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The Generating Functions for Special Pringsheim Continued Fractions

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Abstract
In previous works, some relations between Pringsheim continued fractions and vertices of the paths of minimal length on the suborbital graphs $\mathbf{F}_{u,N}$ were investigated. Then, for special vertices, the relations between these vertices and Fibonacci numbers were examined. On the other hand, Koshy studied relation between recurrence relations of Fibonacci numbers, Pell numbers and generating functions. In this work, it is showed that every vertex on the path of minimal length of suborbital graph $\mathbf{F}_{u,N}$ has a Pringsheim continued fraction. Then, by Koshy's motivation, the generating function of the recurrence relation of these pringsheim continued fractions are examined.

Keywords: Fibonacci Sequence, Generating Functions, Pell Sequence, Continued Fractions, Suborbital Graphs
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1. Introduction

The idea of suborbital graphs is used by Jones, Singerman and Wicks [1] for finite permutation groups, described by Sims [2] for the congruence subgroup $\Gamma_0(N)$ of the modular group Γ. The $\mathbf{G}_{u,N}$ and $\mathbf{F}_{u,N}$ suborbital graphs are consisted at these works. In [3] and [8], the results are extended. The definition of minimal length path for $\mathbf{F}_{u,N}$ suborbital graphs is given by Deger in [4]. In [4], it has been showed that; there is a integer t, which provides $u^2+tu+1 \equiv 0\,(mod\,N)$ congruence equation for $\mathbf{F}_{u,N}$ suborbital graphs. By using integer t, the vertices on the minimal length path of $\mathbf{F}_{u,N}$ suborbital graphs are consisted with Pringsheim continued fractions, according to edge conditions of $\mathbf{F}_{u,N}$ suborbital graphs [3]. When $t=3$, the continued fractions give the even term index Fibonacci numbers [4].

On the other hand, the relations between Fibonacci sequences, Pell sequences and Generating Functions have been examined by Koshy in [5] and [6]. The author has used recurrence relations of Fibonacci and Pell sequences to write generating functions. In this paper, we examine the generating functions for vertices on the minimal length path of $\mathbf{F}_{u,N}$ suborbital graphs. Particularly, for $t=3$ and $t=6$, we find the relations between Fibonacci and Pell numbers with generating functions of $\mathbf{F}_{u,N}$ suborbital graphs. The interval of convergence of the series obtained by generating functions are examined.

1.1 Fibonacci, Lucas, Pell, Pell-Lucas Sequences

The numbers $1,1,2,3,5,8,\ldots$ are called Fibonacci Numbers and the sequence of these numbers is called Fibonacci sequence. The recurrence relation of the Fibonacci sequence is

$$F_n = F_{n-1}+F_{n-2}; n \geq 3$$

with $F_1 = F_2 = 1$; initial conditions for n^{th} Fibonacci number. When initial conditions are chanced, the same relation gives Lucas numbers which are $1,3,4,7,\ldots$. So, the recurrence relation for n^{th} Lucas number is

$$L_n = L_{n-1} + L_{n-2}; n \geq 3$$

with $L_1 = 1, L_2 = 3$ initial conditions.

The numbers $1,2,5,12,\ldots$ are called Pell numbers and the sequence of these numbers is called Pell Sequence. The recurrence relation of n^{th} Pell Number is

$$P_n = 2P_{n-1} + P_{n-2}; n \geq 3$$

with $P_1 = P_2 = 2$; initial conditions . When the initial conditions are chanced, the recurrence relation consists Pell-Lucas numbers which are $1,3,7,17,\ldots$. The recurrence relation for n^{th} Pell-Lucas number is

$$Q_n = 2Q_{n-1} + Q_{n-2}; n \geq 3$$

with $Q_1 = 1, Q_2 = 3$ initial conditions.

Theorem 1.1. *[5]*

$$F_{n-1}F_{n+1} - {F_n}^2 = (-1)^n, n \geq 1.$$

Theorem 1.2. *[5] For all $n \geq 1$, let F_n is the n^{th} Fibonacci number, L_n is the n^{th} Lucas number, P_n is the n^{th} Pell number and Q_n is the n^{th} Pell-Lucas number,then the following identities are written;*

i) $F_{2n} = F_n L_n$

ii) $L_n = F_{n-1} + F_{n+1}$

iii) $5F_n = L_{n-1} + L_{n+1}$

iv) ${L_n}^2 = 5{F_n}^2 + 4(-1)^n$

v) $L_{4n} = 5{F_{2n}}^2 + 2$

vi) $P_n + P_{n-1} = Q_n$

vii) $2Q_n + 3P_n = P_{n+2}$.

1.2 Continued Fractions and Recurrence Relations

For all $m \in \mathbb{Z}^+ \cup \{0\}$,

$$y_0 + \mathrm{K}_{m=1}^{\infty} = y_0 + \cfrac{x_1}{y_1 + \cfrac{x_2}{y_2 + \ddots}} \tag{1.1}$$

is called (infinite) continued fraction, where $x_m \in \mathbb{Z} - \{0\}$ and $y_m \in \mathbb{Z}$ [7]. The n^{th} approximant of the continued fraction is $f_n = y_0 + \mathrm{K}_{m=1}^{n}$. The Möbius transformation of this continued fraction is $T_n(z) = \frac{x_m}{y_m + z}$. If $|y_m| \geq 1 + |x_m|$, we call this transformation Pringsheim transformation and the continued fraction (1.1) is called Pringsheim continued fraction.

The n^{th} numerator X_n and the n^{th} denominator Y_n of a continued fraction $x_0 + \mathrm{K}(x_m/y_m)$ are defined by the recurrence relations

$$\begin{bmatrix} X_n \\ Y_n \end{bmatrix} := y_n \begin{bmatrix} X_{n-1} \\ Y_{n-1} \end{bmatrix} + a_n \begin{bmatrix} X_{n-2} \\ Y_{n-2} \end{bmatrix}, \tag{1.2}$$

where $n = 1,2,3,\ldots$ and initial conditions $X_{-1} := 1, Y_{-1} := 0, X_0 := x_0, Y_0 := 1$. The modified approximant $T_n(z_n)$ can be written as $T_n(z_n) = \frac{X_n + X_{n-1}z_n}{Y_n + Y_{n-1}z_n}$, where $n = 0,1,2,3,\ldots$ and so, for the n^{th} approximant f_n we have $f_n = T_n(0) = \frac{X_n}{Y_n}, f_{n-1} = T_n(\infty) = \frac{X_{n-1}}{Y_{n-1}}$.

1.3 Solving Recurrence Relations

In this section, especially we will introduce a method for solving a large and important class of recurrence relations as following;
Linear Homogeneous Recurrence Relations with Constant Coefficients
Let take

$$y_n = c_1 y_{n-1} + c_2 y_{n-2} + \dots + c_k y_{n-k}, \tag{1.3}$$

recurrence relation, where $c_1, c_2, \dots, c_k \in \mathbb{R}$ and $c_k \neq 0$. The equation (1.3) is called k^{th}-order linear homogeneous recurrence relation with constant coefficients (LHRRWCCs). Here the term linear means that every term on the right-hand side (RHS) of the equation (1.3) contains at most the first power of each predecessor y_i. A recurrence relation is homogeneous if every term on the RHS is a multiple of some y_i, namely the relation is satisfied by the sequence 0, that is, $y_n = 0$ for every n. All coefficients c_i are constants. Since y_n depends on its k immediate predecessors, the order of the recurrence relation is k . Accordingly, to solve a k^{th} order LHRRWCC, we will need k initial conditions, say, $y_0 = c_0, y_1 = c_1, \dots, y_{k-1} = c_k$.
Now, we will examine the second order LHRRWCCs

$$y_n = \alpha y_{n-1} + \beta y_{n-2}, \tag{1.4}$$

where α and β are nonzero solution of the form $c\phi^n$, then $c\phi^n = \alpha c\phi^{n-1} + \beta c\phi^{n-2}$. Since $c\phi^n \neq 0$, this yields $\phi^2 = \alpha\phi + \beta$, that is, $\phi^2 - \alpha\phi - \beta = 0$, so ϕ must be a solution of the characteristic equation

$$x^2 - \alpha x - \beta = 0 \tag{1.5}$$

of the recurrence relation (1.4). The roots of equation (1.4) are called the characteristic roots of recurrence relation (1.5).
The following theorem gives us how characteristic roots help solve LHRRWCCs.

Theorem 1.3. *[5] Let ϕ and δ be the different solutions of the equation $x^2 - \alpha x - \beta = 0$ where $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$. Then every solution of the LHRRWCCs $y_n = \alpha y_{n-1} + \beta y_{n-2}$, where $y_0 = c_0$ and $y_1 = c_1$, is of the form $y_n = A\phi^n + B\delta^n$ for some constants A and B.*

1.4 Generating Functions

Let take $\alpha_0, \alpha_1, \alpha_2, \dots$ sequence, where $\alpha_0, \alpha_1, ..$ are real numbers. Then, the function

$$h(x) = \alpha_0 + \alpha_1 x + a_2 x^2 + \dots + \alpha_n x^n + \dots \tag{1.6}$$

is called the generating function for the sequence α_n. When we want to write the generating function for finite sequence $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n$ where $\alpha_i = 0$ for $i > n$, that is;

$$h(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \dots + \alpha_n x^n. \tag{1.7}$$

In this paper, we will use generating functions as above;

$$\frac{1}{1-\alpha x} = 1 + \alpha x + \alpha^2 x^2 + \dots + \alpha^n x^n + \dots = \sum_{n=0}^{\infty} \alpha^n x^n, \tag{1.8}$$

where α_n is real number for each $n \in \mathbf{N}$.

Example 1.4. *Let use generating functions to solve the Fibonacci recurrence relation $F_n = F_{n-1} + F_{n-2}$, where $F_1 = 1 = F_2$. Here, from the two initial conditions and Fibonacci recurrence relation, we get $F_0 = 0$. Let*

$$h(x) = F_0 + F_1 x + F_2 x^2 + \dots + F_n x^n + \dots$$

be the generating function of the Fibonacci sequence. Because of the orders of F_{n-1} and F_{n-2} are 1 and 2 less than the order of F_n, respectively, find $xh(x)$ and $x^2 h(x)$:

$$xh(x) = F_1 x^2 + F_2 x^3 + F_3 x^4 + \dots + F_{n-1} x^n + \dots$$
$$x^2 h(x) = F_1 x^3 + F_2 x^4 + F_3 x^5 + \dots + F_{n-2} x^n + \dots$$
$$h(x) - xh(x) - x^2 h(x) = F_1 x + (F_2 - F_1) x^2 + (F_3 - F_2 - F_1) x^3 + \dots + (F_n - F_{n-1} - F_{n-2}) x^n + \dots = x$$

since $F_1 = F_2$ *and* $F_n = F_{n-1} + F_{n-2}$. *So;*

$$\begin{aligned}
&(1-x-x^2)h(x) = x \\
&h(x) = \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\left[\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}\right] \\
&\sqrt{5}h(x) = \frac{1}{1-\alpha x} - \frac{1}{1-\beta x} \\
&\qquad\quad = \sum_0^\infty \alpha^n x^n - \sum_0^\infty \beta^n x^n = \sum_0^\infty (\alpha^n - \beta^n)x^n \\
&h(x) = \sum_0^\infty \frac{\alpha^n - \beta^n}{\sqrt{5}} x^n.
\end{aligned}$$

Then, by the equality of generating functions, the Binet formula for F_n *is :*

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$.

Corollary 1.5. *[5] Let* F_n *is the* n^{th} *Fibonacci number;*

$$\frac{1-x}{1-3x+x^2} = \sum_{n=0}^\infty F_{2n+1}x^n. \tag{1.9}$$

Corollary 1.6. *[5] Let* F_n *is the* n^{th} *Fibonacci number;*

$$\frac{x}{1-3x+x^2} = \sum_{n=0}^\infty F_{2n}x^n. \tag{1.10}$$

Corollary 1.7. *[6] Let* P_n *is the* n^{th} *Pell number;*

$$\frac{2x^2}{1-6x^2+x^4} = \sum_{n=0}^\infty P_{2n}x^{2n}. \tag{1.11}$$

Corollary 1.8. *[6] Let* Q_n *is the* n^{th} *Pell-Lucas number;*

$$\frac{1-3x^2}{1-6x^2+x^4} = \sum_{n=0}^\infty Q_{2n}x^{2n}. \tag{1.12}$$

1.5 Suborbital Graphs

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0(modN) \right\}$$

is the subgroup of the well known modular group Γ. So, on $\hat{\mathbb{Q}} := \mathbb{Q} \cup \infty$ we have that $T(\infty) = v \approx w = S(\infty) \Longleftrightarrow T^{-1}S \in \Gamma_0(N)$. This is a Γ equivalence invariant on $\hat{\mathbb{Q}}$ transitively but imprimitively. By this action, we can form suborbital graphs for Γ on $\hat{\mathbb{Q}}$. By this relation, we can write $\frac{r}{s} \approx \frac{x}{y} \Longleftrightarrow ry - sx \equiv 0mod(N)$. For details, see [8],[9],[3].

These ideas firstly were used by Sims [2], and got important in books by Tsukuzu [10], Biggs and White [11] with applying to finite groups. For example $O(\alpha,\alpha) := \{(\gamma,\gamma)|\gamma \in \Omega\}$ is the diagonal of $\Omega \times \Omega$. For $\forall \alpha \in \hat{\mathbb{Q}}$, graph $G(\alpha,\alpha)$ contains only one loup. This graph is self paired and is called trivial suborbital graph. In this work, we study with non-trivial graphs. Because Γ modular group acts on $\hat{\mathbb{Q}}$ by transitively, for $v \in \mathbb{Q}$ every suborbit contains (∞, v). If $v := \frac{u}{N}$ is taken where $N > 1, (u,N) = 1$, this suborbit is showed with $O(u,N)$ and the suborbital graph $G(\infty, v)$ which correspond that suborbit is showed with $G_{u,N}$.

Theorem 1.9. *[1]* $\mathbf{G}_{u,N} = \mathbf{G}_{u',N'}$ *iff* $N = N'$ *and* $u \equiv u' \,(mod\, N)$.

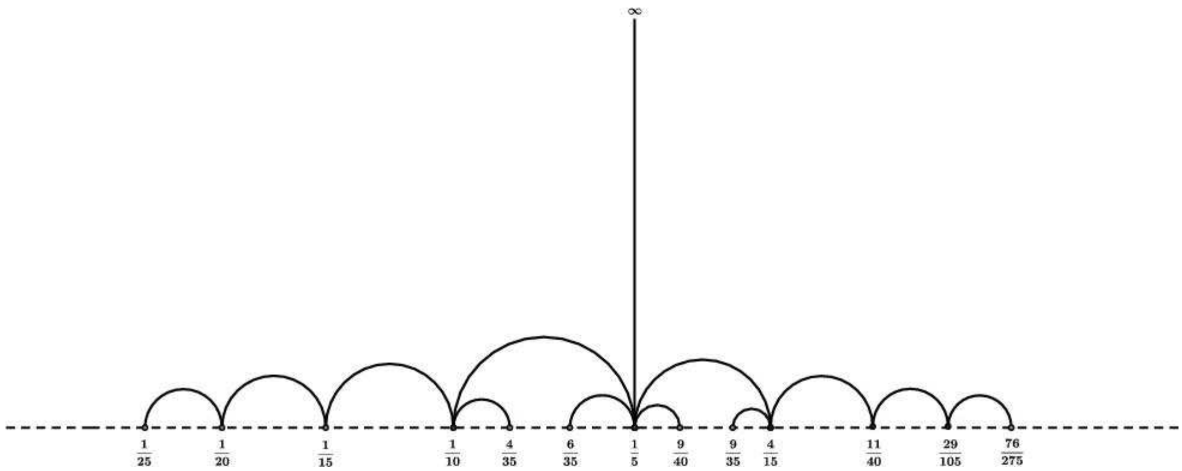

Figure 1.1. Some vertices in the suborbital graph $\mathbf{F}_{1,5}$

Theorem 1.10. *[1] On $G_{u,N}$ suborbital graph there is $\frac{r}{s} \rightarrow \frac{x}{y}$ edge iff $x \equiv (modN), y \equiv (modN)$ and $ry - sx = N$.* ■

Theorem 1.11. *[1] The suborbital graph paired with $\mathbf{G}_{u,N}$ is $\mathbf{G}_{-\bar{u},N}$ where $\bar{u}$ satisfies $u\bar{u} \equiv 1\,(mod\,N)$.*

Theorem 1.12. *[1] $\mathbf{G}_{u,N}$ is self-paired iff $u^2 \equiv -1\,(mod\,N)$.*

We let $\mathbf{F}_{u,N}$ be the subgraph of $\mathbf{G}_{u,N}$ whose vertices for the block $[\infty] = \{x/y \in \hat{\mathbb{Q}} | y \equiv 0 mod(N)\}$ containing ∞. Hence, $\mathbf{G}_{u,N}$ consist of $\Psi(N)$ disjoint copies of $\mathbf{F}_{u,N}$. The following theorem gives us the edge conditions for $\mathbf{F}_{u,N}$ suborbital graphs.

Theorem 1.13. *[1] $\frac{r}{s} \rightarrow \frac{x}{y} \in \mathbf{F}_{u,N}$ if and only if $x \equiv \mp ur\,(mod\,N)$, $ry - sx = \mp N$.*

Definition 1.14. *The main definitions used in our paper:*

(i) Let take different vertices of the graph $\mathbf{F}_{u,N}$ as a sequence $w_0, w_1, \ldots, w_m$. When $m \geq 2$, $w_0 \rightarrow w_1 \rightarrow \ldots \rightarrow w_m \rightarrow w_0$ is called a directed circuit *(*closed path*). If at least one arrow (not all) is overturned in this form, we will called it* undirected circuit*. When $m = 2$, whether the circuit is directed or not,it is called a* triangle*. When $m = 1$, the form $w_0 \rightarrow w_1 \rightarrow w_0$ is called a* self paired edge.

(ii) The configurations $w_0 \rightarrow w_1 \rightarrow \ldots \rightarrow w_m$ and $w_0 \rightarrow w_1 \rightarrow \ldots$ are called a path *and an* infinite path *in $\mathbf{F}_{u,N}$, respectively.*

(iii) When $\frac{r}{s} \xrightarrow{<} \frac{x}{y} \in \mathbf{F}_{u,N}$ $\left(or\, \frac{x}{y} \xleftarrow{<} \frac{r}{s} \in \mathbf{F}_{u,N}\right)$ according to edge conditions from Theorem (1.13), we call $\frac{x}{y}$ the farthest vertex *for $\frac{r}{s}$ vertex, which means that for $\frac{r}{s}$ vertex , there is no vertex which has greater (or smaller) value than $\frac{x}{y}$ in the suborbital graph $\mathbf{F}_{u,N}$.*

(iv) The path $w_0 \rightarrow w_1 \rightarrow \ldots \rightarrow w_m$ is called of minimal length *if and only if $w_i \nleftrightarrow w_j$, where $i < j-1$, $i \in \{0,1,2,3,...,m-2\}$, $j \in \{2,3,...,m\}$ and w_{i+1} must be the farthest vertex which can be joined with the vertex w_i in $\mathbf{F}_{u,N}$.*

(v) If $\mathbf{F}_{u,N}$ does not have any circuits, we call it a forest*. If $\mathbf{F}_{u,N}$ is a connected non-empty graph without circuits ,it is called a* tree.

2. Main Results

Theorem 2.1. *[4] If $(u,N) = 1$, then exist an integer t such that $u^2 + tu + 1 \equiv 0\,(mod\,N)$, for $t \geq 2$.*

On $\mathbf{F}_{u,N}$, $\varphi = \begin{pmatrix} -u & (u^2 - tu + 1)/N \\ -N & u - t \end{pmatrix} \in \Gamma_0(N)$ is a transformation which join the vertices to each other by respectively

$$\infty = \frac{1}{0} \longrightarrow \frac{u}{N} \longrightarrow \frac{u + \frac{1}{t}}{N} \longrightarrow \frac{u + \frac{1}{t - \frac{1}{t}}}{N} \longrightarrow \frac{u + \frac{1}{t - \frac{1}{t - \frac{1}{t}}}}{N} \longrightarrow \cdots$$

on the infinite minimal length path. So, this transformation forms the edges with a continued fractions construction for every edge. Thus, if $u+\frac{x}{y}$ is a vertex on the minimal length path of $\mathbf{F}_{u,N}$, the farthest vertex which can be joined with this vertex is $\varphi\left(\frac{u+\frac{x}{y}}{N}\right)=\frac{u+\frac{y}{ky-x}}{N}$. If the initial vertex is taken $w_0=\frac{u}{N}$, for any $q\in\mathbb{Z}^+$ $w_q=\varphi^q(w_0)$ equality is held. Also, if recurrence relations are used, then the n^{th} vertex which is on the minimal length path of $\mathbf{F}_{u,N}$ is given by ,$\frac{u+T_n(0)}{N}=\frac{u+\frac{X_n}{Y_n}}{N}=\frac{X_{n+1}u-X_n}{X_{n+1}N}$, where for each $n\geq 0, n\in\mathbf{N} x_n:=-1, y_n:=-t$ and $Y_n:=-X_{n+1}$. From matrix relations, we can write

$$\begin{pmatrix} X_{n-1} & X_n \\ -X_n & -X_{n+1} \end{pmatrix}=\begin{pmatrix} 0 & -1 \\ 1 & -t \end{pmatrix}^n.$$

Corollary 2.2. *In right direction, we can write the vertices of the paths of minimal length in $\mathbf{F}_{u,N}$ as above;*

$$\frac{1}{0}\longrightarrow\frac{u}{N}\longrightarrow\frac{u+\frac{1}{t}}{N}\longrightarrow\frac{u+\frac{1}{t-\frac{1}{t}}}{N}\longrightarrow\frac{u+\frac{1}{t-\frac{1}{t-\frac{1}{t}}}}{N}\longrightarrow\cdots.$$

Here, $w_0=\frac{u}{N}$, $w_1=\frac{u+\frac{1}{t}}{N}$, $w_2=\frac{u+\frac{1}{t-\frac{1}{t}}}{N},\cdots$. It is clear that for every vertex we can obtain continued fraction . That is $\cfrac{1}{t-\cfrac{1}{t-\cfrac{1}{t-\ddots}}}$. Now, if we write the Möbius transformation of this continued fraction, we get $T_m(z)=\frac{-1}{-t+z}$, for all m. According to Pringsheim continued fraction's definition; if we take $x_m=-1$ and $y_m=-t$, then $|y_m|\leq 1+|x_m|$ holds. So this continued fraction is Pringsheim continued fraction. Here, we can symbolize this continued fraction with a fraction; for n^{th} vertex $\frac{H_n}{H_{n+1}}$. Thus, we can write a recurrence relation

$$H_n=tH_{n-1}-H_{n-2} \tag{2.1}$$

with $H_1=1, H_2=t$, $n\geq 3$ initial conditions.
If we solve this recurrence relation according to Theorem (1.3), we can obtain above corollary;

Corollary 2.3. *For $n\geq 1$;*

$$H_n=\left(-\frac{1}{2}\right)^n\frac{1}{\sqrt{t^2-4}}\left[(t+\sqrt{t^2-4})^n-(t-\sqrt{t^2-4})^n\right].$$

Lemma 2.4. *From recurrence relation (2.1), the generating function of this relation is*

$$h(x)=\frac{1}{1-tx+x^2}.$$

So;

$$h(x)=\frac{1}{\sqrt{t^2-4}}\sum_{k=0}^{\infty}\left[\left(\frac{2}{t-\sqrt{t^2-4}}\right)^n-\left(\frac{2}{t+\sqrt{t^2-4}}\right)^n\right]x^n.$$

Proof. Let recurrence relation $H_n=tH_{n-1}-H_{n-2}$ where $H_1=1, H_2=t$, $n\geq 3$. So, we shall define the generating function of H_n as following;

$$h(x)=H_0+H_1x+H_2x^2+\ldots+H_nx^n+\cdots$$
$$txh(x)=tH_0x+tH_1x^2+\ldots+tH_{n-1}x^n+tH_nx^{n+1}+\cdots$$
$$x^2h(x)=H_0x^2+\ldots+H_{n-2}x^n+H_{n-1}x^{n+1}+H_nx^{n+2}+\cdots$$
$$h(x)-txh(x)+x^2h(x)=H_0+(H_1-tH_0)x+(H_2-tH_1+H_0)x^2+\cdots+(H_n-tH_{n-1}+H_{n-2})x^n+\cdots=1$$
$$h(x)=\frac{1}{1-tx+x^2}.$$

Now, if we rewrite $h(x) = \frac{1}{1-tx+x^2}$ as a sum of partial fractions, where $\Delta = t^2-4$. Then,

$$\begin{aligned} h(x) &= \frac{1}{\sqrt{t^2-4}} \frac{1}{x-\frac{t+\sqrt{t^2-4}}{2}} - \frac{1}{\sqrt{t^2-4}} \frac{1}{x-\frac{t-\sqrt{t^2-4}}{2}} \\ &= \frac{1}{\sqrt{t^2-4}} \left[\frac{-1}{1-\frac{2x}{t+\sqrt{t^2-4}}} + \frac{1}{1-\frac{2x}{t-\sqrt{t^2-4}}} \right] \end{aligned}$$

holds. From equation (1.8), we have the generating function as;

$$\begin{aligned} h(x) &= \frac{1}{\sqrt{t^2-4}} \left[\sum_{n=0}^{\infty} \left(\frac{2}{t-\sqrt{t^2-4}} \right)^n x^n - \sum_{n=0}^{\infty} \left(\frac{2}{t+\sqrt{t^2-4}} \right)^n x^n \right] \\ &= \frac{1}{\sqrt{t^2-4}} \sum_{n=0}^{\infty} \left[\left(\frac{2}{t-\sqrt{t^2-4}} \right)^n - \left(\frac{2}{t+\sqrt{t^2-4}} \right)^n \right] x^n. \end{aligned}$$

□

Theorem 2.5. *From the $h(x)$ generating function, for $t = 3$;*

$$h(x) = \frac{1}{1-3x+x^2} = \sum_{n=0}^{\infty} F_{2n+2} x^n. \tag{2.2}$$

Proof. When we take $t = 3$ in Lemma 2.4, we can write $h(x) = \frac{1}{1-3x+x^2}$. We rewrite $h(x) = \frac{1}{1-3x+x^2}$ as a sum of partial functions as above;

$$\frac{1}{1-3x+x^2} = \frac{x}{1-3x+x^2} + \frac{1-x}{1-3x+x^2}.$$

From the equation (1.9) and the equation (1.10),

$$\frac{1}{1-3x+x^2} = \sum_{n=0}^{\infty} F_{2n+1} x^n + \sum_{n=0}^{\infty} F_{2n} x^n$$

is written. If we use Fibonacci recurrence relation, we can obtain $g(x) = \frac{1}{1-3x+x^2} = \sum\limits_{n=0}^{\infty} F_{2n+2} x^n$. □

Theorem 2.6. *From the generating function $h(x)$, if $t = 6$, then we have*

$$h(x) = \frac{1}{1-6x+x^2} = \frac{1}{2} \sum_{n=0}^{\infty} P_{2n+2} x^n. \tag{2.3}$$

Proof. If $t = 6$, then from Lemma 2.4, we can write $h(x) = \frac{1}{1-6x+x^2}$. So, if we rewrite $h(x) = \frac{1}{1-6x+x^2}$ as a sum of partial functions as above;

$$\frac{1}{1-6x+x^2} = \frac{1}{2} \left(3 \frac{2x}{1-3x+x^2} + 2 \frac{1-3x}{1-3x+x^2} \right),$$

then from the equation (1.11) and the equation (1.12), we get

$$\frac{1}{1-6x+x^2} = \frac{1}{2} \left(\sum_{n=0}^{\infty} 3P_{2n} x^n + \sum_{n=0}^{\infty} 2Q_{2n} x^n \right).$$

If we use Pell-Lucas identity at Theorem (1.2)(vii), then we have $h(x) = \frac{1}{1-6x+x^2} = \frac{1}{2} \sum\limits_{n=0}^{\infty} P_{2n+2} x^n$. □

Corollary 2.7. *The interval of convergence of* $\sum_{n=0}^{\infty} F_{2n+2}x^n$ *is* $|x| < \frac{1}{\alpha^2}$, *where* $\alpha = \frac{1+\sqrt{5}}{2}$.

Proof. According to golden ratio test of power series and from the equation of $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha$;

$$\lim_{n\to\infty} \left| \frac{F_{2n+4}x^{n+1}}{F_{2n+2}x^n} \right| = |x| \frac{F_{2n+4}}{F_{2n+2}} = |x|\alpha^2$$

is written. For the series to be convergent, it is necessary that $|x| < \frac{1}{\alpha^2}$. □

Corollary 2.8. *The interval of convergence of* $\sum_{n=0}^{\infty} P_{2n+2}x^n$ *is* $|x| < \frac{1}{\gamma^2}$, *where* $\gamma = 1+\sqrt{2}$.

Proof. According to golden ratio test of power series and from the equation of $\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = \gamma$;

$$\lim_{n\to\infty} \left| \frac{P_{2n+4}x^{n+1}}{P_{2n+2}x^n} \right| = |x| \frac{P_{2n+4}}{P_{2n+2}} = |x|\gamma^2$$

is written. For the series to be convergent, it is necessary that $|x| < \frac{1}{\gamma^2}$. □

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