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ON A POWER OF THE COMPANION MATRIX

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Abstract: In this study, we have given some results which have computed power of the companion matrix and the algorithms which are based on these results. So we also have given numerical examples using these algorithms.

Key words: Companion matrix, matrix power
Mathematics Subject Classifications (2000): 15A15, 65F30

KOMPANYAN MATRİSİN KUVVETİ ÜZERİNE

Özet: Bu çalışmada kompanyan matrisin kuvvetini hesaplayan bazı sonuçlar ve bu sonuçların üzerine kurulan algoritmalar verildi. Bu algoritmaları kullanarak numerik örnekler de verildi.

Anahtar kelimeler: Kompanyan matris, matris kuvveti

1. INTRODUCTION

It is well known that matrix A, which is defined as

$$A = (a_{ij}) = [a_1, a_2, \dots , a_N] \; ; \; a_{ij} = \begin{cases} 1 & j = i+1 \\ a_j & i = N, \; j = 1(1)N \\ 0 & others \end{cases} \tag{1}$$

is called the companion matrix (for example, LUTKEPOHL 1996, GOLUB & ORTEGA 1992, ELAYDI 1999, AKIN & BULGAK 1998). The companion matrix and the power of companion matrix have been used in the various science fields. For example, the companion matrix can be used to transform the linear difference (differential) equations in order of N as

$$y(n+N) = a_N y(n+N-1) + \dots + a_1 y(n) \tag{2}$$

to one order system as

$$x(n+1) = Ax(n), \; n\text{-integer number}, \; A - N\times N \text{ matrix.} \tag{3}$$

It is well known that the solution of Cauchy problem of

$$x(n+1) = Ax(n), \; x(0) = x_0$$

is $x(n) = A^n x_0$ (see, ELAYDI 1999, AKIN & BULGAK 1998). Therefore, power of the companion matrix is important.

2. MAIN RESULTS

Now, we give a theorem and two corollaries which give n^{th} power of companion matrix A.

Theorem 1. For matrix A in (1) and n > 1,

$$A^n = \left(a_{ij}^{(n)}\right); \quad a_{ij}^{(n)} = \begin{cases} a_j \times a_{iN}^{(n-1)} & i = 1(1)N,\ j = 1 \\ a_j \times a_{iN}^{(n-1)} + a_{ij-1}^{(n-1)} & i = 1(1)N,\ j = 2(1)N \end{cases},$$

is true, where $a_{ij}^{(1)} = a_{ij}$.

Proof. We give proof of the theorem by induction on the n.

- For $n = 2$, $A^2 = \left(a_{ij}^{(2)}\right); \quad a_{ij}^{(2)} = \begin{cases} a_j \times a_{iN}^{(1)} & i = 1(1)N,\ j = 1 \\ a_j \times a_{iN}^{(1)} + a_{ij-1}^{(1)} & i = 1(1)N,\ j = 2(1)N \end{cases}.$

The elements of matrix $A^2 = \left(a_{ij}^{(2)}\right)$,

"$a_{13}^{(2)} = a_{12}^{(1)} = 1$; $a_{24}^{(2)} = a_{23}^{(1)} = 1$; …; $a_{N-2N}^{(2)} = a_{N-2N-1}^{(1)} = 1$; $a_{N-11}^{(2)} = a_1$; $a_{N-12}^{(2)} = a_2$; $a_{N-13}^{(2)} = a_3$; …; $a_{N-1N}^{(2)} = a_N$; $a_{N1}^{(2)} = a_1 a_N$; $a_{N2}^{(2)} = a_2 a_N + a_1$; $a_{N3}^{(2)} = a_3 a_N + a_2$; …; $a_{Nj}^{(2)} = a_j a_N + a_{j-1}$; … ; $a_{NN}^{(2)} = a_N^2 + a_{N-1}$, and other terms are 0"

is obtained. Therefore, A^2 matrix is written to be

$$A^2 = \begin{pmatrix} 0 & 0 & 1 & \Lambda & 0 \\ 0 & 0 & 0 & 1\Lambda & 0 \\ M & M & M & M & M \\ 0 & 0 & 0 & \Lambda & 1 \\ a_1 & a_2 & a_3 & \Lambda & a_N \\ a_1 a_N & a_2 a_N + a_1 & a_3 a_N + a_2 & \Lambda & a_N^2 + a_{N-1} \end{pmatrix}$$

- Let hypothesis of theorem is true for $n = m$, i.e. let
$$A^m = \left(a_{ij}^{(m)}\right); \quad a_{ij}^{(m)} = \begin{cases} a_j \times a_{iN}^{(m-1)} & i = 1(1)N,\ j = 1 \\ a_j \times a_{iN}^{(m-1)} + a_{ij-1}^{(m-1)} & i = 1(1)N,\ j = 2(1)N \end{cases}.$$
- For $n = m+1$,

$$A^m \times A = \begin{pmatrix} a_1 \times a_{1N}^{(m)} & a_2 \times a_{1N}^{(m)} + a_{11}^{(m)} & \cdots & a_N \times a_{1N}^{(m)} + a_{1N-1}^{(m)} \\ a_1 \times a_{2N}^{(m)} & a_2 \times a_{2N}^{(m)} + a_{21}^{(m)} & \cdots & a_N \times a_{2N}^{(m)} + a_{2N-1}^{(m)} \\ \vdots & \vdots & & \vdots \\ a_1 \times a_{NN}^{(m)} & a_2 \times a_{NN}^{(m)} + a_{N1}^{(m)} & \cdots & a_N \times a_{NN}^{(m)} + a_{NN-1}^{(m)} \end{pmatrix}$$

can be written. Since $a_{ij}^{(m+1)} = \begin{cases} a_j \times a_{iN}^{(m)} & i = 1(1)N,\ j = 1 \\ a_j \times a_{iN}^{(m)} + a_{ij-1}^{(m)} & i = 1(1)N,\ j = 2(1)N \end{cases}$, the matrix A^{m+1}

is obtained as

$$A^{m+1} = A^m \times A = \begin{pmatrix} a_{11}^{(m+1)} & a_{12}^{(m+1)} & \cdots & a_{1N}^{(m+1)} \\ a_{21}^{(m+1)} & a_{22}^{(m+1)} & \cdots & a_{2N}^{(m+1)} \\ & & & \\ a_{N1}^{(m+1)} & a_{N2}^{(m+1)} & \cdots & a_{NN}^{(m+1)} \end{pmatrix}.$$

Thus, proof of the theorem is completed for $n \in \mathbf{N}$ ($n > 1$).

Corollary 1. Under conditions of theorem 1,

$$A^n = \left(a_{ij}^{(n)}\right) = \begin{pmatrix} \left(a_{i+1j}^{(n-1)}\right)_{i=1}^{N-1} \\ \left(a_{Nj}^{(n)}\right) \end{pmatrix}.$$

Proof. Since $a_{ij}^{(n)} = \begin{cases} a_j \times a_{iN}^{(n-1)} & i = 1(1)N,\ j = 1 \\ a_j \times a_{iN}^{(n-1)} + a_{ij-1}^{(n-1)} & i = 1(1)N,\ j = 2(1)N \end{cases}$, we verify that

1) $a_{i1}^{(n)} = a_1 \times a_{iN}^{(n-1)} = a_{i+11}^{(n-1)}$,
2) $a_{ij}^{(n)} = a_j \times a_{iN}^{(n-1)} + a_{ij-1}^{(n-1)} = a_{i+1j}^{(n-1)}$, j=2(1)N

is true for i = 1(1)N-1. We give the proof by induction on the n for n>1.

- For $n = 2$;

1) $a_{i1}^{(2)} = a_1 \times a_{iN}^{(1)} = a_1 \times a_{iN} = \begin{cases} 0 & i = 1(1)N-2 \\ a_1 & i = N-1 \end{cases} = a_{i+11}^{(1)}$ is true.

2) $a_{ij}^{(2)} = a_j \times a_{iN}^{(1)} + a_{ij-1}^{(1)} = a_j \times a_{iN} + a_{ij-1} = \begin{cases} 0 & others \\ 1 & j = i+2 \\ a_j & i = N-1 \end{cases} = a_{i+1j}^{(1)}$ is true.

Thus,

$$A^2=\begin{pmatrix}\left(a_{i+1j}^{(1)}\right)_{i=1}^{N-1}\\ a_{Nj}^{(2)}\end{pmatrix}.$$

- Let hypothesis of theorem is true for $n = m$, i.e. let

$$A^m=\left(a_{ij}^{(m)}\right)=\begin{pmatrix}\left(a_{i+1j}^{(m-1)}\right)_{i=1}^{N-1}\\ \left(a_{Nj}^{(m)}\right)\end{pmatrix};\quad a_{ij}^{(m)}=\begin{cases}a_j\times a_{iN}^{(m-1)} & i=1(1)N,\ j=1\\ a_j\times a_{iN}^{(m-1)}+a_{ij-1}^{(m-1)} & i=1(1)N,\ j=2(1)N\end{cases}.$$

In this case, defined equalities are true for $n \le m$.

- For $n = m+1$;

1) $a_{i1}^{(m+1)}=a_1\times a_{iN}^{(m)}=a_1\times a_{i+1N}^{(m-1)}=a_{i+11}^{(m)}$,
2) $a_{ij}^{(m+1)}=a_j\times a_{iN}^{(m)}+a_{ij-1}^{(m)}=a_j\times a_{i+1N}^{(m-1)}+a_{i+1j-1}^{(m-1)}=a_{i+1j}^{(m)}$, j=2(1)N

are found. Therefore,

$$A^{m+1}=\begin{pmatrix}\left(a_{i+1j}^{(m)}\right)_{i=1}^{N-1}\\ \left(a_{Nj}^{(m+1)}\right)\end{pmatrix}$$

is obtained. Thus, for $n > 1$ corollary is true.

Corollary 2. Under conditions of theorem 1, for $n > 1$, i, j=1(1)N, k =1(1)N-I

$$A^n=\left(a_{ij}^{(n)}\right)=\left\langle a_{i+kj}^{(n-k)}\right\rangle;\quad a_{i+kj}^{(n-k)}=\begin{cases}a_{n-1+ij}^{(1)} & n-k<0\\ a_{Nj}^{(n-N+i)} & n-k\ge 0\end{cases},$$

is true, where $a_{ij}^{(1)}=a_{ij}$ and for $2 \le m \le n$, $a_{ij}^{(m)}=\begin{cases}a_j\times a_{iN}^{(m-1)} & j=1\\ a_j\times a_{iN}^{(m-1)}+a_{ij-1}^{(m-1)} & j=2(1)N\end{cases}$.

Proof. In proof of corollary 1, it is known that $a_{ij}^{(n)}=a_{i+1j}^{(n-1)}$, i = 1(1)N-1, j = 1(1)N. Therefore, it is seen that $a_{ij}^{(n)}=a_{i+kj}^{(n-k)}=a_{Nj}^{(n-N+i)}$, k =1(1)N-i if n-N+$i \ge 1$ and $a_{ij}^{(n)}=a_{i+kj}^{(n-k)}=a_{n-1+ij}^{(1)}$, k=1(1)N-i if n-N+i <1. However, since k = 0 for $i = N$, the equality $a_{ij}^{(n)}=a_{i+kj}^{(n-k)}=a_{Nj}^{(n-N+i)}$ is valid. Thus, corollary is proved.

Note. Theorem 1, Corollary 1 and Corollary 2 are equivalent each other.

3. ALGORITHM

In this section, we give an algorithm which is based on the results. This algorithm which gives n^{th} power of the companion matrix A.

Input. N-order of matrix A, $a_{Nj} = a_j$ (j = 1(1)N) – numbers, n –required power.

Step 1. Take $a_{ij}^{(1)} = a_{ij}$.

Step 2. For $2 \le m \le n$, $a_{ij}^{(m)} = \begin{cases} a_j \times a_{iN}^{(m-1)} & i = 1(1)N, \; j = 1 \\ a_j \times a_{iN}^{(m-1)} + a_{ij-1}^{(m-1)} & i = 1(1)N, \; j = 2(1)N \end{cases}$ is defined.

Step 3. If $n\text{–}N \ge 0$ go to Step 4.

3.1. For $i = 1(1)N\text{-}n$, calculate $a_{ij}^{(n)} = a_{n-1+ij}^{(1)}$.

3.2. For $i = N\text{-}n\text{+}1(1)N$, calculate $a_{ij}^{(n)} = a_{Nj}^{(n-N+i)}$ and then go to Output 1.

Step 4. Take as $a_{ij}^{(n)} = a_{Nj}^{(n-N+i)}$ and calculate $a_{Nj}^{(n-N+i)}$ for i, j = 1(1)N and then go to Output 2.

Output 1. Compose the matrix $A^n = \left(a_{ij}^{(n)}\right) = \left\langle \begin{matrix} \left(a_{n-1+ij}^{(1)}\right)_{i=1}^{N-n} \\ a_{Nj}^{(n-N+i)} \end{matrix} \right\rangle$ and stop.

Output 2. Compose the matrix $A^n = \left(a_{ij}^{(n)}\right) = \left\langle a_{Nj}^{(n-N+i)} \right\rangle$.

4. ILLUSTRATIVE EXAMPLES

Example 1. We calculate 2$^{\text{nd}}$ power of matrix A = [1, -1, 1, 2].

Input. $n = 2$, N=4 and a_1=1, a_2= -1, a_3=1, a_3=2.

Step 1. Take $a_{ij}^{(1)} = a_{ij}$.

Step 2. $a_{ij}^{(2)} = \begin{cases} a_j \times a_{i4}^{(1)} & j = 1 \\ a_j \times a_{i4}^{(1)} + a_{ij-1}^{(1)} & j = 2(1)4 \end{cases}$

Step 3. $n\text{-}N = 2\text{–}4 = -2 < 0$.

3.1. For i = 1,2 ; j = 1(1)4 calculate $a_{ij}^{(2)} = a_{i+1j}^{(1)}$.

$i = 1;\; j = 1(1)4 \Rightarrow a_{1j}^{(2)} = a_{2j}^{(1)}$; $(a_{2j}^{(1)}) = (0\ 0\ 1\ 0)$

$i = 2;\; j = 1(1)4 \Rightarrow a_{2j}^{(2)} = a_{3j}^{(1)}$; $(a_{3j}^{(1)}) = (0\ 0\ 0\ 1)$

3.2. For i = 3, 4 ; j = 1(1)4 calculate $a_{ij}^{(2)} = a_{4j}^{(i-2)}$ and then go to Output 1.

$i = 3; j = 1(1)4 \Rightarrow a_{3j}^{(2)} = a_{4j}^{(1)}$; $(a_{4j}^{(1)}) = (1\ \text{-}1\ 1\ 2)$

$i = 4; j = 1(1)4 \Rightarrow a_{4j}^{(2)} = a_{4j}^{(2)}$; $(a_{4j}^{(2)}) = (1.2\ \ \text{-}1.2\text{+}1\ 1.2\text{-}1\ 2.2\text{+}1) = (2\ \ \text{-}1\ 1\ 5)$

Output 1. $A^2=\left(a_{ij}^{(2)}\right)=\left\langle \begin{matrix}\left(a_{1+ij}^{(1)}\right)_{i=1}^{2} \\ a_{4j}^{(i-2)}\end{matrix} \right\rangle=\begin{pmatrix}0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & 1 & 2 \\ 2 & -1 & 1 & 5\end{pmatrix}.$

Example 2. We calculate 5[th] power of matrix A = [1, -1, 2] with algorithm.

Input. $n = 5$, N=3 and $a_1 = 1$, $a_2 = -1$, $a_3 = 2$.

Step 1. Take $a_{ij}^{(1)}=a_{ij}$.

Step 2. $2 \le m \le 5$, $a_{ij}^{(m)}=\begin{cases} a_j \times a_{i3}^{(m-1)} & j=1 \\ a_j \times a_{i3}^{(m-1)}+a_{ij-1}^{(m-1)} & j=2,3 \end{cases}$ is defined.

Step 3. Since 5–3=2 ≥ 0 go to Step 4.

Step 4. Take as $a_{ij}^{(n)}=a_{3j}^{(2+i)}$ and calculate $a_{3j}^{(2+i)}$ for i, j = 1(1)3 and then go to Output 2.

4.1. For $i=1, j=1(1)3$; $a_{31}^{(3)}=a_1 a_{33}^{(2)}=a_1(a_3 a_{33}^{(1)}+a_{32}^{(1)})=1(2^2-1)=3$

$a_{32}^{(3)}=a_2 a_{33}^{(2)}+a_{31}^{(2)}=a_2(a_3 a_{33}^{(1)}+a_{32}^{(1)})+a_1 a_{33}^{(1)}=-1(2^2-1)+2=-1$

$a_{33}^{(3)}=a_3 a_{33}^{(2)}+a_{32}^{(2)}=a_3(a_3 a_{33}^{(1)}+a_{32}^{(1)})+a_2 a_{33}^{(1)}+a_{31}^{(1)}=2(2^2-1)-2+1=5$

4.2. For $i=2, j=1(1)3$; $a_{31}^{(4)}=a_1 a_{33}^{(3)}=5\times 1=5$, $a_{32}^{(4)}=a_2 a_{33}^{(3)}+a_{31}^{(3)}=-1\times 5+3=-2$,

$a_{33}^{(4)}=a_3 a_{33}^{(3)}+a_{32}^{(3)}=2\times 5-1=9$

4.3. For $i=3, j=1(1)3$; $a_{31}^{(5)}=a_1 a_{33}^{(4)}=9\times 1=9$, $a_{32}^{(5)}=a_2 a_{33}^{(4)}+a_{31}^{(4)}=-1\times 9+5=-4$,

$a_{33}^{(5)}=a_3 a_{33}^{(4)}+a_{32}^{(4)}=2\times 9-2=16$

Output 2. $A^5=\left(a_{ij}^{(5)}\right)=\left\langle a_{3j}^{(2+i)}\right\rangle=\begin{pmatrix}3 & -1 & 5 \\ 5 & -2 & 9 \\ 9 & -4 & 16\end{pmatrix}$ is obtained.

Note. The algorithm given in this study can be applied manually or using computer, but it has been noticed that it is advantageous if computer programming used.

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