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Research Article

ON SOME RESULTS OF WEIGHTED HÖLDER TYPE INEQUALITY ON TIME SCALES

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Abstract: *The concept of time scales has attracted the attention of mathematicians for a quarter-century. The time scales have a very important place in mathematical analysis. Many mathematicians have worked on this subject and they have achieved good results. Inequalities and dynamic equations are at the top of these studies. Inequalities and dynamic equations contributed to the solution of many problems in various branches of science. In this article, some results of weighted Hölder type inequality are presented via* $\diamond_\alpha$*-integral.*

Keywords: *Hölder type inequality, Time scales, Integral inequalities.*

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1. Introduction

Time scales have made a name in many branches of science for the last 30 years. The theory of time scales was initiated by Stefan Hilger [1] in 1988 and it was developed by many mathematicians. They have demonstrated various aspects of integral inequalities [2-32]. The most important examples of time scale studies are differential calculus and inequalities [12]. In 2014, Agarwal, O'Regan, and Saker revealed many features of dynamic inequalities in time scales. Dynamic equations and inequalities have many applications in other disciplines besides mathematics. For example; population dynamics, quantum mechanics, physical problems, wave equations, heat transfer, optical problems, and finance problems [12, 27, 29, 30].

Hölder inequalities have a very important place in harmonic analysis. Many mathematicians have achieved very important results using Hölder inequality [10, 11, 13]. The article aims to demonstrate some results of weighted Hölder's inequality in the two-dimensional case on time scale via the $\diamond_\alpha$-integral.

2. Materials and Method

Now, let's briefly give information about time scales and necessary definitions and notations for our article. The details can be followed from the studies conducted by some researchers [1-31].

$\mathbb{T}$ is a time scale ($\mathbb{T} \neq \emptyset$ and $\mathbb{T} \subset \mathbb{R}$).

Let $\sigma(t), \rho(t)$ be the forward jump operator and the backward jump operator in $\mathbb{T}$. And jump operators are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T}: s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T}: s > t\}$$

for $t \in \mathbb{T}$.

If $\sigma: \mathbb{T} \to \mathbb{T}$, $\sigma(t) > t$, then t is right-scattered and if $\sigma: \mathbb{T} \to \mathbb{T}$, $\sigma(t) = t$, then t is called right-dense. If $\rho: \mathbb{T} \to \mathbb{T}$, $\rho(t) < t$, then t is left-scattered and if $\rho: \mathbb{T} \to \mathbb{T}$, $\rho(t) = t$, then t is called left-dense. Let two mappings $\mu, \vartheta: \mathbb{T} \to \mathbb{R}^+$ such that $\mu(t) = \sigma(t) - t$, $\vartheta(t) = t - \rho(t)$ are called graininess mappings.

Let $g: \mathbb{T} \to \mathbb{R}$ and $g^\sigma: \mathbb{T} \to \mathbb{R}$ by $g^\sigma(t) = g(\sigma(t))$ for $\forall t \in \mathbb{T}$, i.e., $g^\sigma = g \circ \sigma$. And let $g: \mathbb{T} \to \mathbb{R}$ and $g^\rho: \mathbb{T} \to \mathbb{R}$ by $g^\rho(t) = g(\rho(t))$ for $\forall t \in \mathbb{T}$, i.e., $g^\rho = g \circ \rho$.

The generalized derivative $g^\Delta(t)$ of $g: \mathbb{T} \to \mathbb{R}$, becomes $g^\Delta(t) = g'(t)$ when $\mathbb{T} = \mathbb{R}$. And if $\mathbb{T} = \mathbb{Z}$, then $g^\Delta(t)$ reduces to $g^\Delta(t) = \Delta g(t)$.

Definition 2.1. If $G: \mathbb{T} \to \mathbb{R}$ is called a Δ-antiderivative of $g: \mathbb{T} \to \mathbb{R}$, then we define

$$\int_s^t g(\delta)\Delta\delta = G(t) - G(s),$$

for $\forall s, t \in \mathbb{T}$. [12]

Definition 2.2. Let $h: \mathbb{T}_k \to \mathbb{R}$ is called ∇-differentiable at $t \in \mathbb{T}_k$. If $\varepsilon > 0$, then there exists a neighborhood V of t such that

$$\left|h(\rho(t)) - h(s) - h^\nabla(t)(\rho(t) - s)\right| \leq \varepsilon|\rho(t) - s|,$$

for $\forall s \in V$. [14]

Definition 2.3. Let $H: \mathbb{T} \to \mathbb{R}$ is called a ∇-antiderivative of $h: \mathbb{T} \to \mathbb{R}$, then we define

$$\int_s^t h(\delta)\nabla\delta = H(t) - H(s),$$

for $s, t \in \mathbb{T}$. [14]

Let $f(t)$ be differentiable on $\mathbb{T}$ for $\alpha, t \in \mathbb{T}$. Then, we define $f^{\diamond_\alpha}(t)$ by

$$f^{\diamond_\alpha}(t) = \alpha f^\Delta(t) + (1 - \alpha) f^\nabla(t)$$

for $0 \leq \alpha \leq 1$. [14]

Proposition 2.4. If we get $f, h: \mathbb{T} \to \mathbb{R}$, $\diamond_\alpha$-differentiable for $\alpha, t \in \mathbb{T}$, [15] then

(i) $f + h: \mathbb{T} \to \mathbb{R}$ is $\diamond_\alpha$-differentiable for $t \in \mathbb{T}$ with

$$(f + h)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t) + h^{\diamond_\alpha}(t).$$

(ii) Let $k \in \mathbb{R}$, $kf: \mathbb{T} \to \mathbb{R}$ is $\diamond_\alpha$-differentiable for $\alpha, t \in \mathbb{T}$ with

$$(kf)^{\diamond_\alpha}(t) = kf^{\diamond_\alpha}(t).$$

(iii) $fh: \mathbb{T} \rightarrow \mathbb{R}$ is $\diamond_\alpha$-differentiable for $\alpha, t \in \mathbb{T}$ with

$$(fh)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t).h(t) + \alpha f^\sigma(t)h^\Delta(t) + (1-\alpha)f^\rho(t)h^\nabla(t).$$

Definition 2.5. If we get $\alpha, b, t \in \mathbb{T}$ and $f: \mathbb{T} \rightarrow \mathbb{R}$, [15] then

$$\int_b^t f(\tau) \diamond_\alpha \gamma = \alpha \int_b^t f(\gamma)\Delta\gamma + (1-\alpha)\int_b^t f(\gamma)\nabla\gamma$$

for $0 \leq \alpha \leq 1$.

Proposition 2.6. Let $u, v, \alpha, t \in \mathbb{T}$, $c \in \mathbb{R}$ and if $f(\gamma)$, $g(\gamma)$ are $\diamond_\alpha$-integrable on $[u,v]_\mathbb{T}$, then the following statements are valid. [15]

(i) $\int_u^t [f(\gamma) + g(\gamma)] \diamond_\alpha \gamma = \int_u^t f(\gamma) \diamond_\alpha \gamma + \int_u^t g(\gamma) \diamond_\alpha \gamma$,
(ii) $\int_u^t cf(\gamma) \diamond_\alpha \gamma = c\int_u^t f(\gamma) \diamond_\alpha \gamma$,
(iii) $\int_u^t f(\gamma) \diamond_\alpha \gamma = -\int_t^u f(\gamma) \diamond_\alpha \gamma$,
(iv) $\int_u^t f(\gamma) \diamond_\alpha \gamma = \int_u^v f(\gamma) \diamond_\alpha \gamma + \int_v^t f(\gamma) \diamond_\alpha \gamma$,
(v) $\int_u^u f(\gamma) \diamond_\alpha \gamma = 0$.

Lemma 2.7. Let $u, v, \alpha, t \in \mathbb{T}$ with $u < v$. Suppose that $h(\gamma)$ and $g(\gamma)$ are $\diamond_\alpha$-integrable on $[u,v]_\mathbb{T}$, then we have [15]

(i) If $h(\gamma) \geq 0$ for $\forall\gamma \in [u,v]_\mathbb{T}$, then $\int_u^v h(\gamma) \diamond_\alpha \gamma \geq 0$.
(ii) If $h(\gamma) \leq g(\gamma)$ for $\forall\gamma \in [u,v]_\mathbb{T}$, then $\int_u^v h(\gamma) \diamond_\alpha \gamma \leq \int_u^v g(\gamma) \diamond_\alpha \gamma$.
(iii) If $h(\gamma) \geq 0$ for $\forall\gamma \in [u,v]_\mathbb{T}$, then $h(\gamma) = 0$ iff $\int_u^v h(\gamma) \diamond_\alpha \gamma = 0$.

Lemma 2.8. Let h, g be ∇-differentiable two positive functions [31]. For h, g satisfying $g^q l \leq h^p \leq g^q L$ on the $[a,b]$, with $1/p + 1/q = 1$ and $p, q > 1$, we have

$$(l)^{1/pq}\left(\int_a^b h(s)^p \nabla s\right)^{1/p}\left(\int_a^b g(s)^q \nabla s\right)^{1/q} \leq (L)^{1/pq}\int_a^b h(s)g(s)\nabla s.$$

Theorem 2.9. If h is $\diamond_\alpha$-integrable on an interval $I = [a,b]$, then $|h|$ is $\diamond_\alpha$-integrable on I that is [31]

$$\left|\int_I h(s) \diamond_\alpha s\right| \leq \int_I |h(s)| \diamond_\alpha s.$$

3. Results

Theorem 3.1. Let two mappings $h, g \in C_{rd}$ and $h, g: [a,b] \times [a,b] \rightarrow \mathbb{R}$. $\theta(\gamma,\tau), \vartheta(v,\tau) > 0$ weight functions and $\diamond_\alpha$-integrable functions for $a, b \in \mathbb{T}$, we have

$$\int_a^b \int_a^b |h(\gamma,\tau)\theta(\gamma,\tau)g(\gamma,\tau)\vartheta(\gamma,\tau)| \diamond_\alpha \gamma \diamond_\alpha \tau$$

$$\leq \left(\int_a^b \int_a^b |h(\gamma,\tau)\theta(\gamma,\tau)|^p \diamond_\alpha \gamma \diamond_\alpha \tau\right)^{\frac{1}{p}} \left(\int_a^b \int_a^b |g(\gamma,\tau)\vartheta(\gamma,\tau)|^q \diamond_\alpha \gamma \diamond_\alpha \tau\right)^{\frac{1}{q}}$$

where $p > 1,\ q > 1$.

Proof. If $u, v \geq 0$ $(u, v \in \mathbb{R})$ with $1/p + 1/q = 1$ $(p, q > 1)$, then

$$qu^{\frac{1}{q}} + pv^{\frac{1}{p}} \geq pq \tag{1}$$

holds. Assume that

$$\left(\int_a^b \int_a^b |h(\gamma,\tau)\theta(\gamma,\tau)|^p \diamond_\alpha \gamma \diamond_\alpha \tau\right)\left(\int_a^b \int_a^b |g(\gamma,\tau)\vartheta(\gamma,\tau)|^q \diamond_\alpha \gamma \diamond_\alpha \tau\right) \neq 0.$$

and $u(\gamma,\tau), v(\gamma,\tau) \in C_{rd}(\mathbb{R})$ with

$$u(\gamma,\tau) = \frac{|h(\gamma,\tau)\theta(\gamma,\tau)|^p}{\int_a^b \int_a^b |h(\delta_1,\delta_2)\theta(\delta_1,\delta_2)|^p \diamond_\alpha \delta_1 \diamond_\alpha \delta_2},$$

$$v(\gamma,\tau) = \frac{|g(\gamma,\tau)\vartheta(\gamma,\tau)|^q}{\int_a^b \int_a^b |g(\delta_1,\delta_2)\vartheta(\delta_1,\delta_2)|^q \diamond_\alpha \delta_1 \diamond_\alpha \delta_2}.$$

If we apply (1) to functions $u(\gamma,\tau)$ and $v(\gamma,\tau)$ and take integrals from a to b, we get directly Hölder inequalities

$$\int_a^b \int_a^b \frac{|h(\gamma,\tau)\theta(\gamma,\tau)|^p}{\int_a^b \int_a^b |h(\delta_1,\delta_2)\theta(\delta_1,\delta_2)|^p \diamond_\alpha \delta_1 \diamond_\alpha \delta_2} \cdot \frac{|g(\gamma,\tau)\vartheta(\gamma,\tau)|^q}{\int_a^b \int_a^b |g(\delta_1,\delta_2)\vartheta(\delta_1,\delta_2)|^q \diamond_\alpha \delta_1 \diamond_\alpha \delta_2} \diamond_\alpha \gamma \diamond_\alpha \tau$$

$$= \int_a^b \int_a^b u(\gamma,\tau)^{\frac{1}{p}} v(\gamma,\tau)^{\frac{1}{q}} \diamond_\alpha \gamma \diamond_\alpha \tau \leq \int_a^b \int_a^b \left[\frac{u(\gamma,\tau)}{p} + \frac{v(\gamma,\tau)}{q}\right] \diamond_\alpha \gamma \diamond_\alpha \tau$$

$$= \frac{1}{p}\int_a^b \int_a^b \frac{|h(\gamma,\tau)\theta(\gamma,\tau)|^p}{\int_a^b \int_a^b |h(\delta_1,\delta_2)\theta(\delta_1,\delta_2)|^p \diamond_\alpha \delta_1 \diamond_\alpha \delta_2} \diamond_\alpha \gamma \diamond_\alpha \tau$$

$$+ \frac{1}{q}\int_a^b \int_a^b \frac{|g(\gamma,\tau)\vartheta(\gamma,\tau)|^q}{\int_a^b \int_a^b |g(\delta_1,\delta_2)\vartheta(\delta_1,\delta_2)|^q \diamond_\alpha \delta_1 \diamond_\alpha \delta_2} \diamond_\alpha \gamma \diamond_\alpha \tau$$

$$= \frac{1}{p} + \frac{1}{q} = 1.$$

Theorem 3.2. Let $\theta(\gamma,\tau),\vartheta(\gamma,\tau)>0$ be weight functions via $\diamond_\alpha$-integrable. If $M(\gamma,\tau),h(\gamma),g(\tau),\mu(\gamma),\sigma(\tau)$ be non-negative functions with $1/p+1/q=1$ $(p,q>1)$ via $\diamond_\alpha$-integrable, then the following (2) and (3) inequalities are equivalent

$$\int_a^b\int_a^b M(\gamma,\tau)h(\gamma)\theta(\gamma,\tau)g(\tau)\vartheta(\gamma,\tau)\diamond_\alpha\gamma\diamond_\alpha\tau$$

$$\leq\left(\int_a^b \mu(\gamma)^p K(\gamma)(h(\gamma)\theta(\gamma))^p\diamond_\alpha\gamma\right)^{\frac{1}{p}}\left(\int_a^b \sigma(\tau)^q D(\tau)(g(\tau)\vartheta(\tau))^q\diamond_\alpha\tau\right)^{\frac{1}{q}} \tag{2}$$

and

$$\int_a^b \sigma(\tau)^{-p}D(\tau)^{1-p}\left(\int_a^b M(\gamma,\tau)h(\gamma)\theta(\gamma)\diamond_\alpha\gamma\right)^p\diamond_\alpha y\leq\int_a^b \mu(\gamma)^p K(\gamma)(h(\gamma)\theta(\gamma))^p\diamond_\alpha\gamma \tag{3}$$

where $K(\gamma)=\int_a^b\frac{M(\gamma,\tau)}{\sigma(y)^p}\diamond_\alpha\tau$ and $D(y)=\int_a^b\frac{M(\gamma,\tau)}{\mu(\gamma)^q}\diamond_\alpha\gamma$.

Proof. Let's consider the equation below

$$\int_a^b\int_a^b M(\gamma,\tau)h(\gamma)\theta(\gamma,\tau)g(\tau)\vartheta(\gamma,\tau)\diamond_\alpha\gamma\diamond_\alpha\tau$$

$$=\int_a^b\int_a^b M(\gamma,\tau)h(\gamma)\theta(\gamma,\tau)\frac{\mu(\gamma)}{\sigma(\tau)}g(\tau)\vartheta(\gamma,\tau)\frac{\sigma(\tau)}{\mu(\gamma)}\diamond_\alpha\gamma\diamond_\alpha\tau. \tag{4}$$

Now, applying the Hölder inequality to (4), we get

$$\int_a^b\int_a^b M(\gamma,\tau)h(\gamma)g(\tau)\diamond_\alpha\gamma\diamond_\alpha\tau$$

$$\leq\left(\int_a^b \mu(\gamma)^p K(\gamma)(h(\gamma)\theta(\gamma))^p\diamond_\alpha\gamma\right)^{\frac{1}{p}}\left(\int_a^b \sigma(\tau)^q D(\tau)(g(\tau)\vartheta(\tau))^q\diamond_\alpha\tau\right)^{\frac{1}{q}}.$$

Assume that the inequality (2) holds. If

$$g(\tau)=\sigma(\tau)^{-p}D(\tau)^{1-p}\left(\int_a^b M(\gamma,\tau)h(\gamma)\theta(\gamma)\diamond_\alpha\gamma\right)^{p-1}$$

and using (2), we obtain

$$\int_a^b \sigma(\tau)^{-p}D(\tau)^{1-p}\left(\int_a^b M(\gamma,\tau)h(\gamma)\theta(\gamma)\diamond_\alpha\gamma\right)^p\diamond_\alpha\tau$$

$$= \int_a^b \int_a^b M(\gamma,\tau)h(\gamma)\theta(\gamma)g(y) \diamond_\alpha \gamma \diamond_\alpha \tau$$

$$\leq \left(\int_a^b \mu(\gamma)^p K(\gamma)(h(\gamma)\theta(\gamma))^p \diamond_\alpha \gamma\right)^{\frac{1}{p}} \left(\int_a^b \sigma(\tau)^q D(\tau)\big(g(\tau)\vartheta(\tau)\big)^q \diamond_\alpha \tau\right)^{\frac{1}{q}}$$

$$= \left(\int_a^b \mu(\gamma)^p K(\gamma)(h(\gamma)\theta(\gamma))^p \diamond_\alpha \gamma\right)^{\frac{1}{p}} \left(\int_a^b \sigma(\tau)^{-p} D(\tau)^{1-p} \left(\int_a^b M(\gamma,\tau)h(\gamma)\theta(\gamma) \diamond_\alpha \gamma\right)^p \diamond_\alpha \tau\right)^{\frac{1}{q}}.$$

Now, applying Hölder's inequality, we obtain

$$\int_a^b \int_a^b M(\gamma,\tau)h(\gamma)\theta(\gamma)g(\tau)\vartheta(\tau) \diamond_\alpha \gamma \diamond_\alpha \tau$$

$$= \int_a^b \left(\sigma(\tau)^{-1} D\tau^{\frac{-1}{q}} \int_a^b M(\gamma,\tau)h(\gamma)\theta(\gamma) \diamond_\alpha \gamma\right) \sigma(\tau)D(\tau)^{\frac{1}{q}} g(\tau)\vartheta(\tau) \diamond_\alpha \tau$$

$$\leq \left(\int_a^b \sigma(\tau)^{-p} D(\tau)^{1-p} \left(\int_a^b M(\gamma,\tau)h(\gamma)\theta(\gamma) \diamond_\alpha \gamma\right)^p \diamond_\alpha \tau\right)^{\frac{1}{p}} \left(\int_a^b \sigma(\tau)^q D(\tau)(g(\tau)\vartheta(\tau))^q \diamond_\alpha \tau\right)^{\frac{1}{q}}$$

$$\leq \left(\int_a^b \mu(\gamma)^p K(\gamma)(h(\gamma)\theta(\gamma))^p \diamond_\alpha \gamma\right)^{\frac{1}{p}} \left(\int_a^b \sigma(\tau)^q D(\tau)(g(\tau)\vartheta(\tau))^q \diamond_\alpha \tau\right)^{\frac{1}{q}}.$$

4. Discussion

Integration in time scales helps us to achieve many nonlinear integral equations using different inequalities and equations. Many authors have obtained many inequalities and integral equations using various methods [1-32]. The method we use in this article can be applied to other inequalities and equations provided that they meet the requirements. We focused on the concept of weight in inequalities and dynamic equations. In addition, we have obtained some results by using Hölder type inequality in time scales inspired by these studies.

The compliance to Research and Publication Ethics: This work was carried out by obeying research and ethics rules.

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