Turk. J. Math. Comput. Sci.
12(1)(2020) 8–17

https://dergipark.org.tr/en/pub/tjmcs

Fuzzy Soft Bi-Interior Ideals Over Semirings

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Received: 05-08-2019 • Accepted: 20-02-2020

Abstract. In this paper, we introduce the notion of fuzzy soft bi-interior ideals over semirings and study some of their algebraical properties.

2010 AMS Classification: 16Y60,16Y99, 03E72.

Keywords: Bi-interior ideal, Fuzzy soft bi-interior ideal, Semiring.

1. Introduction

Semiring is an algebraic structure which is a common generalization of rings and distributive lattices, was first introduced by Vandiver [28] in 1934 but non-trivial examples of semirings had appeared in the studies on the theory of commutative ideals of rings by Dedekind in 19th century. Semiring is a universal algebra with two binary operations called addition and multiplication, where one of them distributive over the other. Bounded distributive lattices are commutative semirings which are both additively idempotent and multiplicatively idempotent. A natural example of semiring is the set of all natural numbers under usual addition and multiplication of numbers. In particular, if I is the unit interval on the real line, then $(I, \max, \min)$ is a semiring in which 0 is the additive identity and 1 is the multiplicative identity. The theory of rings and the theory of semigroups have considerable impact on the development of the theory of semirings. In structure, semirings lie between semigroups and rings. Additive and multiplicative structures of a semiring play an important role in determining the structure of a semiring. Semiring as the basic algebraic structure was used in the areas of theoretical computer science as well as in the solutions of graph theory, optimization theory and in particular for studying automata, coding theory and formal languages. Semiring theory has many applications in other branches. The notion of ideals was introduced by Dedekind for the theory of algebraic numbers, was generalized by Noether for associative rings. The one and two sided ideals introduced by her, are still central concepts in ring theory. We know that the notion of an one sided ideal of any algebraic structure is a generalization of notion of an ideal. The quasi ideals are generalization of left and right ideals where as the bi-ideals are generalization of quasi ideals. The notion of bi-ideals in semigroups was introduced by Lajos [12]. Iseki [7–9] introduced the concept of quasi ideal for a semiring. Quasi ideals in $\Gamma-$semirings studied by Jagtap and Pawar [10]. Henriksen [6] studied ideals in semirings. As a further generalization of ideals, Steinfeld [26] first introduced the notion of quasi ideals for semigroups and then for rings. We know that the notion of the bi-ideal in semirings is a special case of $(m, n)-$ideal introduced

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by Lajos [12]. The concept of bi-ideals was first introduced by Good and Hughes [4] for a semigroup. Lajos and Szasz [13] introduced the concept of bi-ideals for rings.

The fuzzy set theory was developed by Zadeh [30] in 1965. Many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory etc. The fuzzification of algebraic structure was introduced by Rosenfeld [25] and he introduced the notion of fuzzy subgroups in 1971. Swamy and Swamy [27] studied fuzzy prime ideals in rings in 1988. In 1982, Liu [14] defined and studied fuzzy subrings as well as fuzzy ideals in rings. Mandal [17] studied fuzzy ideals and fuzzy interior ideals in an ordered semiring. Kuroki [11] studied fuzzy interior ideals in semigroups. Murali Krishna Rao [19–24] introduced bi-quasi-ideals, bi-interior ideals in semirings, bi-quasi-ideals and fuzzy bi-quasi - ideals in $\Gamma-$semigroups and $T-$fuzzy ideals in ordered $\Gamma-$semirings. Venkateswarlu et al. [29] introduce the notion of a fuzzy bi-interior ideal of semiring and characterize the regular semiring in terms of fuzzy bi-interior ideal of semiring.

Molodtsov [18] was introduced the concept of soft set theory as a new mathematical tool for dealing with uncertainties, only partially resolves the problem is that objects in universal set often does not precisely satisfy the parameters associated to each of the elements in the set. Then Maji et al. [15] extended soft set theory to fuzzy soft set theory. Aktas and Cagman [2] defined the soft set and soft groups. Majumdar and Samantha [16] extended soft sets to fuzzy soft set. Acar et al. [1], gave the basic concept of soft ring. Ghosh et al. [3] initiated the study of fuzzy soft rings and fuzzy soft ideals. Feng et al. [5] studied soft semirings by using the soft set theory Murali Krishna Rao [24] introduced and studied fuzzy soft ideals and fuzzy soft $k-$ideals over a $\Gamma-$semiring. In this paper, we introduce the notion of fuzzy soft semiring, fuzzy soft ideal, fuzzy soft bi-ideal, fuzzy soft quasi-ideal and fuzzy soft interior ideal over semirings and study some of their algebraical properties.

2. Preliminaries

In this section, we recall some of the fundamental concepts and definitions necessary for this paper, see ([21–24, 29]).

Definition 2.1 ([6]). A set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) will be called a semiring provided

(i) addition is a commutative operation.
(ii) multiplication distributes over addition both from the left and from the right.
(iii) there exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$, for each $x \in S$.

A semiring M is said to be commutative semiring if $xy = yx$, for all $x, y \in M$. An element $a \in M$ is said to be regular element of M if there exist $x \in M$ such that $a = axa$. If every element of semiring M is a regular, then M is said to be regular semiring.

Definition 2.2. A non-empty subset A of semiring M is called

(i) a subsemiring of M if $(A, +)$ is a subsemigroup of $(M, +)$ and $AA \subseteq A$.
(ii) a quasi ideal of M if A is a subsemiring of M and $AM \cap M \subseteq A$.
(iii) a bi-ideal of M if A is a subsemiring of M and $AMA \subseteq A$.
(iv) an interior ideal of M if A is a subsemiring of M and $MAM \subseteq A$.
(v) a left (right) ideal of M if A is a subsemiring of M and $MA \subseteq A (AM \subseteq A)$.
(vi) an ideal if A is a subsemiring of $M, A \subseteq A$ and $MA \subseteq A$.
(vii) a $k-$ideal if A is a subsemiring of $M, AM \subseteq A, MA \subseteq A$ and $x \in M,\ x + y \in A, y \in A$ then $x \in A$.
(viii) a left bi-quasi ideal (right bi-quasi ideal) of M if A is a subsemigroup of $(M, +)$ and $MA \cap LMA \subseteq A$ $(AM \cap AMA \subseteq A)$.
(ix) a bi-quasi ideal of M if it is both a left bi-quasi and a right bi-quasi ideal of M.
(x) a bi-interior ideal of M if A is a subsemiring of M and $MAM \cap AMA \subseteq A$.

Definition 2.3. A semiring M is called

(1) a right bi-quasi simple semiring if M has no right bi-quasi ideal other than M itself.
(2) a left (right) simple semiring if M has no proper left (right) ideal of M.
(3) a bi-quasi simple semiring if M has no proper bi-quasi ideal of M.
(4) simple semring if M has no proper ideals .

Theorem 2.4. *Let M be a semiring. Then the following are hold*

(1) *Every left ideal is a bi-interior ideal of M.*
(2) *Every right ideal is a bi-interior ideal of M.*
(3) *Every quasi ideal is bi-interior ideal of M.*
(4) *Arbitrary intersection of bi-interior of M is also bi-interior ideal of M.*
(5) *Every ideal is a bi-interior ideal of M.*
(6) *If B is a bi-interior ideal of M then BM and MB are bi-interior ideals of M.*

Definition 2.5. Let M be a non-empty set. A mapping $\mu : M \to [0, 1]$ is called a fuzzy subset of M.

Definition 2.6. If μ is a fuzzy subset of M, for $t \in [0, 1]$ then the set $\mu_t = \{x \in M \mid \mu(x) \geq t\}$ is called a level subset of M with respect to a fuzzy subset μ.

Definition 2.7. A fuzzy subset $\mu : M \to [0, 1]$ is a non-empty fuzzy subset if μ is not a constant function.

Definition 2.8. For any two fuzzy subsets λ and μ of M, $\lambda \subseteq \mu$ means $\lambda(x) \leq \mu(x)$, for all $x \in M$.

Definition 2.9. Let A be a non-empty subset of M. The characteristic function of A is a fuzzy subset of M is defined by $\chi_{_A}(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{if } x \notin A. \end{cases}$

Definition 2.10. Let f and g be fuzzy subsets of S. Then $f \cup g, f \cap g$ are fuzzy subsets of S defined by $f \cup g(x) = \max\{f(x), g(x)\}$, $f \cap g(x) = \min\{f(x), g(x)\}$, for all $x \in S$. And $f \circ g$ is defined by

$$f \circ g(z) = \begin{cases} \sup\limits_{z=xy,\ x,y \in S} & \{\min\{f(x), g(y)\}\}, \\ 0 & \text{otherwise} \end{cases}, \text{ for all } z \in S.$$

Definition 2.11. A fuzzy subset μ of semiring M is called

(i) a fuzzy subsemiring of M if it satisfies the following conditions
(a) $\mu(x + y) \geq min\ \{\mu(x), \mu(y)\}$
(b) $\mu(xy) \geq min\ \{\mu(x), \mu(y)\}$, for all $x, y \in M$.
(ii) a fuzzy left (right) ideal of M if it satisfies the following conditions
(a) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$
(b) $\mu(xy) \geq \mu(y)\ (\mu(x))$, for all $x, y \in M$.
(iii) a fuzzy ideal of M if it satisfies the following conditions
(a) $\mu(x + y) \geq min\{\mu(x), \mu(y)\}$
(b) $\mu(xy) \geq max\ \{\mu(x), \mu(y)\}$, for all $x, y \in M$.
(iv) a fuzzy bi-ideal of M if it satisfies the following conditions
(a) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$
(b) $\mu \circ \chi_M \circ \mu \subseteq \mu$, for all $x, y \in M$
(v) a fuzzy quasi ideal of M if it satisfies the following conditions
(a) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$
(b)$\mu \circ \chi_M \cap \chi_M \circ \mu \subseteq \mu$, for all $x, y \in M$
(vi) a fuzzy left (right) bi-quasi ideal of M if it satisfies the following conditions
(a) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$
(b) $\chi_M \circ \mu \cap \mu \circ \chi_M \circ \mu \subseteq \mu(\mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu \subseteq \mu)$, for all $x, y \in M$
A fuzzy subset μ of semiring M is called a fuzzy bi-quasi ideal if it is both a fuzzy left and a fuzzy right bi-quasi ideal of M.

Theorem 2.12. *If μ is a fuzzy quasi-ideal of a regular semiring M then μ is a fuzzy ideal of M.*

Definition 2.13. A fuzzy subset μ of a semiring M is called a fuzzy bi-interior ideal if

(i) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$
(ii) $\chi_M \circ \mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu \subseteq \mu$, for all $x, y \in M$.

Example 2.14. Let Q be the set of all rational numbers, $M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in Q \right\}$. A binary operation is defined as the usual matrix multiplication and $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, 0 \neq b \in Q \right\}$. Then M is a semiring and A is a bi-interior ideal but not a bi-ideal of semiring M.

Define $\mu : M \to [0, 1]$ such that $\mu(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$

Then μ is a fuzzy bi-interior ideal of M.

Theorem 2.15. *M is a regular semiring if and only $AB = A \cap B$, for any right ideal A and left ideal B of semiring M.*

Theorem 2.16. *A semiring M is a regular if and only if $\lambda \circ \mu = \lambda \cap \mu$, for any fuzzy right ideal λ and fuzzy left ideal μ of M.*

Theorem 2.17. *Let M be a regular semiring. Then μ is a fuzzy left (right) bi-quasi ideal of M if and only if μ is a fuzzy quasi ideal of M.*

Theorem 2.18 ([24]). *Let M be a semiring. Then M is a regular semiring if and only if $B = MBM \cap BMB$ for every bi-interior ideal of M.*

Theorem 2.19 ([29]). *Let M be a regular semiring. Then μ is a fuzzy bi-interior ideal of M if and only if μ is a fuzzy quasi ideal of M.*

Definition 2.20. Let U be an initial universe set, E be the set of parameters and $P(U)$ denotes the power set of U. A pair (f, E) is called soft set over U where f is a mapping given by $f : E \to P(U)$.

Definition 2.21. For a soft set (f, A), the set $\{x \in A \mid f(x) \neq \emptyset\}$ is called Support of (f, A) denoted by $Supp(f, A)$. If $Supp(f, A) \neq \emptyset$ then (f, A) is called non null soft set.

Definition 2.22. Let U be an initial universe set, E be the set of parameters and $A \subseteq E$. A pair (f, A) is called fuzzy soft set over U where f is a mapping given by $f : A \to I^U$ where I^U denotes the collection of all fuzzy subsets of U.

Definition 2.23. Let $(f, A), (g, B)$ be fuzzy soft sets over U. Then (f, A) is said to be a fuzzy soft subset of (g, B) denoted by $(f, A) \subseteq (g, B)$ if $A \subseteq B$ and $f(a) \subseteq g(a)$ for all $a \in A$.

Definition 2.24. Let $(f, A), (g, B)$ be fuzzy soft sets. The intersection of fuzzy soft sets (f, A) and (g, B) is denoted by $(f, A) \cap (g, B) = (h, C)$ where $C = A \cup B$ is defined as

$$h_c = \begin{cases} f_c, & \text{if } c \in A \setminus B; \\ g_c, & \text{if } c \in B \setminus A; \\ f_c \cap g_c, & \text{if } c \in A \cap B. \end{cases}$$

Definition 2.25. Let $(f, A), (g, B)$ be fuzzy soft sets over U. Then (f, A) and (g, B) is denoted by $(f, A) \wedge (g, B)$ is defined by $(f, A) \wedge (g, B) = (h, C)$ where $C = A \times B$ $h_c(x) = min\ \{f_a(x), g_b(x)\} = (f_a \cap g_b)(x)$ for all $c = (a, b) \in A \times B$ and $x \in U$.

Definition 2.26. Let M be a semiring, E be a parameter set and $A \subseteq E$. Let f be a mapping given by $f : A \to P(M)$ where $P(M)$ is the power set of M. Then (f, A) is called a soft semiring over M if and only if for each $a \in A$, $f(a)$ is subsemiring of M. i.e. (i) $x, y \in M \Rightarrow x + y \in f(a)$ (ii) $x, y \in M \Rightarrow xy \in f(a)$.

Definition 2.27. Let M be a semiring, E be a parameter set and $A \subseteq E$. Let f be a mapping given by $f : A \to [0, 1]^M$ where $[0, 1]^M$ denotes the collection of all fuzzy subsets of M. Then (f, A) is called a fuzzy soft semiring over M if and only if for each $a \in A$, $f(a) = f_a$ is the fuzzy subsemiring of M. i.e.,(i) $f_a(x + y) \geq min\{f_a(x), f_a(y)\}$ (ii) $f_a(xy) \geq min\{f_a(x), f_a(y)\}$ for all $x, y \in M$.

Definition 2.28. Let M be a semiring, E be a parameter set and $A \subseteq E$. Let f be a mapping given by $f : A \to P(M)$. Then (f, A) is called a soft left (right) ideal over M if and only if for each $a \in A$, $f(a)$ is a left (right) ideal of M. i.e., (i) $x, y \in f(a) \Rightarrow x + y \in f(a)$ (ii) $x \in f(a), r \in M \Rightarrow rx(xr) \in f(a)$ for all $x, y \in M$.

Definition 2.29. Let M be a semiring, E be a parameter set, $A \subseteq E$ and $f : A \to P(M)$. Then (f, A) is called a soft ideal over M if and only if for each $a \in A$, $f(a)$ is an ideal of M. i.e., (i) $x, y \in f(a) \Rightarrow x + y \in f(a)$ (ii) $x \in f(a), r \in M \Rightarrow rx \in f(a)$ and $xr \in f(a)$.

Definition 2.30. Let M be a semiring, E be a parameter set and $A \subseteq E$. Let f be a mapping given by $f : A \to [0,1]^M$ where $[0,1]^M$ denotes the collection of all fuzzy subsets of M. Then (f, A) is called a fuzzy soft left (right) ideal over M if and only if for each $a \in A$, the corresponding fuzzy subset $f_a : M \to [0,1]$ is a fuzzy left (right) ideal of M. i.e., (i) $f_a(x + y) \geq \min\{f_a(x), f_a(y)\}$ (ii) $f_a(xy) \geq f_a(y)(f_a(x))$ for all $x, y \in M$.

Definition 2.31. Let M be a semiring, E be a parameter set and $A \subseteq E$. Let f be a mapping given by $f : A \to [0,1]^M$ where $[0,1]^M$ denotes the collection of all fuzzy subsets of M. Then (f, A) is called a fuzzy soft ideal over M if and only if for each $a \in A$, the corresponding fuzzy subset $f_a : M \to [0,1]$ is a fuzzy ideal of M. i.e., (i) $f_a(x + y) \geq \min\{f_a(x), f_a(y)\}$ (ii) $f_a(xy) \geq \max\{f_a(x), f_a(y)\}$ for all $x, y \in M$.

Definition 2.32. Let M be a semiring, E be a parameter set and $A \subseteq E$. Let f be a mapping given by $f : A \to [0,1]^M$ where $[0,1]^M$ denotes the collection of all fuzzy subsets of M. Then (f, A) is called a fuzzy soft bi- ideal over M if and only if for each $a \in A$, the corresponding fuzzy subset $f_a : M \to [0,1]$ is a fuzzy bi- ideal of M. i.e., (i) $f_a(x + y) \geq \min\{f_a(x), f_a(y)\}$ (ii) $f_a(xy) \geq \max\{f_a(x), f_a(y)\}$ for all $x, y \in M$.

Definition 2.33. Let M be a semiring, E be a parameter set and $A \subseteq E$. Let f be a mapping given by $f : A \to [0,1]^M$ where $[0,1]^M$ denotes the collection of all fuzzy subsets of M. Then (f, A) is called a fuzzy soft interior ideal over M if and only if for each $a \in A$, the corresponding fuzzy subset $f_a : M \to [0,1]$ is a fuzzy interior ideal of M. i.e., (i) $f_a(x + y) \geq \min\{f_a(x), f_a(y)\}$ (ii) $f_a(xy) \geq \max\{f_a(y)\}$ for all $x, y \in M$.

Definition 2.34. Let M be a semiring, E be a parameter set and $A \subseteq E$. Let f be a mapping given by $f : A \to [0,1]^M$ where $[0,1]^M$ denotes the collection of all fuzzy subsets of M. Then (f, A) is called a fuzzy soft quasi ideal over M if and only if for each $a \in A$, the corresponding fuzzy subset $f_a : M \to [0,1]$ is a fuzzy quasi ideal of M. i.e., A fuzzy subset f_a of semiring M is called a fuzzy quasi ideal if

(i) $f_a(x + y) \geq \min\{f_a(x), f_a(y)\}$ (ii)$f_a \circ \chi_S \wedge \chi_S \circ f_a \subseteq f_a$.

Definition 2.35. Let $(f, A), (g, B)$ be fuzzy soft ideals over a semiring M. The product (f, A) and (g, B) is defined as $((f \circ g), C)$ where $C = A \cup B$ and

$$(f \circ g)_c(x) = \begin{cases} f_c(x), & \text{if } c \in A \setminus B; \\ g_c(x), & \text{if } c \in B \setminus A; \\ \sup\limits_{x=ab}\{\min\{f_c(a), g_c(b)\}\}, & \text{if } c \in A \cap B \end{cases}$$

for all $c \in A \cup B$ and $x \in M$.

3. Main Results

In this section, the concept of fuzzy soft bi-interior ideal over semiring is introduced and study their properties.

Definition 3.1. Let M be semiring, E be a parameter set and $A \subseteq E$. Let μ be a mapping given by $\mu : A \to [0,1]^M$, where $[0,1]^M$ denotes the collection of all fuzzy subsets of M. Then (μ, A) is called a fuzzy soft bi-interior ideal over M if and only if for each $a \in A$, the corresponding fuzzy subset

(i) $\mu_a(x + y) \geq \min\{\mu_a(x), \mu_a(y)\}$ (ii) $\chi_M \circ \mu_a \circ \chi_M \cap \mu_a \circ \chi_M \circ \mu_a \subseteq \mu_a$ for all $x, y \in M$.

Theorem 3.2. *Let M be a semiring, E be a parameter set and $A \subseteq E$. If (μ, A) is a fuzzy soft left ideal over M then (μ, A) is a fuzzy soft bi-interior ideal over M.*

Proof. Suppose (μ, A) is a fuzzy soft left ideal over M. Then for each $a \in A$, μ_a is a fuzzy right ideal of M. Let $x \in M$ and μ_a be a fuzzy left ideal of the semiring M and $x \in M$. Then

$$
\begin{aligned}
\chi_M \circ \mu_a(x) &= \sup_{x=cb}\{\min\{\chi_M(c), \mu_a(b)\}\} \\
&= \sup_{x=cb}\{\min\{1, \mu_a(b)\}\} \\
&= \sup_{x=cb}\{\mu_a(b)\} \\
&\leq \sup_{x=cb}\{\mu_a(cb)\} \\
&= \sup_{x=cb}\{\mu_a(x)\} \\
&= \mu(x) \\
\Rightarrow\ \chi_M \circ \mu_a(x) &\leq \mu(x). \\
\mu \circ \chi_M \circ \mu_a(x) &= \sup_{x=uvs}\{\min\{\mu_a(u), \chi_M \circ \mu_a(vs)\}\} \\
&\leq \sup_{x=uvs}\{\min\{\mu_a(u), \mu_a(vs)\}\} \\
&= \mu(x). \\
\text{Now } \chi_M \circ \mu_a \circ \chi_M \cap \mu_a \circ \chi_M \circ \mu_a(x) &= \min\{\chi_M \circ \mu_a \circ \chi_M(x), \mu_a \circ \chi_M \circ \mu_a(x)\} \\
&\leq \min\{\chi_M \circ \mu_a \circ \chi_M(x), \mu_a(x)\} \\
&\leq \mu_a(x).
\end{aligned}
$$

Therefore $\chi_M \circ \mu_a \circ \chi_M \cap \mu_a \circ \chi_M \circ \mu_a(x) \subseteq \mu_a$.
Therefore μ_a is a fuzzy soft bi-interior ideal over M. Hence(μ, A) is a fuzzy soft bi-interior ideal over M. □

Corollary 3.3. *Let M be a semiring, E be a parameter set and $A \subseteq E$. If (μ, A) is a fuzzy soft right ideal over M then (μ, A) is a fuzzy soft bi-interior ideal over M.*

Corollary 3.4. *Let M be a semiring, E be a parameter set and $A \subseteq E$. If (μ, A) is a fuzzy soft ideal over M then (μ, A) is a fuzzy soft bi-interior ideal over M.*

Theorem 3.5. *Let M be a semiring, μ be a non-empty fuzzy subset of M. A fuzzy subset μ is a fuzzy bi-interior ideal of a semiring M if and only if the level subset μ_t of μ is a bi-interior ideal of a semiring M for every $t \in [0, 1]$, where $\mu_t \neq \phi$.*

Proof. Let M be a semiring and μ be a non-empty fuzzy subset of M. Suppose μ is a fuzzy bi-interior ideal of the semiring M, $\mu_t \neq \phi, t \in [0, 1]$ and $a, b \in \mu_t$. Then

$$
\begin{aligned}
&\mu(a) \geq t, \mu(b) \geq t \\
\Rightarrow &\mu(a + b) \geq \min\{\mu(a), \mu(b)\} \geq t \\
\Rightarrow &a + b \in \mu_t.
\end{aligned}
$$

Let $x \in M\mu_t\, M \cap \mu_t\, M\mu_t$. Then $x = bau = cde$, where $b, u, d \in M, a, c, e \in \mu_t$. Then
$\chi_M \circ \mu \circ \chi_M(x) \geq t$ and $\mu \circ \chi_M \circ \mu(x) \geq t \Rightarrow \mu(x) \geq t$. Therefore $x \in \mu_t$.
Hence μ_t is a bi-interior ideal of M.
Conversely suppose that μ_t is a bi-interior ideal of the semiring M, for all $t \in Im(\mu)$.
Let $x, y \in M, \mu(x) = t_1, \mu(y) = t_2$ and $t_1 \geq t_2$. Then $x, y \in \mu_{t_2}$

$$
\begin{aligned}
&\Rightarrow x + y \in \mu_{t_2} \text{ and } xy \in \mu_{t_2} \\
&\Rightarrow \mu(x + y) \geq t_2 = \min\{t_1, t_2\} = \min\{\mu(x), \mu(y)\}. \\
\text{Therefore } &\mu(x + y) \geq t_2 = \min\{\mu(x), \mu(y)\}.
\end{aligned}
$$

We have $M\, \mu_t\, M \cap \mu_t\, M\mu_t \subseteq \mu_t$, for all $t \in Im(\mu)$. Suppose $t = \min\{Im(\mu)\}$. Then $M\, \mu_t\, M \cap \mu_t\, M\mu_t \subseteq \mu_t$.
Therefore $\chi_M \circ \mu \circ \chi_M \cap \mu \circ \chi_M \circ \mu \subseteq \mu$. Hence μ is a fuzzy bi-interior ideal of the semiring M. □

Theorem 3.6. *Let M be a semiring, E be a parameter set and $A \subseteq E$. Then (I, A) is a soft bi-interior ideal over M if and only if (χ_I, A) is a fuzzy soft bi-interior ideal over M.*

Proof. Suppose (I, A) is a soft bi-interior ideal over M. Then for each $a \in A$, I_a is a bi-interior ideal of the semiring M. Obviously χ_I is a fuzzy subsemiring of M. We have $MI_aM \cap I_aMI_a \subseteq I_a$. Then

$$\begin{aligned} &\chi_M \circ \chi_I \circ \chi_M \cap \chi_I \circ \chi_M \circ \chi_I \subseteq \chi_I \\ &= \chi_{M\ I_a\ M \cap I_a\ M\ I_a} \\ &\subseteq \chi_I. \end{aligned}$$

Therefore χ_I is a fuzzy bi-interior ideal of the semiring M.
Hence (χ_I, A) is a fuzzy soft bi-interior ideal of the semiring M.

Conversely suppose that (χ_I, A) is a fuzzy soft bi-interior ideal over the semiring M. $\{\chi_I\}_a$ is a fuzzy bi-interior ideal of M. Then I is a subsemiring of M. We have

$$\begin{aligned} &\chi_M \circ \chi_I \circ \chi_M \cap \chi_I \circ \chi_M \circ \chi_I \subseteq \chi_I \\ &\Rightarrow \chi_{M\ I_a\ M} \cap \chi_{I_a\ M\ I_a} \subseteq \{\chi_I\}_a \\ &\Rightarrow \chi_{M\ I_a\ M \cap I_a\ M\ I_a} \subseteq \chi_I. \\ &\Rightarrow M\ I_a\ M \cap I_a\ M\ I_a \subseteq I_a. \end{aligned}$$

Therefore I_a is a bi-interior ideal of the semiring M. Hence (I, A) is a soft bi-interior ideal over M. □

Theorem 3.7. *If μ and λ are fuzzy bi-interior ideals of a semiring M then $\mu \cap \lambda$ is a fuzzy bi-interior ideal of a semiring M.*

Proof. Let μ and λ be fuzzy bi-interior ideals of the semiring M and $x, y \in M$. Then

$$\begin{aligned} \mu \cap \lambda(x+y) &= \min\{\mu(x+y), \lambda(x+y)\} \\ &\geq \min\{\min\{\mu(x), \mu(y)\}, \min\{\lambda(x), \lambda(y)\}\} \\ &= \min\{\min\{\mu(x), \lambda(x)\}, \min\{\mu(y), \lambda(y)\}\} \\ &= \min\{\mu \cap \lambda(x), \mu \cap \lambda(y)\} \\ \chi_M \circ \mu \cap \lambda(x) &= \sup_{x=ab}\{\min\{\chi_M(a), \mu \cap \lambda(b)\}\} \\ &= \sup_{x=ab}\{\min\{\chi_M(a), \min\{\mu(b), \lambda(b)\}\} \\ &= \sup_{x=ab}\{\min\{\min\{\chi_M(a), \mu(b)\}, \min\{\chi_M(a), \lambda(b)\}\} \\ &= \min\{\sup_{x=ab}\{\min\{\chi_M(a), \mu(b)\}, \sup_{x=ab}\{\min\{\chi_M(a), \lambda(b)\}\} \\ &= \min\{\chi_M \circ \mu(x).\chi_M \circ \lambda(x)\} \\ &= \chi_M \circ \mu \cap \chi_M \circ \lambda(x). \end{aligned}$$

Therefore $\chi_M \circ \mu \cap \lambda = \chi_M \circ \mu \cap \chi_M \circ \lambda$.

$$\begin{aligned} \mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda(x) &= \sup_{x=abc}\{\min\{\mu \cap \lambda(a), \chi_M \circ \mu \cap \lambda(b\ c)\}\} \\ &= \sup_{x=abc}\{\min\{\mu \cap \lambda(a), \chi_M \circ \mu \cap \chi_M \circ \lambda(b\ c)\}\} \\ &= \sup_{x=abc}\{\min\{\min\{\mu(a), \lambda(a)\}, \min\{\chi_M \circ \mu(bc), \chi_M \circ \lambda(b\ c)\}\} \\ &= \sup_{x=abc}\{\min\{\min\{\mu(a), \chi_M \circ \mu(bc)\}, \min\{\lambda(a), \chi_M \circ \lambda(b\ c)\}\} \\ &= \min\{\mu \circ \chi_M \circ \mu(x), \lambda \circ \chi_M \circ \lambda(x)\} \\ &= \mu \circ \chi_M \circ \mu \cap \lambda \circ \chi_M \circ \lambda(x). \end{aligned}$$

Therefore $\mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda = \mu \circ \chi_M \circ \mu \cap \lambda \circ \chi_M \circ \lambda$.
Similarly $\chi_M \circ \mu \cap \lambda \circ \chi_M = \chi_M \circ \mu \circ \chi_M \cap \chi_M \circ \lambda \circ \chi_M$.

Hence $\chi_M \circ \mu \cap \lambda \circ \chi_M \cap \mu \cap \lambda \circ \chi_M \circ \mu \cap \lambda = (\chi_M \circ \mu \circ \chi_M) \cap (\mu \circ \chi_M \circ \mu) \cap (\chi_M \circ \lambda \circ \chi_M) \cap (\lambda \circ \chi_M \circ \lambda) \subseteq \mu \cap \lambda$.
Hence $\mu \cap \lambda$ is a fuzzy bi-interior ideal of M. □

Theorem 3.8. *Let (f, A) and (g, B) be fuzzy soft bi-interiors over semiring M. Then $(f, A) \cap (g, B)$ is a fuzzy soft bi-interior ideal over M.*

Proof. By definition 2.24, we have $(f, A) \cap (g, B) = (h, C)$, where $C = A \cup B$.
Case (i) : $h_c = f_c$ if $c \in A \setminus B$. Then h_c is a fuzzy bi- ideal of M since (f, A) is a fuzzy soft bi-interior ideal over M.
Case (ii) : If $c \in B \setminus A$ then $h_c = g_c$. Therefore h_c is a fuzzy bi-interior ideal of M since $(g, B)$is a fuzzy soft bi-interior ideal over M.
Case (iii) : If $c \in A \cap B$, and $x, y \in M$ then $h_c = f_c \cap g_c$.
Hence by Theorem 3.7, h_c is a fuzzy bi-interior ideal of M. Thus $(f, A) \cap (g, B)$ is a fuzzy soft bi- interior ideal over M. □

Theorem 3.9. *Let (f, A) and (g, B) be fuzzy soft bi-interior ideals over semiring M. Then $(f, A) \wedge (g, B)$ is a fuzzy soft bi-interior ideal over M.*

Proof. By Definition 2.25, $(f, A) \wedge (g, B) = (h, C)$, where $C = A \times B$.
Let $c = (a, b) \in C = A \times B$ and $x, y \in M$. Then

$$\begin{aligned}
h_c(x + y) &= f_a(x + y) \wedge g_b(x + y) \\
&= \min\{f_a(x + y), g_b(x + y)\} \\
&\geq \min\{\min\{f_a(x), f_a(y)\}, \min\{g_b(x), g_b(y)\}\} \\
&= \min\{\min\{f_a(x), g_b(x)\}, \min\{f_a(y), g_b(y)\}\} \\
&= \min\{f_a \wedge g_b(x), f_a \wedge g_b(y)\} \\
&= \min\{h_c(x), h_c(y)\}.
\end{aligned}$$

Hence by Theorem 3.7, h_c is a fuzzy bi-interior ideal of M. Hence h_c is a fuzzy soft bi-interior ideal over M.
Therefore $(h, A \times B)$ is a fuzzy soft bi-interior ideal over M. □

Similarly, we can prove this following theorems.

Theorem 3.10. *Let (f, A) and (g, B) be fuzzy soft bi-interiors over semiring M. Then the product (f, A) and (g, B) is a fuzzy soft bi-interior ideal over M.*

Definition 3.11. A fuzzy set μ of a semiring M is said to be normal fuzzy ideal if μ is a fuzzy ideal of M and $\mu(0) = 1$.

Definition 3.12. Let (f, A) be fuzzy soft ideal over a semiring M. Then (f, A) is said to be normal fuzzy soft semiring if f_a is normal fuzzy ideal of semiring over M, for all $a \in A$.

Theorem 3.13. *If (f, A) is a fuzzy soft left ideal over semiring M and for each $a \in A$, f_a^+ is defined by $f_a^+(x) = f_a(x) + 1 - f_a(0)$, for all $x \in M$ then (f^+, A) is a normal fuzzy soft bi-interior ideal over semiring M and (f, A) is subset of (f^+, A).*

Proof. Let (f, A) be a fuzzy soft left ideal over semiring M and for each $a \in A$, f_a^+ is defined by $f_a^+(x) = f_a(x) + 1 - f_a(0)$, for all $x \in M$ and $x, y \in M$ and $a \in A$. Then

$$\begin{aligned}
f_a^+(x + y) &= f_a(x + y) + 1 - f_a(0) \\
&\geq min\{f_a(x), f_a(y)\} + 1 - f_a(0) \\
&= min\{f_a(x) + 1 - f_a(0), f_a(y) + 1 - f_a(0)\} \\
&= min\{f_a^+(x), f_a^+(y)\} \\
f_a^+(xy) &= f_a(xy) + 1 - f_a(0) \\
&\geq max\{f_a(x), f_a(y)\} + 1 - f_a(0) \\
&= max\{f_a(x) + 1 - f_a(0), f_a(y) + 1 - f_a(0)\} \\
&= max\{f_a^+(x), f_a^+(y)\}.
\end{aligned}$$

Therefore f_a^+ is fuzzy left ideal of M. Hence (f^+, A) is a soft fuzzy left ideal over semiring M.
By Theorem 3.2, (f^+, A) is a soft fuzzy bi-interior ideal over M.

If $x = 0$ then $f_a^+(0) = 1$ and $f_a \subseteq f_a^+$. Hence (f^+, A) is a normal fuzzy soft bi-interior ideal over M and (f, A) is a subset of (f^+, A). □

Theorem 3.14. *Let M be a regular semiring, E be a parameter set and $A \subseteq E$. Then (I, A) is a fuzzy soft bi-interior ideal over M if and only if (f, A) is a fuzzy soft quasi ideal over M.*

Proof. Let M be a regular semiring. Suppose (μ, A) is a fuzzy soft bi-interior ideal over M. Then for each $a \in A$, μ_a is a fuzzy bi-interior ideal of M. Let $x \in M$. Then

$$\chi_M \circ \mu_a \circ \chi_M \cap \mu_M \circ \chi_M \circ \mu_a \subseteq \mu_a.$$

Suppose $\chi_M \circ \mu_a(x) > \mu_a(x)$.

Since M is a regular, there exist $y \in M$ such that $x = x\,y\,x$.

$$\text{Then } \mu_a \circ \chi_M \circ \mu_a(x) = \sup_{x=xyx} \{\min\{\mu_a(x), \chi_M \circ \mu_a(yx)\}\ ) \\ > \sup_{x=xyx} \{\min\{\mu_a(x), \mu_a(yx)\}\}, \\ = \mu(x),$$

which is a contradiction. Therefore $\mu_a \circ \chi_M \cap \chi_M \circ \mu_a \subseteq \mu_a$.
Therefore μ_a is a fuzzy quasi ideal over M. Hence(μ, A) is a fuzzy soft quasi ideal over M. By Theorem 2.19, converse is true. □

Acknowledgements

The authors express sincere thanks to the esteemed referee(s) for their careful readings, valuable suggestions and comments, which helped them to improve the presentation of the paper.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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