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The Forward Kinematics of Rolling Contact of Timelike Curves Lying on Timelike Surfaces

Mehmet Aydınalp[1]*, Mustafa Kazaz[1], Hasan Hüseyin Uğurlu[2]

[1]Manisa Celal Bayar University, Faculty of Arts and Sciences, Department of Mathematics, 45140 Manisa, TURKEY
[2]Gazi University, Faculty of Education, Department of Secondary Education Science and Mathematics Teaching, Mathematics Teaching Program, 06560 Ankara, TURKEY
*aydinalp64@gmail.com
*Orcid: 0000-0002-5601-866X

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Abstract

The aim of the present study is to investigate the forward kinematics of spin-rolling contact motion of one timelike surface on another timelike surface along their timelike trajectory curves in Lorentzian 3-space. This study does not take sliding motion into consideration but applies a new Darboux frame method to establish the kinematics of spin-rolling motion.

Keywords: Darboux frame; forward kinematics; Lorentzian 3-space; pure-rolling; rolling contact; spin-rolling.

1. Introduction

Rolling contact has been widely used in robotics and engineering in order to manipulate the configurations (positions and orientations). Rolling contact is also associated with surface geometry. Each surface has trajectory curves that we can determine arbitrarily and during the rolling contact motion, these curves trace each other. When there is no sliding in rolling contact, non-integrable kinematic constraints occur. It calls for that the arc lengths of the trajectory curves are equal to each other [10].

The kinematics of rolling contact motion has two categories: spin-rolling motion and pure-rolling motion [8]. On the other hand, there occur two geometric constraints for rolling contact. One is that the unit normal vectors of the two surfaces are coincident at the contact point. The other is that the contact points have the same velocity. Namely, the two contact trajectory curves are tangent to each other and have the same rolling rate. In this study, one of the surfaces is assumed to be fixed surface and the other is assumed to be moving surface which rolls on fixed one. In this sense, a moving surface undergoes spin-rolling motion or pure-rolling motion under these two geometric constraints. Furthermore, when a moving surface undergoes pure-rolling motion, one more constraint is needed. We can mention for this constraint that the two contact trajectory curves have the same geodesic curvature, that is, the angular velocity component ω_3 in the direction of the unit normal vector $\boldsymbol{n}$ to the surface is zero. Therefore, the contact trajectory curves are not arbitrary [9]. For this reason, a pure-rolling motion has two degrees of freedom (2DOFs) and instantaneous rotation axis passes through the contact point in all cases and this axis is parallel to the common tangent plane of the two surfaces. Spin-rolling motion, which is also known as twist-rolling motion, has three degrees of freedom (3DOFs), that is, a moving surface has the angular velocity components: ω_1 and ω_2 about the axes $\boldsymbol{T}$ and $\boldsymbol{g}$ on the tangent plane, respectively, and ω_3 about the common normal axis $\boldsymbol{n}$ at the contact point. The instantaneous rotation axis of a moving surface has no obligation to be parallel to the common tangent plane, namely, it may be in any direction [8].

The contact kinematics can be categorized into two main parts: forward and inverse kinematics. The forward kinematics is used for observing how the moving surface can roll in a period of time, when the geometry of the trajectory curves on surfaces are

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calculated. The inverse kinematics is used for obtaining the rolling direction, rolling rate and compensatory spin rate, when the geometry of two surfaces and the desired motion of the moving surface is determined [9,10].
The kinematics of a point contact between rigid bodies have attracted many researchers. Neimark and Fufaev [17] were the first to establish the moving frame along the lines of curvature in order to derive the velocity equation of spin-rolling motion. Cai and Roth [4,5] investigated instantaneous time-based kinematics of rigid objects in point contact, both in planar and spatial cases including sliding and pure-rolling motion and they aimed to measure the relative motion at the point of contact. Montana [15] studied the kinematics of sliding-spin-rolling motion and derived a differential-geometric model of the rolling constraint between general bodies. Li and Canny [13] used Montana's contact equations to study the existence of an admissible path between two configurations in the case of pure rolling and, if it does, then how to find it. Sarkar et al. [21] extended Montana's definition but with a different approach by obtaining the acceleration equations and they proved the obvious dependence on Christoffel symbols and they simplified the derivative of the metric tensor. Marigo and Bicchi [14] obtained similar equations with Montana's contact equations by applying a different approach that allowed an analysis of admissibility of a pure-rolling contact. Agrachev and Sachkov [1] solved the controllability problem of a pair of pure-rolling rigid bodies. Chelouah and Chitour [6] presented two procedures to analyze the motion-planning problem when one manifold was a plane and the other was a convex surface. Chitour et al. [7] studied the pure-rolling of a pair of smooth convex objects. Tchon [22] identified the property of repeatability of inverse kinematics algorithms for mobile manipulators and formulated a necessary and sufficient condition under repeatability. Tchon and Jakubiak [23] designed an extended Jacobian repeatable inverse kinematics algorithm for doubly nonholonomic mobile manipulators based on the concept of endogenous configuration space. Cui and Dai [8] studied the forward kinematics of non-sliding spin-rolling motion by establishing a new Darboux frame method and then Cui [9] investigated the kinematics of sliding-rolling motion of two contact surfaces. Cui and Dai [10] also investigated the inverse kinematics of rolling contact by using polynomial formulation when the desired angular velocity, the geometry of two surfaces and the coordinates of the contact point on each surface were given in Euclidean 3-space. Then, they obtained admissible rolling motion between two contact surfaces. For the fundamental concepts of kinematics, see [3, 12, 16].

This study is organized as follows:
In Section 2, we present basic concepts in Lorentzian 3-space.

In Section 3, we investigate the forward kinematics of spin-rolling motion without sliding of one timelike surface on another timelike surface by applying a new moving-frame method. Initially, we give the Darboux-frame-based translational velocity equation of an arbitrary point in Lorentzian 3-space. Then, we get a new equation of angular velocity with respect to the rolling speed and two sets of geometric invariants containing the geodesic curvature, the normal curvature, and the geodesic torsion, that is, $\{k_g, k_n, \tau_g\}$ and $\{\bar{k}_g, \bar{k}_n, \bar{\tau}_g\}$. We determine the instantaneous kinematics of a timelike moving surface by applying the translational velocity and the angular velocity equations. Then, we present two examples that give spin-rolling motion and pure-rolling motion of two timelike surfaces without sliding, respectively.

In Section 4, we conclude the study.

2. Material and Methods

In this section, we give basic concepts related to Lorentzian 3-space. For more details, we refer to ref. [2, 19, 20, 24].

The Lorentzian space IR_1^3 is the real vector space IR_1^3 endowed with the standard flat metric given by

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = u_1 v_1 + u_2 v_2 - u_3 v_3,$$

where $\boldsymbol{u} = (u_1, u_2, u_3)$, $\boldsymbol{v} = (v_1, v_2, v_3)$ are two vectors in IR_1^3. An arbitrary vector $\boldsymbol{u} = (u_1, u_2, u_3)$ in IR_1^3 can have one of three Lorentzian causal characters with respect to this metric. If $\langle \boldsymbol{u}, \boldsymbol{u} \rangle > 0$ or $\boldsymbol{u} = 0$ then $\boldsymbol{u}$ is called a spacelike vector; if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle < 0$ then $\boldsymbol{u}$ is called a timelike vector; if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0$ and $\boldsymbol{u} \neq 0$ then $\boldsymbol{u}$ is called a null (lightlike) vector [19]. We should note that a timelike vector is future pointing or past pointing if the first compound of the vector is positive or negative, respectively. The norm of a vector $\boldsymbol{u} \in IR_1^3$ is given by $\|\boldsymbol{u}\| = \sqrt{|\langle \boldsymbol{u}, \boldsymbol{u} \rangle|}$. If the vector $\boldsymbol{u}$ is a spacelike vector, then $\|\boldsymbol{u}\|^2 = \langle \boldsymbol{u}, \boldsymbol{u} \rangle$; if $\boldsymbol{u}$ is a timelike vector, then $\|\boldsymbol{u}\|^2 = -\langle \boldsymbol{u}, \boldsymbol{u} \rangle$ [24].

Let $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$ be two vectors in IR_1^3. Then Lorentzian vector product of $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined by

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$\boldsymbol{u}\times\boldsymbol{v}=\left(u_3v_2-u_2v_3,u_1v_3-u_3v_1,u_1v_2-u_2v_1\right)$ [24].

Definition 2.1. — [2, 20]

(*i*) Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be future pointing (or past pointing) timelike vectors in IR_1^3. Then there is a unique real number $\theta\geq 0$ such that $\langle\boldsymbol{u},\boldsymbol{v}\rangle=-\|\boldsymbol{u}\|\|\boldsymbol{v}\|\cosh\theta$, and this number is called the *hyperbolic angle* between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

(*ii*) Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be spacelike vectors in IR_1^3 and they span a timelike vector subspace. Then there is a unique real number $\theta\geq 0$ such that $\langle\boldsymbol{u},\boldsymbol{v}\rangle=\|\boldsymbol{u}\|\|\boldsymbol{v}\|\cosh\theta$, and this number is called the *central angle* between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

(*iii*) Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be spacelike vectors in IR_1^3 and they span a spacelike vector subspace. Then there is a unique real number $\theta\geq 0$ such that $\left|\langle\boldsymbol{u},\boldsymbol{v}\rangle\right|=\|\boldsymbol{u}\|\|\boldsymbol{v}\|\cos\theta$, and this number is called the *spacelike angle* between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

(*iv*) Let $\boldsymbol{u}$ be a spacelike vector and $\boldsymbol{v}$ be a timelike vector in IR_1^3. Then there is a unique real number $\theta\geq 0$ such that $\left|\langle\boldsymbol{u},\boldsymbol{v}\rangle\right|=\|\boldsymbol{u}\|\|\boldsymbol{v}\|\sinh\theta$, and this number is called the *Lorentzian timelike angle* between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

An arbitrary curve $\alpha=\alpha(s)$ in IR_1^3 can locally be spacelike, timelike, or lightlike (null), if all of its velocity vectors $d\alpha/ds$ are spacelike, timelike, or lightlike (null), respectively. A surface in Lorentzian space IR_1^3 is called a spacelike (timelike, or lightlike) surface if the normal vector of the surface is a timelike (spacelike, or lightlike) vector, respectively [19].

The spheres in the space IR_1^3 are three types and can be defined as follows:

The Lorentzian and hyperbolic unit spheres are given by

$$S_1^2=\left\{\boldsymbol{u}=(u_1,u_2,u_3)\in IR_1^3:\langle\boldsymbol{u},\boldsymbol{u}\rangle=1\right\}$$

and

$$H_0^2=\left\{\boldsymbol{u}=(u_1,u_2,u_3)\in IR_1^3:\langle\boldsymbol{u},\boldsymbol{u}\rangle=-1\right\}$$

respectively, where S_1^2 is a timelike surface and H_0^2 is a spacelike surface. Lastly, the light cone is given by

$$\Lambda^2=\left\{\boldsymbol{u}=(u_1,u_2,u_3)\in IR_1^3:\langle\boldsymbol{u},\boldsymbol{u}\rangle=0\right\}$$

where Λ^2 is a lightlike surface [19, 24].

Let S be a timelike surface and $\alpha=\alpha(s)$ be any arbitrary curve lying on the surface S. Then, the curve α has causal characters, namely, α is either spacelike, timelike or lightlike.

When α is a timelike curve, Darboux frame denoted by $(\boldsymbol{T},\boldsymbol{g},\boldsymbol{n})$ of α is a solid perpendicular trihedron in IR_1^3 associated with each point $M\in\alpha$, where $\boldsymbol{T}$ is the unit tangent timelike vector to the curve α, $\boldsymbol{n}$ is the unit normal spacelike vector to the timelike surface S and $\boldsymbol{g}=-\boldsymbol{n}\times\boldsymbol{T}$ (that is, $\boldsymbol{g}$ is tangential to S which is also a spacelike vector) at the point M. Then, the Lorentzian vector product and inner product of the unit vectors are given as follows:

$$\boldsymbol{T}\times\boldsymbol{g}=-\boldsymbol{n},\ \ \boldsymbol{g}\times\boldsymbol{n}=\boldsymbol{T},\ \ -\boldsymbol{n}\times\boldsymbol{T}=\boldsymbol{g}$$

and

$$\langle\boldsymbol{T},\boldsymbol{T}\rangle=-1,\ \langle\boldsymbol{g},\boldsymbol{g}\rangle=1,\ \langle\boldsymbol{n},\boldsymbol{n}\rangle=1.$$

Let s be arc length of the timelike curve α. In this case, the derivative formulae (the equations of motion) of the Darboux frame (trihedron) is given by

$$\frac{d\boldsymbol{m}}{ds}=\boldsymbol{T},\quad \frac{d}{ds}\begin{bmatrix}\boldsymbol{T}\\\boldsymbol{g}\\\boldsymbol{n}\end{bmatrix}=\begin{bmatrix}0 & k_g & k_n\\ k_g & 0 & -\tau_g\\ k_n & \tau_g & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{T}\\\boldsymbol{g}\\\boldsymbol{n}\end{bmatrix}$$

where the vector $\boldsymbol{m}$ is the position vector of the point M that depends on the choice of the coordinate system. Further, the position vector corresponding to an arbitrary trajectory curve on a surface in IR_1^3 can have three causal characters. Hence, we express that $\boldsymbol{m}$ is either spacelike, timelike or lightlike position vector. The components of the vector $\boldsymbol{m}$ are obtained from the measurement along the axes of the coordinate system. In these formulae, k_g, k_n and τ_g are called the geodesic curvature, the normal curvature and the geodesic torsion, respectively. It can be easily shown that the geodesic curvature k_g, the normal curvature k_n and the geodesic torsion τ_g of the timelike curve α are given by

$$k_g=\langle d\boldsymbol{T}/ds,\boldsymbol{g}\rangle,\ k_n=\langle d\boldsymbol{T}/ds,\boldsymbol{n}\rangle,$$

$$\tau_g=-\langle d\boldsymbol{g}/ds,\boldsymbol{n}\rangle.$$

The Darboux instantaneous rotation vector of the Darboux trihedron is given by

$$\boldsymbol{\omega}=\tau_g\boldsymbol{T}+k_n\boldsymbol{g}-k_g\boldsymbol{n}\ \text{[24]}.$$

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Then, for a timelike curve $\alpha(s)$ lying on a timelike surface S, the followings are well-known [24]:

(*i*) $\alpha(s)$ is a geodesic curve $\Leftrightarrow k_g = 0$,

(*ii*) $\alpha(s)$ is an asymptotic line $\Leftrightarrow k_n = 0$,

(*iii*) $\alpha(s)$ is a principal line $\Leftrightarrow \tau_g = 0$.

3. The Forward Kinematics of Rolling Contact Motion

In this section, we study the forward kinematics of rolling contact of timelike surfaces along their timelike trajectory curves by adopting the Darboux frame method in Lorentzian 3-space. The main contribution of this section is that a new equation of the angular velocity of the spin-rolling motion of a timelike moving surface is generated. The new equation is specified with regards to three unit vectors and geometric invariants, which are arc lengths of the timelike contact trajectory curves and the induced curvatures of the two timelike surfaces.

3.1. The Kinematics of Spin-Rolling Motion

In this subsection, we present the geometric kinematics of the spin-rolling motion of two contact timelike surfaces along their timelike trajectory curves. Note that during the rolling motion, both of the two timelike surfaces have the coincident spacelike unit normal vectors at the contact point. When a timelike moving surface relative to fixed surface undergo spin-rolling motion without sliding as in Figure 1, the moving surface entirely maintains its timelike surface character at every moment. This means that the timelike moving surface has the spacelike unit normal not only at the contact point, but also at every point on the moving surface during the rolling contact.

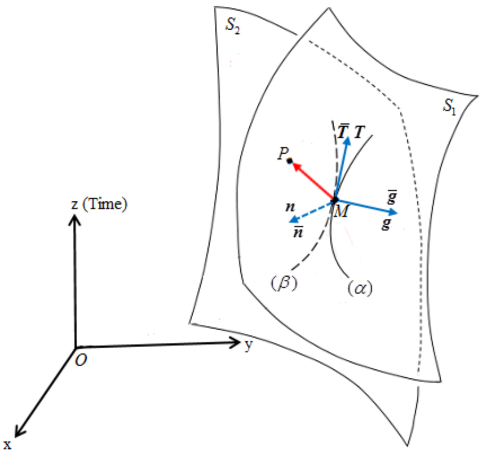

Figure 1. A timelike moving surface S_2 spin-rolling on a timelike fixed surface S_1 along timelike curves β and α.

Now, assume that α and β are timelike contact trajectory curves on timelike surfaces S_1 and S_2, respectively. Let us denote the Darboux frames (the right handed orthonormal frames) attached to the contact point M of α and β as $(\boldsymbol{T},\boldsymbol{g},\boldsymbol{n})$ and $(\bar{\boldsymbol{T}},\bar{\boldsymbol{g}},\bar{\boldsymbol{n}})$, respectively. The vectors $\boldsymbol{T}$, $\boldsymbol{g}$, $\boldsymbol{n}$ and $\bar{\boldsymbol{T}}$, $\bar{\boldsymbol{g}}$, $\bar{\boldsymbol{n}}$ are the unit vectors of the timelike fixed and timelike moving surfaces, respectively and there is not any intrinsic coordinate system for these vectors. By rolling constraints, the vectors $\boldsymbol{T}$ and $\bar{\boldsymbol{T}}$ are always collinear and, hereby, are $\boldsymbol{n}$ and $\bar{\boldsymbol{n}}$. For this reason, the two frames are coincident, as shown in Figure 1, where $\boldsymbol{n}$ points outward of the surface S_1, and $\bar{\boldsymbol{n}}$ points inward of the surface S_2. Suppose the arc lengths of the curves α and β are s and $\bar{s}$, respectively. Then the derivative formulas of the Darboux frames $(\boldsymbol{T},\boldsymbol{g},\boldsymbol{n})$ and $(\bar{\boldsymbol{T}},\bar{\boldsymbol{g}},\bar{\boldsymbol{n}})$ are

$$\frac{d\boldsymbol{m}}{ds}=\boldsymbol{T},\quad \frac{d}{ds}\begin{bmatrix}\boldsymbol{T}\\ \boldsymbol{g}\\ \boldsymbol{n}\end{bmatrix}=\begin{bmatrix}0 & k_g & k_n\\ k_g & 0 & -\tau_g\\ k_n & \tau_g & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{T}\\ \boldsymbol{g}\\ \boldsymbol{n}\end{bmatrix}$$

and

$$\frac{d\bar{\boldsymbol{m}}}{d\bar{s}}=\bar{\boldsymbol{T}},\quad \frac{d}{d\bar{s}}\begin{bmatrix}\bar{\boldsymbol{T}}\\ \bar{\boldsymbol{g}}\\ \bar{\boldsymbol{n}}\end{bmatrix}=\begin{bmatrix}0 & \bar{k}_g & \bar{k}_n\\ \bar{k}_g & 0 & -\bar{\tau}_g\\ \bar{k}_n & \bar{\tau}_g & 0\end{bmatrix}\begin{bmatrix}\bar{\boldsymbol{T}}\\ \bar{\boldsymbol{g}}\\ \bar{\boldsymbol{n}}\end{bmatrix},$$

where $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$ are the position vectors of the point M with respect to the Darboux frames $(\boldsymbol{T},\boldsymbol{g},\boldsymbol{n})$ and $(\bar{\boldsymbol{T}},\bar{\boldsymbol{g}},\bar{\boldsymbol{n}})$, respectively. Both $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$ have three causal characters. Now, assume that P is an arbitrary point on S_2. Then the (spacelike, timelike or lightlike) position vector, denoted by $\bar{\boldsymbol{p}}$, of the point P in the frame $(\bar{\boldsymbol{T}},\bar{\boldsymbol{g}},\bar{\boldsymbol{n}})$ can be given as

$$\bar{\boldsymbol{p}}=\bar{\boldsymbol{m}}+\bar{\lambda}_1\bar{\boldsymbol{T}}+\bar{\lambda}_2\bar{\boldsymbol{g}}+\bar{\lambda}_3\bar{\boldsymbol{n}}.$$

By differentiating $\bar{\boldsymbol{p}}$ with respect to $\bar{s}$ we have

$$\begin{aligned}\frac{d\bar{\boldsymbol{p}}}{d\bar{s}}=&\left(1+\frac{d\bar{\lambda}_1}{d\bar{s}}+\bar{\lambda}_2\bar{k}_g+\bar{\lambda}_3\bar{k}_n\right)\bar{\boldsymbol{T}}\\ &+\left(\frac{d\bar{\lambda}_2}{d\bar{s}}+\bar{\lambda}_1\bar{k}_g+\bar{\lambda}_3\bar{\tau}_g\right)\bar{\boldsymbol{g}}\\ &+\left(\frac{d\bar{\lambda}_3}{d\bar{s}}+\bar{\lambda}_1\bar{k}_n-\bar{\lambda}_2\bar{\tau}_g\right)\bar{\boldsymbol{n}},\end{aligned}\qquad \textbf{(3.1)}$$

where $\bar{k}_g$, $\bar{k}_n$ and $\bar{\tau}_g$ are the geodesic curvature, the normal curvature, and the geodesic torsion at point M of the timelike curve β, respectively. Since P is a fixed point on the surface of S_2, then $\frac{d\boldsymbol{p}}{d\bar{s}}=0$. By putting this equality into equation (3.1) yields

$$\frac{d\bar{\lambda}_1}{d\bar{s}}=-\bar{\lambda}_2\bar{k}_g-\bar{\lambda}_3\bar{k}_n-1,\quad \frac{d\bar{\lambda}_2}{d\bar{s}}=-\bar{\lambda}_1\bar{k}_g-\bar{\lambda}_3\bar{\tau}_g,$$
$$\frac{d\bar{\lambda}_3}{d\bar{s}}=-\bar{\lambda}_1\bar{k}_n+\bar{\lambda}_2\bar{\tau}_g.$$

Let $\boldsymbol{p}$ denote the (spacelike, timelike or lightlike) position vector of the point P in the frame $(\boldsymbol{T},\boldsymbol{g},\boldsymbol{n})$. Then

$$\boldsymbol{p}=\boldsymbol{m}+\lambda_1\boldsymbol{T}+\lambda_2\boldsymbol{g}+\lambda_3\boldsymbol{n}.$$

By differentiating $\boldsymbol{p}$ with respect to s we have

$$\begin{aligned}\frac{d\boldsymbol{p}}{ds}&=\left(1+\frac{d\lambda_1}{ds}+\lambda_2 k_g+\lambda_3 k_n\right)\boldsymbol{T}\\&+\left(\frac{d\lambda_2}{ds}+\lambda_1 k_g+\lambda_3\tau_g\right)\boldsymbol{g}\\&+\left(\frac{d\lambda_3}{ds}+\lambda_1 k_n-\lambda_2\tau_g\right)\boldsymbol{n},\end{aligned}\tag{3.2}$$

where k_g, k_n and τ_g are the geodesic curvature, the normal curvature, and the geodesic torsion at point M of the timelike curve α, respectively. The vector $\boldsymbol{p}$ has three causal characters and, thus, $\frac{d\boldsymbol{p}}{ds}$ has three causal characters. By the constraints for rolling contact, two timelike contact trajectory curves have the same arc lengths at the contact point. Since the Darboux frames $(\boldsymbol{T},\boldsymbol{g},\boldsymbol{n})$ and $(\bar{\boldsymbol{T}},\bar{\boldsymbol{g}},\bar{\boldsymbol{n}})$ are coincident at the contact point, it follows that

$$\lambda_1=\bar{\lambda}_1,\ \lambda_2=\bar{\lambda}_2,\ \lambda_3=\bar{\lambda}_3$$

and consequently

$$\frac{d\lambda_1}{ds}=\frac{d\bar{\lambda}_1}{d\bar{s}},\ \frac{d\lambda_2}{ds}=\frac{d\bar{\lambda}_2}{d\bar{s}},\ \frac{d\lambda_3}{ds}=\frac{d\bar{\lambda}_3}{d\bar{s}}\tag{3.3}$$

By substituting Eqs. (3.1) and (3.3) into Eq. (3.2), we have

$$\begin{aligned}\frac{d\boldsymbol{p}}{ds}&=\left(-\lambda_2 k_g^*-\lambda_3 k_n^*\right)\boldsymbol{T}+\left(-\lambda_1 k_g^*-\lambda_3\tau_g^*\right)\boldsymbol{g}\\&+\left(-\lambda_1 k_n^*+\lambda_2\tau_g^*\right)\boldsymbol{n},\end{aligned}\tag{3.4}$$

where

$$k_g^*=\bar{k}_g-k_g,\ k_n^*=\bar{k}_n-k_n\ \tau_g^*=\bar{\tau}_g-\tau_g.\tag{3.5}$$

The scalars k_g^*, k_n^* and τ_g^* are called induced geodesic curvature, induced normal curvature, and induced geodesic torsion, respectively. In Euclidean 3-space, for the Darboux trihedron and the induced curvatures, see [9,11].

3.2. Darboux Frame Based Equation of Spin-Rolling Motion

From the Eq. (3.4), the velocity of an arbitrary point P on the timelike moving surface S_2 in terms of time t can be obtained as

$$\begin{aligned}\boldsymbol{v}_P=\frac{d\boldsymbol{p}}{ds}\frac{ds}{dt}&=\sigma(-\lambda_2 k_g^*-\lambda_3 k_n^*)\boldsymbol{T}\\&+\sigma(-\lambda_1 k_g^*-\lambda_3\tau_g^*)\boldsymbol{g}\\&+\sigma(-\lambda_1 k_n^*+\lambda_2\tau_g^*)\boldsymbol{n}\end{aligned}\tag{3.6}$$

where $\sigma=ds/dt$ is the magnitude of the rolling velocity. Note that $\boldsymbol{v}_P$ has three causal characters. This equation gives the Darboux frame based translational velocity equation of an arbitrary point. Assume that the angular velocity of S_2 relative to the fixed surface S_1 is

$$\boldsymbol{\omega}=\omega_x\boldsymbol{T}+\omega_y\boldsymbol{g}+\omega_z\boldsymbol{n}.\tag{3.7}$$

When the vector $\boldsymbol{r}_{MP}=\lambda_1\boldsymbol{T}+\lambda_2\boldsymbol{g}+\lambda_3\boldsymbol{n}$ is also given, then the velocity $\boldsymbol{v}_P=\boldsymbol{\omega}\times\boldsymbol{r}_{MP}$ of the point P is obtained as

$$\begin{aligned}\boldsymbol{v}_P&=(\lambda_3\omega_y-\lambda_2\omega_z)\boldsymbol{T}+(\lambda_3\omega_x-\lambda_1\omega_z)\boldsymbol{g}\\&+(\lambda_1\omega_y-\lambda_2\omega_x)\boldsymbol{n}.\end{aligned}\tag{3.8}$$

By comparing Eq. (3.6) with Eq. (3.8), we have

$$\omega_x=-\sigma\tau_g^*,\quad \omega_y=-\sigma k_n^*,\quad \omega_z=\sigma k_g^*$$

and by putting these equalities into (3.7), we obtain the angular velocity of S_2 as

$$\boldsymbol{\omega}=\sigma(-\tau_g^*\boldsymbol{T}-k_n^*\boldsymbol{g}+k_g^*\boldsymbol{n})\tag{3.9}$$

In Eq. (3.9), the angular velocity $\boldsymbol{\omega}$ has three terms. The first two terms give the pure-rolling velocity about an axis in the timelike tangent plane at the contact point and the third term gives the velocity of spin motion about the spacelike unit normal direction at the contact point in Lorentzian 3-space. Then, the pure-rolling velocity is given by $-\sigma\tau_g^*\boldsymbol{T}-\sigma k_n^*\boldsymbol{g}$ and the velocity of spin motion is given by $\sigma k_g^* \boldsymbol{n}$. As a result, the timelike moving surface can follow the desired trajectory timelike curve on the timelike fixed surface by the help of these three terms. We should note that a pure-rolling motion does not have spin-rolling motion in the direction of the unit spacelike normal of the timelike surfaces. Then we give the following results:

Corollary 3.1. When two timelike surfaces undergo pure-rolling motion in Lorentzian 3-space, the geodesic curvatures of the two corresponding contact trajectory timelike curves have to be equal, that is, $k_g = \bar{k}_g$.

Corollary 3.2. Suppose the contact trajectory timelike curves α and β are geodesic curves on timelike surfaces S_1 and S_2, respectively. Then the rolling motion is a pure-rolling motion in Lorentzian 3-space.

3.3. Examples

In this subsection, two examples are given to understand the rolling contact motion of two timelike surfaces along their timelike trajectory curves in Lorentzian 3-space. The first example demonstrates how the spin-rolling motion of a Lorentzian unit cylinder on a timelike plane occurs. The second example demonstrates the pure-rolling motion of a Lorentzian unit sphere on a timelike cylinder of radius $\frac{1}{2}$.

3.3.1. Spin-rolling Motion of a Lorentzian Unit Cylinder on a Timelike Plane

Assume that a Lorentzian unit cylinder (surface S_2) rolls without sliding on a timelike plane (surface S_1) at a contact point M along the timelike curves α and β as in Figure 2. Suppose timelike curves α and β are parameterized by arc lengths s and $\bar{s}$, respectively. Let denote the Darboux frames (the right handed orthonormal frames) attached to the contact point M of the curves α and β as $(\boldsymbol{T}, \boldsymbol{g}, \boldsymbol{n})$ and $(\bar{\boldsymbol{T}}, \bar{\boldsymbol{g}}, \bar{\boldsymbol{n}})$, respectively. Let the parametrization of timelike plane be

$$x(u,v) = (v, 1, u),$$

and let α be a timelike ellipse lying on the timelike plane parameterized as

$$\alpha(t) = x\big(u(t), v(t)\big) = (-1+\cosh t, 1, 2\sinh t).$$

The derivative of arc length s with respect to t is

$$\frac{ds}{dt} = \sqrt{-\left\langle \frac{d\boldsymbol{\alpha}}{dt}, \frac{d\boldsymbol{\alpha}}{dt} \right\rangle} = \sqrt{1+3\cosh^2 t}.$$

The unit timelike tangent vector $\boldsymbol{T} = d\boldsymbol{\alpha}/ds$ of the curve α is obtained as

$$\boldsymbol{T} = \frac{1}{\sqrt{1+3\cosh^2 t}}(\sinh t, 0, 2\cosh t). \qquad \textbf{(3.10)}$$

We consider that the unit spacelike normal vector $\boldsymbol{n}$ points outward and it is obtained as

$$\boldsymbol{n} = \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{\left\| \boldsymbol{x}_u \times \boldsymbol{x}_v \right\|} = (0, -1, 0). \qquad \textbf{(3.11)}$$

The unit spacelike vector $\boldsymbol{g} = -\boldsymbol{n} \times \boldsymbol{T}$ (tangential to the surface S_2) is obtained as

$$\boldsymbol{g} = \frac{1}{\sqrt{1+3\cosh^2 t}}(-2\cosh t, 0, -\sinh t). \qquad \textbf{(3.12)}$$

From the Lorentzian inner product of (3.10)−(3.12) and their derivatives with respect to s, the geodesic curvature, the normal curvature, and the geodesic torsion of the timelike curve α are obtained as

$$\left.\begin{aligned} k_g &= \langle d\boldsymbol{T}/dt, \boldsymbol{g} \rangle \Big/ \frac{ds}{dt} = \frac{-2}{(1+3\cosh^2 t)^{3/2}}, \\ k_n &= \langle d\boldsymbol{T}/dt, \boldsymbol{n} \rangle \Big/ \frac{ds}{dt} = 0, \\ \tau_g &= -\langle d\boldsymbol{g}/dt, \boldsymbol{n} \rangle \Big/ \frac{ds}{dt} = 0, \end{aligned}\right\} \qquad \textbf{(3.13)}$$

respectively. Then, the curve α is both a principal line and an asymptotic line. Let the parameterization of the Lorentzian unit cylinder (see Figure 2) be

$$y(\bar{u}, \bar{v}) = (-\bar{v}, \cosh \bar{u}, \sinh \bar{u}).$$

Suppose β is a timelike $\bar{u}-$parametric curve (namely, a Lorentzian unit circle) lying on the Lorentzian unit cylinder parameterized as

$$\beta(\bar{u}) = y(\bar{u}, 0) = (0, \cosh \bar{u}, \sinh \bar{u})$$

where $\bar{v} = \bar{v}_0 = 0$. Since the differentiation of arc length $\bar{s}$ of the curve β with respect to $\bar{u}$ is $\frac{d\bar{s}}{d\bar{u}} = \left\| \frac{d\boldsymbol{\beta}}{d\bar{u}} \right\| = 1$, it can be clearly seen that β is a unit-speed curve. The unit timelike tangent vector $\bar{\boldsymbol{T}}$ of the curve β is given by

$$\bar{\boldsymbol{T}} = (0, \sinh \bar{u}, \cosh \bar{u}) \qquad \textbf{(3.14)}$$

Assume that the unit spacelike normal vector $\bar{\boldsymbol{n}}$ points inward and it is obtained as

$$\bar{\boldsymbol{n}} = -\frac{\boldsymbol{y}_{\bar{u}} \times \boldsymbol{y}_{\bar{v}}}{\left\| \boldsymbol{y}_{\bar{u}} \times \boldsymbol{y}_{\bar{v}} \right\|} = (0, -\cosh \bar{u}, -\sinh \bar{u}) \qquad \textbf{(3.15)}$$

The unit spacelike vector $\bar{\boldsymbol{g}}$, which is tangential to the surface S_2, is obtained as

$$\bar{\boldsymbol{g}} = -\bar{\boldsymbol{n}} \times \bar{\boldsymbol{T}} = (-1, 0, 0). \qquad \textbf{(3.16)}$$

From the Lorentzian inner product of (3.14) − (3.16) and their derivatives with respect to $\bar{s}$, the geodesic curvature, the normal curvature, and the geodesic torsion of the timelike curve β are obtained as

$$\left.\begin{aligned} \bar{k}_g &= \left\langle d\bar{\boldsymbol{T}}/d\bar{u}, \bar{\boldsymbol{g}} \right\rangle \Big/ \frac{d\bar{s}}{d\bar{u}} = 0, \\ \bar{k}_n &= \left\langle d\bar{\boldsymbol{T}}/d\bar{u}, \bar{\boldsymbol{n}} \right\rangle \Big/ \frac{d\bar{s}}{d\bar{u}} = -1, \\ \bar{\tau}_g &= -\left\langle d\bar{\boldsymbol{g}}/d\bar{u}, \bar{\boldsymbol{n}} \right\rangle \Big/ \frac{d\bar{s}}{d\bar{u}} = 0, \end{aligned}\right\} \qquad \textbf{(3.17)}$$

respectively. Then the curve β is both a principal line and a geodesic line. Substituting (3.13) and (3.17) into (3.5), we have $k_g^* = \dfrac{2}{(1+3\cosh^2 t)^{3/2}}$, $k_n^* = -1$, $\tau_g^* = 0$. Then, from (3.9), the angular velocity of the Lorentzian unit cylinder is obtained as

$$\boldsymbol{\omega} = \sigma \left(\boldsymbol{g} + \frac{2}{(1+3\cosh^2 t)^{3/2}} \boldsymbol{n} \right) \qquad \textbf{(3.18)}$$

The coordinate of the center point P of the unit cylinder in the frame $(\boldsymbol{T}, \boldsymbol{g}, \boldsymbol{n})$ at a point M is origin. From Darboux frame based translation equation (3.6) and (3.8), the velocity of point P is

$$\boldsymbol{v}_P = \boldsymbol{\omega} \times \boldsymbol{r}_{MP} = \sigma \boldsymbol{T},$$

where $\boldsymbol{r}_{MP} = \boldsymbol{n}$. After the information is used from the velocity equation to control the timelike moving surface to follow the desired trajectory timelike curve α lying on the fixed timelike surface, a brief discussion is provided. The moving surface has 2 DOFs. At any instant, the first term σ of (3.18) gives the angular velocity about the axis that is perpendicular to the Lorentzian unit cylinder. The second term gives the information about how fast the unit cylinder spins to follow the curve α. In Figure 2 (a) and Figure 2 (b), we clearly show how the spin-rolling motion occurs for a Lorentzian unit cylinder on a timelike plane at $t = 0$ and $t = 1$, respectively. In this example, we see that the Lorentzian unit cylinder can always undergo spin-rolling motion on the timelike plane, that is, the spin-rolling motion is not restricted.

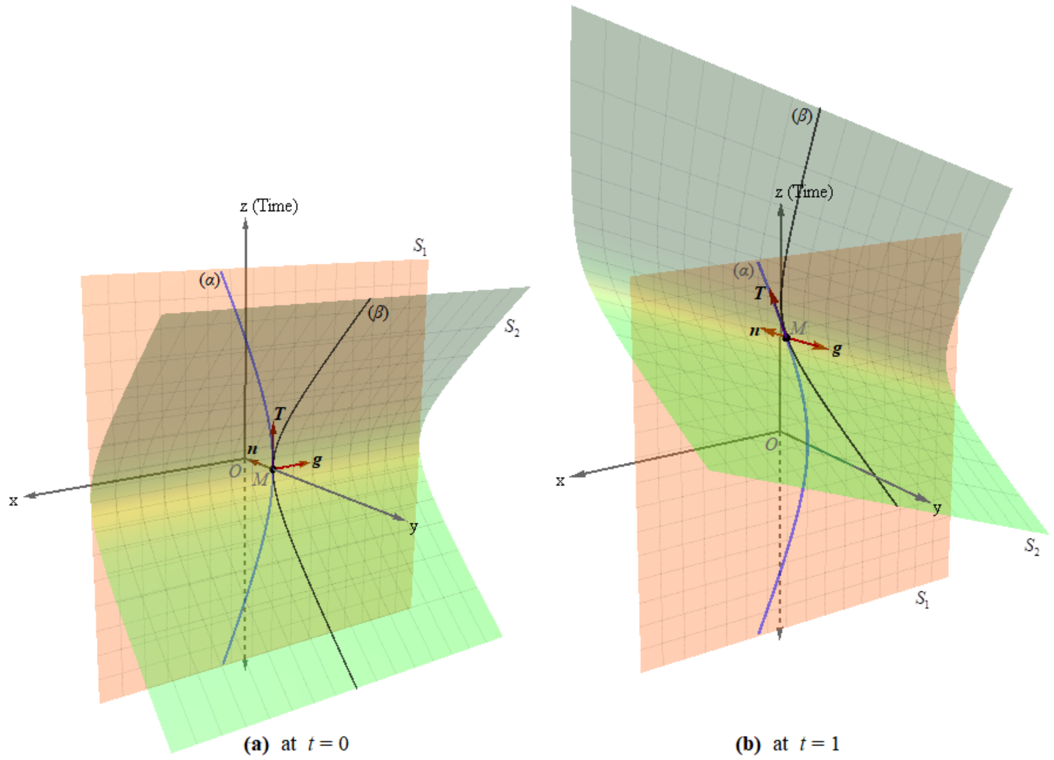

Figure 2. Spin-rolling of a Lorentzian unit cylinder on a timelike plane along timelike curves β and α.

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3.3.2. Pure-rolling Motion of a Lorentzian Unit Sphere on a Timelike Cylinder of Radius $1/2$

Suppose a Lorentzian unit sphere S_1^2 (surface S_2) rolls without sliding on a timelike cylinder of radius $1/2$ (surface S_1) at a contact point M along timelike curves α and β as in Figure 3.

Assume that the parametric equation of the timelike cylinder with radius $\frac{1}{2}$ is given by

$$x(u,v)=\left(\frac{1+\cos v}{2},\frac{u}{\sqrt{3}}-\frac{\sin v}{\sqrt{3}},\frac{2u}{\sqrt{3}}-\frac{\sin v}{2\sqrt{3}}\right)$$

which is generated by rotating the surface $x_1(u,v)=\left(\frac{1+\cos v}{2},-\frac{\sin v}{2},u\right)$ around $x-$axis (spacelike axis) with the hyperbolic angle $\cosh^{-1}\left(\frac{2}{\sqrt{3}}\right)$ in the negative direction, and let $\alpha = x\big(u(t),v(t)\big)=x\big(t,t\big)$ be a timelike helix curve lying on the timelike cylinder parameterized as

$$\alpha(t)=\left(\frac{1+\cos t}{2},\frac{t}{\sqrt{3}}-\frac{\sin t}{\sqrt{3}},\frac{2t}{\sqrt{3}}-\frac{\sin t}{2\sqrt{3}}\right).$$

The derivative of arc length s with respect to t is obtained as

$$\frac{ds}{dt}=\sqrt{\left|\left\langle\frac{d\boldsymbol{\alpha}}{dt},\frac{d\boldsymbol{\alpha}}{dt}\right\rangle\right|}=\frac{\sqrt{3}}{2}.$$

The unit timelike tangent vector $\boldsymbol{T}=\boldsymbol{T}(t,t)$ of the curve α is obtained as

$$\boldsymbol{T}=\frac{2}{\sqrt{3}}\left(\frac{-\sin t}{2},\frac{1}{\sqrt{3}}-\frac{\cos t}{\sqrt{3}},\frac{2}{\sqrt{3}}-\frac{\cos t}{2\sqrt{3}}\right) \tag{3.19}$$

Consider that the unit spacelike normal vector $\boldsymbol{n}=\boldsymbol{n}(t,t)$ is outward. In this case, we have

$$\boldsymbol{n}=\left(-\cos t,\frac{2\sin t}{\sqrt{3}},\frac{\sin t}{\sqrt{3}}\right) \tag{3.20}$$

The unit spacelike vector $\boldsymbol{g}=-\boldsymbol{n}\times\boldsymbol{T}=\boldsymbol{g}(t,t)$ (tangential to the surface S_1) is obtained as

$$\boldsymbol{g}=\frac{2}{\sqrt{3}}\left(\sin t,\frac{-1+4\cos t}{2\sqrt{3}},\frac{-1+\cos t}{\sqrt{3}}\right) \tag{3.21}$$

From the Lorentzian inner product of (3.19)$-$(3.21) and their derivatives with respect to s, the geodesic curvature, the normal curvature, and the geodesic torsion of the timelike helix curve α lying on a timelike cylinder are obtained as

$$k_g=0,\ k_n=\frac{2}{3},\ \tau_g=\frac{4}{3}, \tag{3.22}$$

respectively. Then, the curve α is a geodesic curve.

Now, let parameterize the Lorentzian unit sphere S_1^2 as

$$y(\bar{u},\bar{v})=(\cos\bar{v}\cosh\bar{u},\sin\bar{v}\cosh\bar{u},\sinh\bar{u}).$$

Suppose β is a timelike $\bar{u}-$parametric curve (that is, a Lorentzian unit circle) lying on S_1^2 parameterized as

$$\beta(\bar{u})=y(\bar{u},0)=(\cosh\bar{u},0,\sinh\bar{u}).$$

where $\bar{v}=0$. Since the derivative of $\bar{s}$ with respect to $\bar{u}$ is $\frac{d\bar{s}}{d\bar{u}}=\left\|\frac{d\boldsymbol{\beta}}{d\bar{u}}\right\|=1$, it is clearly seen that β is a unit-speed curve. The unit timelike tangent vector $\bar{\boldsymbol{T}}$ of the curve β is given by

$$\bar{\boldsymbol{T}}=\frac{d\boldsymbol{\beta}}{d\bar{u}}=(\sinh\bar{u},0,\cosh\bar{u}). \tag{3.23}$$

Assume that the unit spacelike normal vector $\bar{\boldsymbol{n}}$ of S_1^2 is inward (points origin). Then we have

$$\bar{\boldsymbol{n}}=-\frac{\boldsymbol{y}_{\bar{u}}\times\boldsymbol{y}_{\bar{v}}}{\left\|\boldsymbol{y}_{\bar{u}}\times\boldsymbol{y}_{\bar{v}}\right\|}=(-\cosh\bar{u},0,-\sinh\bar{u}). \tag{3.24}$$

The unit spacelike vector $\bar{\boldsymbol{g}}=-\bar{\boldsymbol{n}}\times\bar{\boldsymbol{T}}$, which is tangential to the surface S_2, is obtained as

$$\bar{\boldsymbol{g}}=\big(0,1,0\big). \tag{3.25}$$

From the Lorentzian inner product of (3.23)$-$(3.25) and their derivatives with respect to $\bar{s}$, the geodesic curvature, the normal curvature, and the geodesic torsion of the timelike curve β are obtained as

$$\bar{k}_g=0,\ \bar{k}_n=-1,\ \bar{\tau}_g=0, \tag{3.26}$$

respectively. It is clearly seen that the curve β is both a principal line and a geodesic line. Substituting (3.22) and (3.26) into (3.5), we have $k_g^*=0$, $k_n^*=-\frac{5}{3}$, $\tau_g^*=-\frac{4}{3}$. From (3.9), the angular velocity of the Lorentzian unit sphere is obtained as

$$\boldsymbol{\omega} = \sigma\left(\frac{4}{3}\boldsymbol{T} + \frac{5}{3}\boldsymbol{g}\right).$$

In Figure 3 (a) and Figure 3 (b), we clearly show how the pure-rolling motion occurs for a Lorentzian unit sphere on a timelike cylinder of radius $1/2$ at $t=0$ and $t=1$, respectively. In this study, we consider that $x-$ and $y-$ axes are spacelike axes, and $z-$axis is timelike axis. For this reason, the rolling contact motion is not restricted in Lorentzian 3-space. In this example, we obviously observe that the Lorentzian unit sphere can always undergo pure-rolling motion, namely, there is not any restriction for this rolling contact.

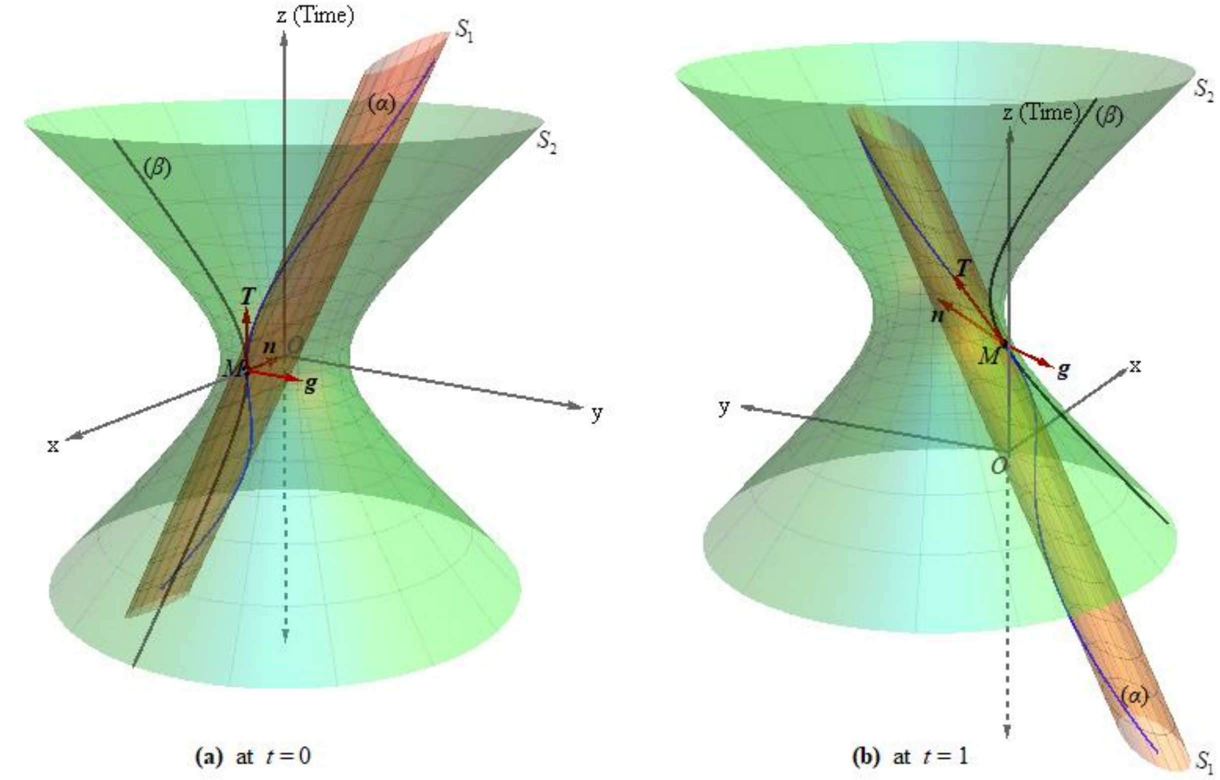

Figure 3. Pure-rolling of a Lorentzian unit sphere on a timelike cylinder along timelike curves β and α.

Consequently, we can see that this method is expressed with regards to geometric invariants that can be easily applied to arbitrary timelike parametric surfaces and timelike curves.

4. Conclusion

In this study, we established a new Darboux frame method to investigate the forward kinematics of the instantaneous spin-rolling motion and pure-rolling motion between the timelike moving surface and the timelike fixed surface through the contact point in Lorentzian 3-space. We remarked that the moving surface always maintains its causal character during a rolling motion. The forward kinematics of the moving surface was determined by the magnitude of rolling velocity σ and induced curvatures k_g^*, k_n^*, and τ_g^*. The result was given with regards to geometric invariants that can be easily generalized to arbitrary timelike parametric surfaces and timelike contact curves.

Author's Contributions

Ethics

There are no ethical issues after the publication of this manuscript.

References

[1]. Agrachev, AA, Sachkov, YL. An intrinsic approach to the control of rolling bodies, in Proc. 38th IEEE Conf. Decis. Control, Phoenix, AZ, USA; 1999, pp 431–435.

[2]. Birman, GS, Nomizu, K. 1984. Trigonometry in Lorentzian Geometry, *Ann. Math. Month.*, 91(9): 543-549.

[3]. Bottema, O, Roth, B. Theoretical Kinematics; North-Holland Publ. Co.: Amsterdam, 1979; pp 556.

[4]. Cai, C, Roth, B. On the spatial motion of rigid bodies with point contact, in Proc. IEEE Conf. Robot. Autom.; 1987, pp 686–695.

[5]. Cai, C, Roth, B. 1986. On the planar motion of rigid bodies with point contact, *Mech. Mach. Theory*, 21: 453–466.

[6]. Chelouah, A, Chitour, Y. 2003. On the motion planning of rolling surfaces, *Forum Math*., 15(5): 727–758.

[7]. Chitour, Y, Marigo, A, Piccoli, B. 2005. Quantization of the rolling-body problem with applications to motion planning, *Syst. Control Lett*., 54(10): 999–1013.

[8]. Cui, L, Dai, JS. 2010. A Darboux-Frame-Based Formulation of Spin-Rolling Motion of Rigid Objects With Point Contact, *IEEE Trans. Rob*., 26(2): 383–388.

[9]. Cui, L. Differential Geometry Based Kinematics of Sliding-Rolling Contact and Its Use for Multifingered Hands, Ph.D. thesis, King's College London, University of London, London, UK, 2010.

[10]. Cui, L, Dai, JS. 2015. A Polynomial Formulation of Inverse Kinematics of Rolling Contact, *ASME J. Mech. Rob*., **7**(4): 041003_041001-041009.

[11]. Do Carmo, MP. Differential Geometry of Curves and Surfaces; Prentice-Hall: Englewood Cliffs, New Jersey, 1976.

[12]. Karger, A, Novak, J. Space Kinematics and Lie Groups; STNL Publishers of Technical Lit.: Prague, Czechoslovakia, 1978.

[13]. Li, ZX, Canny, J. 1990. Motion of two rigid bodies with rolling constraint, *IEEE Trans. Robot. Autom*., 6(1): 62–72.

[14]. Marigo, A, Bicchi, A. 2000. Rolling bodies with regular surface: Controllability theory and application, *IEEE Trans. Autom. Control*, 45(9): 1586–1599.

[15]. Montana, DJ. 1995. The kinematics of multi-fingered manipulation, *IEEE Trans. Robot. Autom*., 11(4): 491–503.

[16]. Müller, HR. Kinematik Dersleri; Ankara Üniversitesi Fen Fakültesi Yayınları, 1963.

[17]. Neimark, JI, Fufaev, NA. Dynamics of Nonholonomic Systems; Providence, RI: Amer. Math. Soc., 1972.

[18]. Nelson, EW, Best CL, McLean, WG. Schaum's Outline of Theory and Problems of Engineering Mechanics, Statics and Dynamics (5th Ed.); McGraw-Hill: New York, 1997.

[19]. O'Neill, B. Semi-Riemannian Geometry with Applications to Relativity; Academic Press: London, 1983.

[20]. Ratcliffe, JG. Foundations of Hyperbolic Manifolds; Springer: New York, 2006.

[21]. Sarkar, N, Kumar, V, Yun, X. 1996. Velocity and Acceleration Analysis of Contact Between Three-Dimensional Rigid Bodies, *ASME J. Appl. Mech*., 63(4): 974–984.

[22]. Tchon, K. 2002. Repeatability of inverse kinematics algorithms for mobile manipulators, *IEEE Trans. Autom. Control*, 47(8): 1376– 1380.

[23]. Tchon, K, Jakubiak, J. An extended Jacobian inverse kinematics algorithm for doubly nonholonomic mobile manipulators, in Proc. IEEE Int. Conf. Robot. Autom., Barcelona, Spain; 2005, pp 1548–1553.

[24]. Uğurlu HH, Çalışkan, A. Darboux Ani Dönme Vektörleri ile Spacelike ve Timelike Yüzeyler Geometrisi; Celal Bayar Üniversitesi Yayınları: Manisa, 2012.