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New generalization of reverse Minkowski's inequality for fractional integral

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Abstract

The realizations of inequalities which containing the fractional integral and differential operators is considered to be important due to its wide implementations among authors. In this research, we introduce some new fractional integral inequalities of Minkowski's type by using Riemann-Liouville fractional integral operator. We replace the constants appears on Minkowski's inequality by two positive functions. Further, we establish some new fractional inequalities related to the reverse Minkowski type inequalities via Riemann-Liouville fractional integral. Using this fractional integral operator, some special cases of reverse Minkowski type are also discussed.

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1. Introduction

During their unrelenting effort in the development of mathematics, mathematicians in the past few decades have expanded the concept of classical calculus of derivatives and integrals for integer orders to the fractional calculus, which is a generalized form of classical integrals and derivatives in case of non-integer order. Recently, the fractional calculus theory has get more attention due to its important applications in various fields such as computer networking, biology, physics, fluid dynamics, signal processing, image processing, control theory and other fields. One of the widespread approaches among authors is the use

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of fractional integrals and derivatives operators. As a consequence many different kinds of fractional integrals and derivatives have been realized, such as the Riemann-Liouville, Weyl types, Liouville, Hadamard, Katugampola and some other types can be found found in Kilbas et.al. [13].

In classical integral and differential equations, mathematical inequalities play a responsible role and in the last few years, a number of useful and important mathematical inequalities invented by many authors. Inequalities which involve fractional integrals and derivatives are crucial in the study of a number of differential and integral equations, among which we mention [1], [2], [3], [5], [17]. One of the most renowned and important integral inequality is given by Hermann Minkowski. In the last few decades, this inequality has received considerable attention from many authors and several articles have appeared in the literature. In (2010), Dahmani [9], gave the reverse Minkowski and Hermite-Hadamard inequalities by mean of Riemann–Liouville fractional integral. In (2010), Erhan et al. [16], studied the inequalities of reverse Minkowski and Hermite-Hadamard involving two functions using the classical Riemann integral. It should be noted that in (2006), Lazhar [6], also presented a work related to the Hardy's inequality and the reverse Minkowski's inequality. In (2013), Chinchane with Pachpatte [8] and Taf with Brahim [19], established the reverse Minkowski's inequality via Hadamard fractional integral. Recently, in (2018), Vanterler et al. [21], presented the reverse Minkowski inequalities and some other related inequalities by mean of Katugampola fractional integral. In (2019), Rahman et al. [15], employed the generalized proportional fractional integral operators to establish the reverse Minkowski's inequality and other fractional inequalities.Very recently, in (2020), Aljaaidi and Pachpatte [4], presented the reverse Minkowski inequalities via ψ-Riemann-Liouville fractional integral operators. More survey of some of the earlier and recent developments related to the inequality mentioned above can be found in [7], [10], [12], [14], [18], [20].

Our purpose in this paper is to use Riemann-Liouville fractional integral operator to introduce some new fractional integral inequalities of Minkowski's type in case of functional bounds. Moreover, we establish some new fractional inequalities related to the reverse Minkowski's type inequality via Riemann-Liouville fractional integral operator. The paper is organized as follows: In second section, we recollect some notations, definitions, results and preliminary facts which are used throughout this paper. In third section, we give our main results of reverse Minkowski's inequality with functional bounds. In fourth section, we present some other related results involving Riemann-Liouville fractional integral operator.

2. Basic Definitions and Tools

Now, in this section, we give some basic definitions and properties of fractional integrals used to obtain and discuss our new results.

Definition 2.1. *[13] Let $\delta > 0$ and f be an integrable functions on $[a, b]$ with $a \geq 0$. The notation $\mathcal{I}^{\delta}_{a+}$ and $\mathcal{I}^{\delta}_{b-}$ are called respectively the left and right-sided Riemann-Liouville fractional integrals and defined by*

$$\mathcal{I}^{\delta}_{a+} f(x) = \frac{1}{\Gamma(\delta)} \int_a^x (x-\tau)^{\delta-1} f(\tau)\, d\tau, x > a \tag{1}$$

and

$$\mathcal{I}^{\delta}_{b-} f(x) = \frac{1}{\Gamma(\delta)} \int_x^b (\tau - x)^{\delta-1} f(\tau)\, d\tau, x < b, \tag{2}$$

where, $\Gamma(\delta) = \int_0^\infty e^{-u} u^{\delta-1} du$ is a Gamma function and $\mathcal{I}^{0}_{a+} f(x) = \mathcal{I}^{0}_{b-} f(x) = f(x)$.

In present paper, we use only the left-sided fractional integrals (1) to obtain and discuss our results. For the convenience of establishing the results, we use the expression $\mathcal{I}^{\delta}$ to denote the left-sided Riemann-Liouville fractional integral operator $\mathcal{I}^{\delta}_{a+}$ at $a = 0$.

Theorem 2.2. *[6] Let $n \geq 1$. Assume that there exist two positive functions f, g defined on $L^n[a, b]$ with $n \in [1, \infty)$. If $0 < q \leq \frac{f(x)}{g(x)} \leq Q, \forall\, x \in [a, b]$ where $q, Q \in \mathbb{R}^*_+$, then we have*

$$\left(\int_a^b f^n(x)\, dx\right)^{\frac{1}{n}} + \left(\int_a^b g^n(x)\, dx\right)^{\frac{1}{n}} \leq \frac{1 + Q(q+2)}{(q+1)(Q+1)} \left(\int_a^b (f+g)^n(x)\, dx\right)^{\frac{1}{n}}. \tag{3}$$

Theorem 2.3. *[6] Let $n \geq 1$. Assume that there exist two positive functions f, g defined on $L^n[a,b]$ with $n \in [1,\infty)$. If $0 < q \leq \frac{f(x)}{g(x)} \leq Q, \forall\ x \in [a,b]$ where $q, Q \in \mathbb{R}_+^*$, then we have*

$$\left(\int_a^b f^n(x)\,dx\right)^{\frac{2}{n}} + \left(\int_a^b g^n(x)\,dx\right)^{\frac{2}{n}} \geq \left(\frac{(q+1)(Q+1)}{Q} - 2\right)\left(\int_a^b f^n(x)\,dx\right)^{\frac{1}{n}}\left(\int_a^b g^n(x)\,dx\right)^{\frac{1}{n}}. \quad (4)$$

For the inequalities (3) and (4), Dahmani in [9] established a fractional versions inequalities as follows

Theorem 2.4. *Let $\delta > 0$, $n \geq 1$. Assume that there exist two positive functions f, g defined on $[0,\infty)$ such that for all $x > 0$, $\mathcal{I}^\delta f^n(x) < \infty$, $\mathcal{I}^\delta g^n(x) < \infty$. If $0 < q \leq \frac{f(\tau)}{g(\tau)} \leq Q, \forall\ \tau \in [0,x]$, then we have*

$$\left(\mathcal{I}^\delta f^n(x)\right)^{\frac{1}{n}} + \left(\mathcal{I}^\delta g^n(x)\right)^{\frac{1}{n}} \leq \frac{1 + Q(q+2)}{(q+1)(Q+1)}\left(\mathcal{I}^\delta (f+g)^n(x)\right)^{\frac{1}{n}}. \quad (5)$$

Theorem 2.5. *Let $\delta > 0$, $n \geq 1$. Assume that there exist two positive functions f, g defined on $[0,\infty)$,such that for all $x > 0$, $\mathcal{I}^\delta f^n(x) < \infty$, $\mathcal{I}^\delta g^n(x) < \infty$. If $0 < q \leq \frac{f(\tau)}{g(\tau)} \leq Q, \forall\ \tau \in [0,x]$, then we have*

$$\left(\mathcal{I}^\delta f^n(x)\right)^{\frac{2}{n}} + \left(\mathcal{I}^\delta g^n(x)\right)^{\frac{2}{n}} \geq \left(\frac{(q+1)(Q+1)}{Q} - 2\right)\left(\mathcal{I}^\delta f^n(x)\right)^{\frac{1}{n}}\left(\mathcal{I}^\delta g^n(x)\right)^{\frac{1}{n}}. \quad (6)$$

Note that, the corresponding results of all results discussed in this paper for the right-sided Riemann-Liouville fractional integrals (2) can be obtained by same arguments.

3. Reverse Minkowski's inequalities for fractional integral

Here, we are ready to give our generalization of the reverse Minkowski inequalities for fractional integrals in case of functional bounds.

Theorem 3.1. *Let f, g be two positive functions on $[0,\infty)$, such that $\mathcal{I}^\delta f^n(x), \mathcal{I}^\delta g^n(x) < \infty$, $\forall x \in [0,\infty)$. Assume that there exist two positive functions u, v such that $0 < u(\rho) \leq \frac{f(\tau)}{g(\tau)} \leq v(\rho)$, $\tau, \rho \in [0,x]$. Then for all $\delta > 0$, $n \geq 1$, we have*

$$\begin{aligned}&\left[\mathcal{I}^\delta f^n(x)\right]^{\frac{1}{n}} + \left[\mathcal{I}^\delta g^n(x)\right]^{\frac{1}{n}} \\ &\leq \frac{\Gamma(\delta+1)}{x^\delta}\left\{\mathcal{I}^\delta\left(\frac{v(x)}{v(x)+1}\right) + \mathcal{I}^\delta\left(\frac{1}{u(x)+1}\right)\right\}\left[\mathcal{I}^\delta (f+g)^n(x)\right]^{\frac{1}{n}}. \quad (7)\end{aligned}$$

Proof. Using the condition $\frac{f(\tau)}{g(\tau)} \leq v(\rho)$; $\tau, \rho \in [0,x]$, we can write

$$[v(\rho)+1]^n f^n(\tau) \leq v^n(\rho)(f+g)^n(\tau) \quad (8)$$

and by using the condition $u(\rho) \leq \frac{f(\tau)}{g(\tau)}$; $\tau, \rho \in [0,x]$, we can write

$$\left(1 + \frac{1}{u(\rho)}\right)^n g^n(\tau) \leq \left(\frac{1}{u(\rho)}\right)^n (f+g)^n(\tau). \quad (9)$$

Multiplying both sides of (8) and both sides of (9) by $\frac{1}{\Gamma(\delta)}(x-\tau)^{\delta-1}$, $\tau \in (0,x)$ and integrating the resulting inequalities with respect to τ over $(0,x)$, we get respectively

$$\frac{[v(\rho)+1]^n}{\Gamma(\delta)}\int_0^x (x-\tau)^{\delta-1} f^n(\tau)\,d\tau \leq \frac{v^n(\rho)}{\Gamma(\delta)}\int_0^x (x-\tau)^{\delta-1}(f+g)^n(\tau)\,d\tau. \quad (10)$$

and

$$\begin{aligned}&\left(1+\frac{1}{u\left(\rho\right)}\right)^{n}\frac{1}{\Gamma\left(\delta\right)}\int_{0}^{x}\left(x-\tau\right)^{\delta-1}g^{n}\left(\tau\right)d\tau\\&\leq\left(\frac{1}{u\left(\rho\right)}\right)^{n}\frac{1}{\Gamma\left(\delta\right)}\int_{0}^{x}\left(x-\tau\right)^{\delta-1}\left(f+g\right)^{n}\left(\tau\right)d\tau.\end{aligned}\tag{11}$$

So we have

$$\left[v\left(\rho\right)+1\right]^{n}\mathcal{I}^{\delta}f^{n}\left(x\right)\leq v^{n}\left(\rho\right)\mathcal{I}^{\delta}\left(f+g\right)^{n}\left(x\right)$$

and

$$\left(1+\frac{1}{u\left(\rho\right)}\right)^{n}\mathcal{I}^{\delta}g^{n}\left(x\right)\leq\left(\frac{1}{u\left(\rho\right)}\right)^{n}\mathcal{I}^{\delta}\left(f+g\right)^{n}\left(x\right),$$

which can be written as

$$\left[\mathcal{I}^{\delta}f^{n}\left(x\right)\right]^{\frac{1}{n}}\leq\frac{v\left(\rho\right)}{v\left(\rho\right)+1}\left[\mathcal{I}^{\delta}\left(f+g\right)^{n}\left(x\right)\right]^{\frac{1}{n}}\tag{12}$$

and

$$\left[\mathcal{I}^{\delta}g^{n}\left(x\right)\right]^{\frac{1}{n}}\leq\left(\frac{1}{u\left(\rho\right)+1}\right)\left[\mathcal{I}^{\delta}\left(f+g\right)^{n}\left(x\right)\right]^{\frac{1}{n}}.\tag{13}$$

Now, multiplying by $\frac{1}{\Gamma(\delta)}\left(x-\rho\right)^{\delta-1}$, $\rho\in\left(0,x\right)$ both sides of (12) and (13), then integrating the resulting inequalities with respect to ρ over $\left(0,x\right)$, we obtain respectively

$$\left[\mathcal{I}^{\delta}f^{n}\left(x\right)\right]^{\frac{1}{n}}\frac{x^{\delta}}{\Gamma\left(\delta+1\right)}\leq\left[\mathcal{I}^{\delta}\left(f+g\right)^{n}\left(x\right)\right]^{\frac{1}{n}}\frac{1}{\Gamma\left(\delta\right)}\int_{0}^{x}\left(x-\rho\right)^{\delta-1}\left(\frac{v\left(\rho\right)}{v\left(\rho\right)+1}\right)d\rho$$

and

$$\left[\mathcal{I}^{\delta}g^{n}\left(x\right)\right]^{\frac{1}{n}}\frac{x^{\delta}}{\Gamma\left(\delta+1\right)}\leq\left[\mathcal{I}^{\delta}\left(f+g\right)^{n}\left(x\right)\right]^{\frac{1}{n}}\frac{1}{\Gamma\left(\delta\right)}\int_{0}^{x}\left(x-\rho\right)^{\delta-1}\left(\frac{1}{u\left(\rho\right)+1}\right)d\rho,$$

which yields

$$\left[\mathcal{I}^{\delta}f^{n}\left(x\right)\right]^{\frac{1}{n}}\leq\frac{\Gamma\left(\delta+1\right)}{x^{\delta}}\mathcal{I}^{\delta}\left(\frac{v\left(x\right)}{v\left(x\right)+1}\right)\left[\mathcal{I}^{\delta}\left(f+g\right)^{n}\left(x\right)\right]^{\frac{1}{n}}\tag{14}$$

and

$$\left[\mathcal{I}^{\delta}g^{n}\left(x\right)\right]^{\frac{1}{n}}\leq\frac{\Gamma\left(\delta+1\right)}{x^{\delta}}\mathcal{I}^{\delta}\left(\frac{1}{u\left(x\right)+1}\right)\left[\mathcal{I}^{\delta}\left(f+g\right)^{n}\left(x\right)\right]^{\frac{1}{n}}.\tag{15}$$

Hence, the required inequality (7) can be obtained by adding the inequalities (14) and (15), which completes the proof. □

In the following corollary, we apply Theorem (3.1) for two parameters

Corollary 3.2. *Let f, g be two positive functions on $\left[0,\infty\right)$, such that $\mathcal{I}^{\delta}f^{n}\left(x\right),\mathcal{I}^{\delta}g^{n}\left(x\right)<\infty$, $\forall x\in\left[0,\infty\right)$. Suppose that there exist two positive functions u,v such that $0<u\left(\rho\right)\leq\frac{f(\tau)}{g(\tau)}\leq v\left(\rho\right)$, $\tau,\rho\in\left[0,x\right]$. Then for all $\delta>0,\gamma>0$, $n\geq1$, we have*

$$\begin{aligned}&\left[\mathcal{I}^{\delta}f^{n}\left(x\right)\right]^{\frac{1}{n}}+\left[\mathcal{I}^{\delta}g^{n}\left(x\right)\right]^{\frac{1}{n}}\\&\leq\frac{\Gamma\left(\gamma+1\right)}{x^{\gamma}}\left\{\mathcal{I}^{\gamma}\left(\frac{v\left(x\right)}{v\left(x\right)+1}\right)+\mathcal{I}^{\gamma}\left(\frac{1}{u\left(x\right)+1}\right)\right\}\left[\mathcal{I}^{\delta}\left(f+g\right)^{n}\left(x\right)\right]^{\frac{1}{n}}.\end{aligned}\tag{16}$$

Proof. The proof follows by multiplying both sides of (12) and (13) by $\frac{1}{\Gamma(\gamma)}\left(x-\rho\right)^{\gamma-1}$, $\rho\in\left(0,x\right)$ and integrating the resulting inequalities with respect to ρ over $\left(0,x\right)$, then the proof can be completed with same argument as in Theorem3.1. □

The next Theorem is follows

Theorem 3.3. *Let f, g be two positive functions on $[0,\infty)$, such that $\mathcal{I}^{\delta} f^{n}(x), \mathcal{I}^{\delta} g^{n}(x) < \infty, \forall x \in [0,\infty)$. Suppose that there exist two positive functions u, v such that $0 < u(\rho) \leq \frac{f(\tau)}{g(\tau)} \leq v(\rho)$, $\tau, \rho \in [0,x]$. Then for all $\delta > 0$, $n \geq 1$, we have*

$$\left[\mathcal{I}^{\delta} f^{n}(x)\right]^{\frac{2}{n}} + \left[\mathcal{I}^{\delta} g^{n}(x)\right]^{\frac{2}{n}} \\ \geq \left[\frac{\Gamma^{2}(\delta+1)}{x^{2\delta}} \mathcal{I}^{\delta}\left(\frac{v(x)+1}{v(x)}\right) \mathcal{I}^{\delta}(u(x)+1) - 2\right] \left[\mathcal{I}^{\delta} f^{n}(x)\right]^{\frac{1}{n}} \left[\mathcal{I}^{\delta} g^{n}(x)\right]^{\frac{1}{n}}. \tag{17}$$

Proof. Rewriting the inequalities (12) and (13) respectively as

$$\left(\frac{v(\rho)+1}{v(\rho)}\right)\left[\mathcal{I}^{\delta} f^{n}(x)\right]^{\frac{1}{n}} \leq \left[\mathcal{I}^{\delta}(f+g)^{n}(x)\right]^{\frac{1}{n}} \tag{18}$$

and

$$(u(\rho)+1)\left[\mathcal{I}^{\delta} g^{n}(x)\right]^{\frac{1}{n}} \leq \left[\mathcal{I}^{\delta}(f+g)^{n}(x)\right]^{\frac{1}{n}}. \tag{19}$$

Multiplying by $\frac{1}{\Gamma(\delta)}(x-\rho)^{\delta-1}$, $\rho \in (0,x)$ both sides of (18) and (19), then integrating the resulting inequalities with respect to ρ over $(0,x)$, we obtain respectively

$$\frac{\Gamma(\delta+1)}{x^{\delta}} \mathcal{I}^{\delta}\left(\frac{v(x)+1}{v(x)}\right)\left[\mathcal{I}^{\delta} f^{n}(x)\right]^{\frac{1}{n}} \leq \left[\mathcal{I}^{\delta}(f+g)^{n}(x)\right]^{\frac{1}{n}} \tag{20}$$

and

$$\frac{\Gamma(\delta+1)}{x^{\delta}} \mathcal{I}^{\delta}(u(x)+1)\left[\mathcal{I}^{\delta} g^{n}(x)\right]^{\frac{1}{n}} \leq \left[\mathcal{I}^{\delta}(f+g)^{n}(x)\right]^{\frac{1}{n}}. \tag{21}$$

Now, carrying out the multiplication between the inequalities (14) and (15), we have

$$\frac{\Gamma^{2}(\delta+1)}{x^{2\delta}} \mathcal{I}^{\delta}\left(\frac{v(x)+1}{v(x)}\right) \mathcal{I}^{\delta}(u(x)+1)\left[\mathcal{I}^{\delta} f^{n}(x)\right]^{\frac{1}{n}}\left[\mathcal{I}^{\delta} g^{n}(x)\right]^{\frac{1}{n}} \leq \left(\left[\mathcal{I}^{\delta}(f+g)^{n}(x)\right]^{\frac{1}{n}}\right)^{2}. \tag{22}$$

Applying Minkowski inequality to the right hand side of (22), we obtain

$$\left(\left[\mathcal{I}^{\delta}(f+g)^{n}(x)\right]^{\frac{1}{n}}\right)^{2} \leq \left(\left[\mathcal{I}^{\delta} f^{n}(x)\right]^{\frac{1}{n}} + \left[\mathcal{I}^{\delta} g^{n}(x)\right]^{\frac{1}{n}}\right)^{2}.$$

It follows that

$$\left(\left[\mathcal{I}^{\delta}(f+g)^{n}(x)\right]^{\frac{1}{n}}\right)^{2} \leq \left[\mathcal{I}^{\delta} f^{n}(x)\right]^{\frac{2}{n}} + \left[\mathcal{I}^{\delta} g^{n}(x)\right]^{\frac{2}{n}} + 2\left[\mathcal{I}^{\delta} f^{n}(x)\right]^{\frac{1}{n}}\left[\mathcal{I}^{\delta} g^{n}(x)\right]^{\frac{1}{n}}. \tag{23}$$

Using (22) and (23), we get the desired inequality (17). Then Theorem (3.3) is thus proved. □

Remark 3.4. *(i) If we put $u(x) = q$ and $v(x) = Q$, then Theorem (3.1) reduce to Theorem (2.1) and Theorem (3.3) reduce to Theorem (2.3) obtained by Dahmani in [9]. (ii) Applying Theorem (3.3) for $\delta = 1$ with $u(x) = q$ and $v(x) = Q$, then we obtain Theorem (1.2) on $[0,x]$ obtained by Bougoffa in [6] and Theorem (2.2) on $[0,x]$ obtained by Set et. al. in [16].*

4. Other related fractional integral inequalities

This part is dedicated for some new fractional inequalities which is related to reverse Minkowski inequalities with functional bounds.

Theorem 4.1. *Let $\delta > 0$, $n, m \geq 1$, $n, m \in \mathbb{R}_+^*$ where $\frac{1}{n} + \frac{1}{m} = 1$. Suppose that f, g be two positive functions on $[0, \infty)$, such that $\mathcal{I}^\delta f^n(x), \mathcal{I}^\delta g^n(x) < \infty$, $\forall x \in [0, \infty)$ and there exist two positive functions u, v such that $0 < u(\rho) \leq \frac{f(\tau)}{g(\tau)} \leq v(\rho)$, $\tau, \rho \in [0, x]$. Then we have*

$$\begin{aligned}\mathcal{I}^\delta(fg)(x) &\leq \frac{2^{p-1}\Gamma(\delta+1)}{nx^\delta}\mathcal{I}^\delta\left(\frac{v^n(x)}{(v(x)+1)^n}\right)\mathcal{I}^\delta\left[f^n(x)+g^n(x)\right] \\ &\quad+\frac{2^{p-1}\Gamma(\delta+1)}{mx^\delta}\mathcal{I}^\delta\left(\frac{1}{(1+u(x))^m}\right)\mathcal{I}^\delta\left[f^m(x)+g^m(x)\right]. \end{aligned} \tag{24}$$

Proof. Using the condition $\frac{f(\tau)}{g(\tau)} \leq v(\rho)$; $\tau, \rho \in [0, x]$, we can write

$$(v(\rho)+1)^n f^n(\tau) \leq v^n(\rho)(f+g)^n(\tau) \tag{25}$$

and by using the condition $u(\rho) \leq \frac{f(\tau)}{g(\tau)}$; $\tau, \rho \in [0, x]$, we can write

$$(1+u(\rho))^m g^m(\tau) \leq (f+g)^m(\tau). \tag{26}$$

Multiplying both sides of (25) and both sides of (26) by $\frac{1}{\Gamma(\delta)}(x-\tau)^{\delta-1}$, $\tau \in (0, x)$ and integrating the resulting inequalities with respect to τ over $(0, x)$, we get respectively

$$\mathcal{I}^\delta f^n(x) \leq \left(\frac{v^n(\rho)}{(v(\rho)+1)^n}\right)\mathcal{I}^\delta(f+g)^n(x) \tag{27}$$

and

$$\mathcal{I}^\delta g^m(x) \leq \left(\frac{1}{(1+u(\rho))^m}\right)\mathcal{I}^\delta(f+g)^m(x). \tag{28}$$

Again, multiplying by $\frac{1}{\Gamma(\delta)}(x-\rho)^{\delta-1}$, $\rho \in (0, x)$ both sides of (27) and (28), then integrating the resulting inequalities with respect to ρ over $(0, x)$, we obtain respectively

$$\mathcal{I}^\delta f^n(x) \leq \frac{\Gamma(\delta+1)}{x^\delta}\mathcal{I}^\delta\left(\frac{v^n(x)}{(v(x)+1)^n}\right)\mathcal{I}^\delta(f+g)^n(x) \tag{29}$$

and

$$\mathcal{I}^\delta g^m(x) \leq \frac{\Gamma(\delta+1)}{x^\delta}\mathcal{I}^\delta\left(\frac{1}{(1+u(x))^m}\right)\mathcal{I}^\delta(f+g)^m(x). \tag{30}$$

Now, Considering Young's inequality, [11]

$$f(\tau)g(\tau) \leq \frac{f^n(\tau)}{n} + \frac{g^m(\tau)}{m}. \tag{31}$$

Multiplying both sides of (31) by $\frac{1}{\Gamma(\delta)}(x-\tau)^{\delta-1}$, $\tau \in (0, x)$ and integrating the resulting inequalities with respect to τ over $(0, x)$, we obtain

$$\mathcal{I}^\delta(fg)(x) \leq \frac{1}{n}\mathcal{I}^\delta f^n(x) + \frac{1}{m}\mathcal{I}^\delta g^m(x). \tag{32}$$

Thus, using the inequalities (29), (30) and (32), we have

$$\begin{aligned}\mathcal{I}^\delta(fg)(x) &\leq \frac{1}{n}\mathcal{I}^\delta f^n(x) + \frac{1}{m}\mathcal{I}^\delta g^m(x) \\ &\leq \frac{\Gamma(\delta+1)}{nx^\delta}\mathcal{I}^\delta\left(\frac{v^n(x)}{(v(x)+1)^n}\right)\mathcal{I}^\delta(f+g)^n(x) \\ &\quad+\frac{\Gamma(\delta+1)}{mx^\delta}\mathcal{I}^\delta\left(\frac{1}{(1+u(x))^m}\right)\mathcal{I}^\delta(f+g)^m(x). \end{aligned} \tag{33}$$

By using the following inequality $(k+h)^p \le 2^{p-1}(k^p+h^p)$, $p>1$, $k,h\ge 0$, we can write

$$\mathcal{I}^{\delta}(f+g)^n(x) \le 2^{p-1}\mathcal{I}^{\delta}\left[f^n(x)+g^n(x)\right] \tag{34}$$

and

$$\mathcal{I}^{\delta}(f+g)^m(x) \le 2^{p-1}\mathcal{I}^{\delta}\left[f^m(x)+g^m(x)\right]. \tag{35}$$

Thus, replacing the inequalities (34) and (35) in (33), we get the required inequality (24). □

Theorem 4.2. *Let f, g be two positive functions on $[0,\infty)$, such that $\mathcal{I}^{\delta}f^n(x), \mathcal{I}^{\delta}g^n(x)<\infty$, $\forall x\in[0,\infty)$. Assume that there exist two positive functions u,v such that $0<k<u(\rho)\le\frac{f(\tau)}{g(\tau)}\le v(\rho)$, $\tau,\rho\in[0,x]$. Then for all $\delta>0$, $n\ge 1$, we have*

$$\begin{aligned}
&\frac{\Gamma(\delta+1)}{x^{\delta}}\left\{\mathcal{I}^{\delta}\left(\frac{1}{(v(x)-k)}\right)+\mathcal{I}^{\delta}\left(\frac{u(x)}{u(x)-k}\right)\right\}\left(\mathcal{I}^{\delta}(f(x)-kg(x))^n\right)^{\frac{1}{n}}\\
&\le\left(\mathcal{I}^{\delta}f^n(x)\right)^{\frac{1}{n}}+\left(\mathcal{I}^{\delta}g^n(x)\right)^{\frac{1}{n}}\\
&\le\frac{\Gamma(\delta+1)}{x^{\delta}}\left\{\mathcal{I}^{\delta}\left(\frac{v(x)}{v(x)-k}\right)+\mathcal{I}^{\delta}\left(\frac{1}{(u(x)-k)}\right)\right\}\left(\mathcal{I}^{\delta}(f(x)-kg(x))^n\right)^{\frac{1}{n}}.
\end{aligned}\tag{36}$$

Proof. Using the condition $0<k<u(\rho)\le\frac{f(\tau)}{g(\tau)}\le v(\rho)$, we can write

$$u(\rho)\le\frac{f(\tau)}{g(\tau)}\le v(\rho)\Longrightarrow u(\rho)-k\le\frac{f(\tau)}{g(\tau)}-k\le v(\rho)-k,$$

which yields

$$u(\rho)-k\le\frac{f(\tau)-kg(\tau)}{g(\tau)}\le v(\rho)-k,$$

it follows that

$$\frac{(f(\tau)-kg(\tau))^n}{(v(\rho)-k)^n}\le g^n(\tau)\le\frac{(f(\tau)-kg(\tau))^n}{(u(\rho)-k)^n}. \tag{37}$$

Also, we have

$$u(\rho)\le\frac{f(\tau)}{g(\tau)}\le v(\rho)\Longrightarrow\frac{1}{k}-\frac{1}{u(\rho)}\le\frac{1}{k}-\frac{g(\tau)}{f(\tau)}\le\frac{1}{k}-\frac{1}{v(\rho)},$$

which yields

$$\frac{u(\rho)-k}{u(\rho)}\le\frac{f(\tau)-kg(\tau)}{f(\tau)}\le\frac{v(\rho)-k}{v(\rho)},$$

it follows that

$$\left(\frac{u(\rho)}{u(\rho)-k}\right)^n(f(\tau)-kg(\tau))^n\le f^n(\tau)\le\left(\frac{v(\rho)}{v(\rho)-k}\right)^n(f(\tau)-kg(\tau))^n. \tag{38}$$

Multiplying both sides of each of (37) and (38) by $\frac{1}{\Gamma(\delta)}(x-\tau)^{\delta-1}$, $\tau\in(0,x)$ and integrating the resulting inequalities with respect to τ over $(0,x)$, we obtain respectively

$$\begin{aligned}
&\frac{1}{\Gamma(\delta)(v(\rho)-k)^n}\int_0^x(x-\tau)^{\delta-1}(f(\tau)-kg(\tau))^n\,d\tau\\
&\le\frac{1}{\Gamma(\delta)}\int_0^x(x-\tau)^{\delta-1}g^n(\tau)\,d\tau\\
&\le\frac{1}{\Gamma(\delta)(u(\rho)-k)^n}\int_0^x(x-\tau)^{\delta-1}(f(\tau)-kg(\tau))^n\,d\tau
\end{aligned}$$

and

$$\begin{aligned}
&\left(\frac{u(\rho)}{u(\rho)-k}\right)^{n}\frac{1}{\Gamma(\delta)}\int_{0}^{x}(x-\tau)^{\delta-1}(f(\tau)-kg(\tau))^{n}\,d\tau\\
&\leq\frac{1}{\Gamma(\delta)}\int_{0}^{x}(x-\tau)^{\delta-1}f^{n}(\tau)\,d\tau\\
&\leq\left(\frac{v(\rho)}{v(\rho)-k}\right)^{n}\frac{1}{\Gamma(\delta)}\int_{0}^{x}(x-\tau)^{\delta-1}(f(\tau)-kg(\tau))^{n}\,d\tau,
\end{aligned}$$

it follows that

$$\left(\frac{1}{(v(\rho)-k)^{n}}\right)\mathcal{I}^{\delta}(f(x)-kg(x))^{n}\leq\mathcal{I}^{\delta}g^{n}(x)\leq\left(\frac{1}{(u(\rho)-k)^{n}}\right)\mathcal{I}^{\delta}(f(x)-kg(x))^{n}\tag{39}$$

and

$$\left(\frac{u(\rho)}{u(\rho)-k}\right)^{n}\mathcal{I}^{\delta}(f(x)-kg(x))^{n}\leq\mathcal{I}^{\delta}f^{n}(x)\leq\left(\frac{v(\rho)}{v(\rho)-k}\right)^{n}\mathcal{I}^{\delta}(f(x)-kg(x))^{n}.\tag{40}$$

which are respectively equivalent to

$$\left(\frac{1}{(v(\rho)-k)}\right)\left(\mathcal{I}^{\delta}(f(x)-kg(x))^{n}\right)^{\frac{1}{n}}\leq\left(\mathcal{I}^{\delta}g^{n}(x)\right)^{\frac{1}{n}}\leq\left(\frac{1}{(u(\rho)-k)}\right)\left(\mathcal{I}^{\delta}(f(x)-kg(x))^{n}\right)^{\frac{1}{n}}\tag{41}$$

and

$$\left(\frac{u(\rho)}{u(\rho)-k}\right)\left(\mathcal{I}^{\delta}(f(x)-kg(x))^{n}\right)^{\frac{1}{n}}\leq\left(\mathcal{I}^{\delta}f^{n}(x)\right)^{\frac{1}{n}}\leq\left(\frac{v(\rho)}{v(\rho)-k}\right)\left(\mathcal{I}^{\delta}(f(x)-kg(x))^{n}\right)^{\frac{1}{n}}.\tag{42}$$

Now, multiplying by $\frac{1}{\Gamma(\delta)}(x-\rho)^{\delta-1}$, $\rho\in(0,x)$ both sides of (41) and (42), then integrating the resulting inequalities with respect to ρ over $(0,x)$, we obtain respectively

$$\begin{aligned}
&\mathcal{I}^{\delta}\left(\frac{1}{(v(x)-k)}\right)\left(\mathcal{I}^{\delta}(f(x)-kg(x))^{n}\right)^{\frac{1}{n}}\\
&\leq\frac{x^{\delta}}{\Gamma(\delta+1)}\left(\mathcal{I}^{\delta}g^{n}(x)\right)^{\frac{1}{n}}\\
&\leq\mathcal{I}^{\delta}\left(\frac{1}{(u(x)-k)}\right)\left(\mathcal{I}^{\delta}(f(x)-kg(x))^{n}\right)^{\frac{1}{n}}
\end{aligned}\tag{43}$$

and

$$\begin{aligned}
&\mathcal{I}^{\delta}\left(\frac{u(x)}{u(x)-k}\right)\left(\mathcal{I}^{\delta}(f(x)-kg(x))^{n}\right)^{\frac{1}{n}}\\
&\leq\frac{x^{\delta}}{\Gamma(\delta+1)}\left(\mathcal{I}^{\delta}f^{n}(x)\right)^{\frac{1}{n}}\\
&\leq\mathcal{I}^{\delta}\left(\frac{v(x)}{v(x)-k}\right)\left(\mathcal{I}^{\delta}(f(x)-kg(x))^{n}\right)^{\frac{1}{n}}.
\end{aligned}\tag{44}$$

The addition of (43) and (44), ends the proof. □

Theorem 4.3. *Let f, g be two positive functions on $[0,\infty)$, such that $\mathcal{I}^{\delta}f^{n}(x),\mathcal{I}^{\delta}g^{n}(x)<\infty$, $\forall x\in[0,\infty)$. Suppose that there exist two positive functions u,v such that $0<u(\tau)\leq\frac{f(\tau)}{g(\tau)}\leq v(\tau)$, $\tau\in[0,x]$. Then for all $\delta>0$, $n\geq1$, we have*

$$\left(\mathcal{I}^{\delta}f^{n}(x)\right)^{\frac{1}{n}}+\left(\mathcal{I}^{\delta}g^{n}(x)\right)^{\frac{1}{n}}\leq2\left(\mathcal{I}^{\delta}h^{n}(f(x),g(x))\right)^{\frac{1}{n}},\tag{45}$$

where h is an integrable function defined on $[0,\infty)$, as

$$h(f(x),g(x))=\max\left\{\left[\left(1+\frac{v(x)}{u(x)}\right)f(x)-v(x)g(x)\right],\frac{(v(x)+u(x))g(x)-f(x)}{u(x)}\right\}.$$

Proof. Taking $0 < u(\tau) \leq \frac{f(\tau)}{g(\tau)} \leq v(\tau)$, $\tau \in [0, x]$, we have

$$u(\tau) \leq v(\tau) + u(\tau) - \frac{f(\tau)}{g(\tau)} \tag{46}$$

and

$$v(\tau) + u(\tau) - \frac{f(\tau)}{g(\tau)} \leq v(\tau). \tag{47}$$

Using (46) and (47), we can write

$$g(\tau) \leq \frac{(v(\tau) + u(\tau))\, g(\tau) - f(\tau)}{u(\tau)} \leq h(f(\tau), g(\tau)), \tag{48}$$

where h is an integrable function defined on $[0, \infty)$, as

$$h(f(\tau), g(\tau)) = \max\left\{ \left[\left(1 + \frac{v(\tau)}{u(\tau)}\right) f(\tau) - v(\tau)\, g(\tau)\right], \frac{(v(\tau) + u(\tau))\, g(\tau) - f(\tau)}{u(\tau)} \right\},$$

it follows that

$$g^n(\tau) \leq h^n(f(\tau), g(\tau)). \tag{49}$$

In other hand, under the given hypothesis $\frac{1}{v(\tau)} \leq \frac{g(\tau)}{f(\tau)} \leq \frac{1}{u(\tau)}$, we can write

$$\frac{1}{v(\tau)} \leq \frac{1}{v(\tau)} + \frac{1}{u(\tau)} - \frac{g(\tau)}{f(\tau)} \tag{50}$$

and

$$\frac{1}{v(\tau)} + \frac{1}{u(\tau)} - \frac{g(\tau)}{f(\tau)} \leq \frac{1}{u(\tau)}. \tag{51}$$

From (50) and (51), we get

$$\frac{1}{v(\tau)} \leq \frac{\left(\frac{1}{v(\tau)} + \frac{1}{u(\tau)}\right) f(\tau) - g(\tau)}{f(\tau)} \leq \frac{1}{u(\tau)},$$

so we have

$$\begin{aligned} f(\tau) &\leq v(\tau)\left(\frac{1}{v(\tau)} + \frac{1}{u(\tau)}\right) f(\tau) - v(\tau)\, g(\tau) \\ &= \left(1 + \frac{v(\tau)}{u(\tau)}\right) f(\tau) - v(\tau)\, g(\tau) \\ &\leq \max\left\{ \left[\left(1 + \frac{v(\tau)}{u(\tau)}\right) f(\tau) - v(\tau)\, g(\tau)\right], \frac{(v(\tau) + u(\tau))\, g(\tau) - f(\tau)}{u(\tau)} \right\} \\ &= h(f(\tau), g(\tau)), \end{aligned}$$

which yields

$$f^n(\tau) \leq h^n(f(\tau), g(\tau)). \tag{52}$$

Now, multiplying both sides of each of (49) and (52) by $\frac{1}{\Gamma(\delta)}(x - \tau)^{\delta - 1}$, $\tau \in (0, x)$ and integrating the resulting inequalities with respect to τ over $(0, x)$, we obtain respectively

$$\left(\mathcal{I}^{\delta} g^n(x)\right)^{\frac{1}{n}} \leq \left(\mathcal{I}^{\delta} h^n(f(x), g(x))\right)^{\frac{1}{n}} \tag{53}$$

and

$$\left(\mathcal{I}^{\delta} f^n(x)\right)^{\frac{1}{n}} \leq \left(\mathcal{I}^{\delta} h^n(f(x), g(x))\right)^{\frac{1}{n}}. \tag{54}$$

Hence, the addition of (53) and (54) completes the proof. □

Conclusion 4.4. *In this paper, we have presented the reverse Minkowski's inequalities in new generalization through replacing the constants which appear as borders on Minkowski's inequality by two positive functions. Our work produces functional bounds analogues of many pre-existing results in the literature. We have also presented some other related inequalities for reverse Minkowski type inequalities. The definitions and a few advantages of the used fractional integral over the other fractional integrals are presented in the literature .*

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