then we have*

$$rad(\Gamma_1 + \Gamma_2) = 1.$$

Proof. We easily see that the vertex x^p of Γ_1 is adjacent to all other vertices of Γ_1. Moreover, by the definition of join operation this vertex is adjacent to all vertices of Γ_2. Finally, we get $rad(\Gamma_1 + \Gamma_2) = 1$ □

The girth of a graph G is the length of a shortest cycle in G. If the graph G does not contain any cycle, then the girth is taken as infinite [10] .

Theorem 2.3. *Let $S_M^1 = \{0, x,^2, ..., x^p\}$ and $S_M^2 = \{0, y, y^2, ..., y^s\}$ be two monogenic semigroups. Then the girth of the graph $\Gamma_1 + \Gamma_2$ is 3.*

Proof. By the definition of join operation, we have $x^i \sim x^p$ for all $1 \leq i \leq p - 1$ and $y^j \sim y^s$ for all $1 \leq j \leq s - 1$. Also we have $x^i \sim y^j$ for all $1 \leq i \leq p$ and $1 \leq j \leq s$. That is $x^i \sim x^p \sim y^j \sim x^i$ for every $i = 1, 2, ..., p$ and $j = 1, 2, ..., s$. Then the girth of the graph $\Gamma_1 + \Gamma_2$ is 3. □

The number of edges incident at i in G is called the degree of the vertex i in G and is denoted by $d_G(i)$. Among all degrees, the maximum $\Delta(G)$ (or the minimum $\delta(G)$) degrees of G is the number of the largest (or smallest) degree in G [10].

Theorem 2.4. *Let $S_M^1 = \{0, x, x^2, ..., x^p\}$ and $S_M^2 = \{0, y, y^2, ..., y^s\}$ be two monogenic semigroups. Then*

$$\Delta(\Gamma_1 + \Gamma_2) = p + s - 1.$$

Proof. Let us take the vertex x^p of Γ_1. It is clear that $x^p \sim x^i$ for any $1 \leq i \leq p - 1$. By the definition join operation we have $x^p \sim y^j$ for any $1 \leq j \leq s$. So we get $deg(x^p) = p - 1 + s$. Now let us take the vertex y^s of Γ_2. By similar steps we get $deg(y^s) = s - 1 + p$. Thus, we have $\Delta(\Gamma_1 + \Gamma_2) = p + s - 1$. □

Theorem 2.5. *Let $S_M^1 = \{0, x, x^2, ..., x^p\}$ and $S_M^2 = \{0, y, y^2, ..., y^s\}$ be two monogenic semigroups. Then*

$$\delta(\Gamma_1 + \Gamma_2) = min\{p + 1, s + 1\}.$$

Proof. Among the vertices of $V(\Gamma_1 + \Gamma_2)$, let us take vertices $x \in V(\Gamma_1)$ and $y \in V(\Gamma_2)$. The vertex x is adjacent to the vertex x^p in Γ_1 and all vertices in Γ_2. Also the vertex y is adjacent to the vertex y^s in Γ_2 and all vertices in Γ_1. That is, $deg(x) = s + 1$ and $deg(y) = p + 1$. Thus we get $\delta(\Gamma_1 + \Gamma_2) = min\{p + 1, s + 1\}$. So as desired. □

The degree sequence of a graph is the non-increasing sequence of the degrees of the vertices. The irregularity index is equal to the number of distinct elements in the degree sequence. The degree sequence and irregularity index of the graph G are denoted by $DS(G)$ and $t(G)$, respectively [13].

Theorem 2.6 ([1]). *Let S_M be a monogenic semigroup. Then*

$$DS(\Gamma(S_M)) = \{1, 2, 3, ..., \left\lfloor \frac{p}{2} \right\rfloor - 1, \left\lfloor \frac{p}{2} \right\rfloor, \left\lfloor \frac{p}{2} \right\rfloor, \left\lfloor \frac{p}{2} \right\rfloor + 1, \left\lfloor \frac{p}{2} \right\rfloor + 2, ..., p - 2, p - 1\}$$

and $t(\Gamma(S_M)) = p - 1$, respectively.

Theorem 2.7. *Let $S_M^1 = \{0, x, x^2, ..., x^p\}$ and $S_M^2 = 0, y, y^2, ..., y^s$ be two monogenic semigroups with $p \geq s$. Then the degree sequence of $G = \Gamma_1 + \Gamma_2$ is given by*

$$\begin{aligned} DS(G) \quad = \quad & \{1 + s, 2 + s, 3 + s, ..., \left\lfloor \frac{p}{2} \right\rfloor - 1 + s, \left\lfloor \frac{p}{2} \right\rfloor + s, \left\lfloor \frac{p}{2} \right\rfloor + s, \left\lfloor \frac{p}{2} \right\rfloor + 1 + s, \left\lfloor \frac{p}{2} \right\rfloor + 2 + s, ..., p - 2 + s, p - 1 + s, 1 + p, \\ & 2 + p, 3 + p, ..., \left\lfloor \frac{s}{2} \right\rfloor - 1 + p, \left\lfloor \frac{s}{2} \right\rfloor + p, \left\lfloor \frac{s}{2} \right\rfloor + p, \left\lfloor \frac{s}{2} \right\rfloor + 1 + p, \left\lfloor \frac{s}{2} \right\rfloor + 2 + p, ..., s - 2 + p, s - 1 + p\} \end{aligned}$$

and $t(\Gamma_1 + \Gamma_2) = p - 1$, respectively.

Proof. By the definition of join product and Theorem 2.6, we get the degree sequence of $\Gamma_1 + \Gamma_2$ as follows:

$$\begin{aligned} DS(G) \quad = \quad & \{1 + s, 2 + s, 3 + s, ..., \left\lfloor \frac{p}{2} \right\rfloor - 1 + s, \left\lfloor \frac{p}{2} \right\rfloor + s, \left\lfloor \frac{p}{2} \right\rfloor + s, \left\lfloor \frac{p}{2} \right\rfloor + 1 + s, \left\lfloor \frac{p}{2} \right\rfloor + 2 + s, ..., p - 2 + s, p - 1 + s, 1 + p, \\ & 2 + p, 3 + p, ..., \left\lfloor \frac{s}{2} \right\rfloor - 1 + p, \left\lfloor \frac{s}{2} \right\rfloor + p, \left\lfloor \frac{s}{2} \right\rfloor + p, \left\lfloor \frac{s}{2} \right\rfloor + 1 + p, \left\lfloor \frac{s}{2} \right\rfloor + 2 + p, ..., s - 2 + p, s - 1 + p\} \end{aligned}$$

Moreover, since $p \geq s$, the irregularity index of the graph $\Gamma_1 + \Gamma_2$ is $t(\Gamma_1 + \Gamma_2) = (p - 1 + s) - (1 + s) + 1 = p - 1$, as required. □

A matching M in a graph G is a subset of edges no two of which have a common vertex. The maximum number of edges in a matching of a graph G is called the matching number of G and denoted $\beta(G)$. A matching is perfect if every vertex in G is incident to some edge in the matching [10].

Theorem 2.8. *Let $S_M = \{0, x, x^2, ..., x^p\}$ be a monogenic semigroups. Then the matching number of $G = \Gamma(S_M)$ is given by $\beta(G) = p - \left\lceil \frac{p}{2} \right\rceil$.*

Proof. By the definition of $\Gamma(S_M)$, we get $x^k x^{p+1-k} \in E(\Gamma(S_M))$ for $\left\lceil \frac{p}{2} \right\rceil + 1 \leq k \leq p$. Then a maximum matching in G is $M = \{x^p x, x^{p-1} x^2, ..., x^{\left\lceil \frac{p}{2} \right\rceil + 1} x^{p - \left\lceil \frac{p}{2} \right\rceil}\}$. Finally, we get $\beta(G) = p - \left\lceil \frac{p}{2} \right\rceil$. □

Remark 2.9. If p is even, the graph $\Gamma(S_M)$ is the graph with perfect matchings.

Theorem 2.10. *Let $S_M^1 = \{0, x, x^2, ..., x^p\}$ and $S_M^2 = \{0, y, y^2, ..., y^s\}$ be two monogenic semigroups with $p \geq s$. Then the matching number of $G = \Gamma_1 + \Gamma_2$ is given by*

$$\beta(G) = \begin{cases} \frac{p+s}{2}, & \text{both } p \text{ and } q \text{ are odd or even} \\ \frac{p+s-1}{2}, & \text{otherwise} \end{cases}$$

Proof. Let $S_M^1 = \{0, x, x^2, ...x^p\}$ and $S_M^2 = \{0, y, y^2, ..., y^s\}$ be two monogenic semigroups with $p \geq s$.

Case 1. Let p and s be even. In this case, by the definition of join product and theorem 2.8 we get $x^k x^{p+1-k} \in E(\Gamma_1)$, $\left\lceil \frac{p}{2} \right\rceil + 1 \leq k \leq p$ and $y^k y^{s+1-k} \in E(\Gamma_2)$, $\left\lceil \frac{s}{2} \right\rceil + 1 \leq k \leq s$. Then a maximum matching in G is $M = \{x^p x, x^{p-1} x^2, ..., x^{\left\lceil \frac{p}{2} \right\rceil + 1} x^{p - \left\lceil \frac{p}{2} \right\rceil}, y^s y, y^{s-1} y^2, ..., y^{\left\lceil \frac{s}{2} \right\rceil + 1} y^{s - \left\lceil \frac{s}{2} \right\rceil})\}$. Finally, we get $\beta(G) = p + s - \left\lceil \frac{p}{2} \right\rceil - \left\lceil \frac{s}{2} \right\rceil = p + s - p/2 - s/2 = (p + s)/2$.

Case 2. Let p and s be odd. By similar argument, we get

$$M = \{x^p x, x^{p-1} x^2, ..., x^{\left\lceil \frac{p}{2} \right\rceil + 1} x^{p - \left\lceil \frac{p}{2} \right\rceil}, y^s y, y^{s-1})y^2, ..., y^{\left\lceil \frac{s}{2} \right\rceil + 1} y^{s - \left\lceil \frac{s}{2} \right\rceil}, x^{\frac{p+1}{2}} y^{\frac{s+1}{2}}\}.$$

Then we have

$$\beta(G) = p + s - \left\lceil \frac{p}{2} \right\rceil - \left\lceil \frac{s}{2} \right\rceil + 1 = p + s - (p + 1)/2 - (s + 1)/2 + 1 = (p + s)/2.$$

Case 3. Let p and s be even and odd, respectively. Then a maximum matching in G is
$M = \{x^p x, x^{p-1} x^2, ..., x^{\lceil \frac{p}{2} \rceil +1} x^{p-\lceil \frac{p}{2} \rceil}, y^s y, y^{s-1} y^2, ..., y^{\lceil \frac{s}{2} \rceil +1} y^{s-\lceil \frac{s}{2} \rceil}\}$. Finally , we get $\beta(G) = p + s - \lceil \frac{p}{2} \rceil - \lceil \frac{s}{2} \rceil = p + s - \frac{p}{2} - \frac{s+1}{2} = \frac{p+s-1}{2}$. Proof is completed. $\qquad \square$

A subset A of the vertex set $V(G)$ of a graph is collect the domination set if every vertex $V(G) \setminus A$ is joined to at least one vertex of A by an edge. The domination number $\gamma(G)$ is the number of vertices in the smallest dominating set for G [10].

Theorem 2.11. Let $S_M^1 = \{0, x, x^2, ..., x^p\}$ and $S_M^2 = \{0, y, y^2, ..., y^s\}$ be two monogenic semigroups. Then

$$\gamma(\Gamma_1 + \Gamma_2) = 1.$$

Proof. By the definition of join operation, the vertex x^p is adjacent to all the other vertices. That is, $x^p \sim x^i$ and $x^p \sim y^j$ for $i = 1, 2, ..., p - 1$ and $j = 1, 2, ..., s$. So the dominating set consist of only the element x^p. Then the domination number of the graph $\Gamma_1 + \Gamma_1$ is 1. $\qquad \square$

The chromatic number of a graph G is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color. The chromatic number of a graph G is denoted $\chi(G)$ [11].

Theorem 2.12 ([7]). *The chromatic number of $\Gamma(S_M)$ is equal to $\chi(\Gamma(S_M)) = 1 + \lceil \frac{p-1}{2} \rceil$.*

We now give chromatic number of $\Gamma(S_M^1) + \Gamma(S_M^2)$.

Theorem 2.13. Let $S_M^1 = \{0, x, x^2, ..., x^p\}$ and $S_M^2 = \{0, y, y^2, ..., y^s\}$ be two monogenic semigroups. Then

$$\chi(\Gamma_1 + \Gamma_2) = 2 + \left\lceil \frac{p-1}{2} \right\rceil + \left\lceil \frac{s-1}{2} \right\rceil.$$

Proof. By Theorem 2.12, we have $1 + \lceil \frac{p-1}{2} \rceil$ of chromatic number of the graph Γ_1. Also we have $1 + \lceil \frac{s-1}{2} \rceil$ of chromatic number of the graph Γ_2. Since all vertices in the graph Γ_1 are adjacent all vertices of the graph Γ_2, we cannot use the colors we use in Γ_1 to color the vertices of the graph Γ_2. Then we get

$$\chi(\Gamma_1 + \Gamma_2) = 2 + \left\lceil \frac{p-1}{2} \right\rceil + \left\lceil \frac{s-1}{2} \right\rceil.$$

Proof is completed. $\qquad \square$

A clique of a graph G is a complete subgraph of G. The clique number of a graph G, denoted $\omega(G)$, is the maximum number of vertices in a complete subgraph of G.

Theorem 2.14. Let $S_M^1 = \{0, x, x^2, ..., x^p\}$ and $S_M^2 = \{0, y, y^2, ..., y^s\}$ be two monogenic semigroups. Then

$$\omega(\Gamma_1 + \Gamma_2) = 2 + \left\lceil \frac{p-1}{2} \right\rceil + \left\lceil \frac{s-1}{2} \right\rceil.$$

Proof. Let $A = \{x^{\lceil \frac{p}{2} \rceil}, x^{\lceil \frac{p}{2} \rceil +1}, ..., x^p\}$ be subset of the set $V(\Gamma_1)$ and $B = \{y^{\lceil \frac{s}{2} \rceil}, y^{\lceil \frac{p}{2} \rceil +1}, ..., y^s\}$ be subset of the set $V(\Gamma_2)$. Furthermore, all of the vertices in sets A and B are adjacent to each other. Then we get

$$\omega(\Gamma_1 + \Gamma_2) = |A| + |B| = (p - \left\lceil \frac{p}{2} \right\rceil + 1) + (s - \left\lceil \frac{s}{2} \right\rceil + 1).$$

Finally, the fact that $p - \lceil \frac{p}{2} \rceil = \lceil \frac{p-1}{2} \rceil$ and $q - \lceil \frac{q}{2} \rceil = \lceil \frac{q-1}{2} \rceil$, we get $\omega(\Gamma_1 + \Gamma_2) = 2 + \lceil \frac{p-1}{2} \rceil + \lceil \frac{s-1}{2} \rceil$. So proof is completed. $\qquad \square$
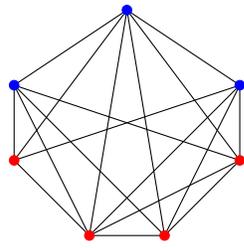
Figure 1: The graph $\Gamma_1 + \Gamma_2$

Example 2.15. Let us consider the semigroups $S_M^1 = \{x, x^2, x^3, x^4\}$ and $S_M^2 = \{y, y^2, y^3\}$. Then the graph $\Gamma_1 + \Gamma_2$ is given as drawn in Figure 1. Then we have

1. $diam(\Gamma_1 + \Gamma_2) = 2$ (by Theorem 2.1.

2. $rad(\Gamma_1 + \Gamma_2) = 1$ (by Theorem 2.2).

3. $girth(\Gamma_1 + \Gamma_2) = 3$ (by Theorem 2.3).

4. $\Delta(\Gamma_1 + \Gamma_2) = 6$ (by Theorem 2.4).

5. $\delta(\Gamma_1 + \Gamma_2) = 4$ (by Theorem 2.5).

6. $DS(\Gamma_1 + \Gamma_2) = \{4, 5, 5, 5, 5, 6, 6\}$ (by Theorem 2.7).

7. $t(\Gamma_1 + \Gamma_2) = 3$ (by Theorem 2.7).

8. $\beta(\Gamma_1 + \Gamma_2) = 3$ (by Theorem 2.10).

9. $\gamma(\Gamma_1 + \Gamma_2) = 1$ (by Theorem 2.11).

10. $\chi(\Gamma_1 + \Gamma_2) = 5$ (by Theorem 2.13).

11. $\omega(\Gamma_1 + \Gamma_2) = 5$ (by Theorem 2.14).

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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