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ON RULED NON-DEGENERATE SURFACES WITH DARBOUX FRAME IN MINKOWSKI 3-SPACE

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Abstract. In this paper, ruled non-degenerate surfaces with respect to Darboux frame are studied. Characterization of them which are related to the geodesic torsion, the normal curvature and the geodesic curvature with respect to Darboux frame are examined. Furthermore, some special cases of non-null rulings are demonstrated according to Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ with Darboux frame $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$. Finally, the integral invariants of these surfaces are examined.

Keywords: Ruled surface, Darboux frame, Lorentz 3-space, integral invariants.

AMS Subject Classification: 53A35, 53A25.

1. Introduction

The curve and surface theories are popular topics in differential geometry so a ruled surface is the important subject of differential geometry. A ruled surface can always be easily parameterized. These surfaces can be described by moving a straight line along a chosen curve. Therefore, the equation of the ruled surface can be written as

$$X(s,v) = \alpha(s) + v\mathbf{e}(s), \|\mathbf{e}(s)\| = 1$$

where (α) is curve which is called the *base curve* of the ruled surface and the curve $\mathbf{e}$ is also called the *spherical indicatrix vector* of the ruled surface. The ruled surface have been studied for centuries by geometers. The geometry and theory of ruled surfaces are widely used in sciences for instance Computer-Aided Manufacturing (CAM), Computer-Aided Geometric Design (CAGD), architectural design and kinematics.

In the literature, many properties of ruled surfaces have been examined in Euclidean and non-Euclidean spaces. B. Ravani and T. S. Ku have generalized the theory of Bertrand offsets of curves for ruled surfaces with geodesic Frenet frame, [1]. A. Turgut and H.

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H. Hacısalihoğlu have investigated spacelike and timelike ruled surfaces and given some theorems related to the distribution parameter in Minkowski 3-space, [2, 3, 4]. Y. H. Kim and D. W. Yoon have classified ruled surfaces in terms of the second Gaussian curvature, the mean curvature and the Gaussian curvature in Minkowski 3-space, [5]. E. Kasap and N. Kuruoğlu have studied Bertrand offsets of ruled surfaces with geodesic Frenet frame in Minkowski 3-space, [6]. Y. H. Kim and D. W. Yoon have investigated non-developable ruled surfaces in Lorentz-Minkowski space, [7]. The involute-evolute offsets and Mannheim offsets of ruled surfaces with geodesic Frenet frame are studied in [8, 9]. N. Yüksel have defined the ruled surfaces according to Bishop Frame in Minkowski 3-Space, [10]. C. Ekici and H. Öztürk have given timelike ruled surfaces and they obtained some theorems related to the geodesic Frenet curvature and the second fundamental form, [11]. S. Kızıltuğ and A. Çakmak have studied developable ruled surfaces with Darboux frame in Minkowski 3-space, [12]. The ruled surfaces with Darboux frame (RSDF) are investigated and Bertrand offsets of RSDF are defined by G. Y. Şentürk and S. Yüce, [13, 14]. D. W. Yoon has classified evolute offsets of a ruled surface with constant Gaussian curvature and mean curvature and investigated linear Weingarten evolute offsets in Minkowski 3-space, [15].

In this study, we study on ruled non-degenerate surfaces with Darboux frame in $\mathbb{E}_1^3$. We take the relation between the Darboux frame $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$ and the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ of base curve to write characteristic properties and integral invariants of ruled non-degenerate surfaces which related to the geodesic torsion, the normal and the geodesic curvatures.

2. Preliminaries

2.1. **Differential geometry in $\mathbb{R}_1^3$.** Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be vectors in $\mathbb{R}^3$. The Lorentzian inner product of x and y is defined to be the real number

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3.$$

The vector space $\mathbb{R}^3$ with the Lorentzian inner product is called Minkowski (Lorentz) 3-space and is denoted by $\mathbb{E}_1^3$, [16, 17].

Let x be a vector in $\mathbb{R}^3$. The sign of $\langle x, x\rangle$ determines the type of x. In particular if $\langle x, x\rangle > 0$ or $x = 0$, then x is spacelike; and if $\langle x, x\rangle < 0$, then x is timelike; and if $\langle x, x\rangle = 0$, then x is null (lightlike). A timelike vector x said to be positive (resp. negative) if and only if $x_1 > 0$. The Lorentzian norm of x is defined $\|x\| = \sqrt{|\langle x, x\rangle|}$.

Two vectors x, y in $\mathbb{R}_1^3$ are Lorentzian orthogonal if and only if $\langle x, y\rangle = 0$.

Theorem 2.1. *Let x and y be nonzero Lorentzian orthogonal vectors in $\mathbb{R}_1^3$. If x is timelike, then y is spacelike,* [17]

$\{x, y, z\}$ in $\mathbb{R}_1^3$ are Lorentzian orthonormal if and only if $\|x\|^2 = -1$ and $\langle x, y\rangle = \langle x, z\rangle = \langle z, y\rangle = 0$ and $\|y\|^2 = \|z\|^2 = 1$, [17]. For any vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{R}_1^3$, the Lorentzian vector product of x and y is defined by, [16, 17]

$$x \times y = \begin{vmatrix} -e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_3y_2 - x_2y_3, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1),$$

where

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases} \quad e_i = (\delta_{i1}, \delta_{i2}, \delta_{i3}), e_1 \times e_2 = e_3, e_2 \times e_3 = -e_1, e_3 \times e_1 = e_2.$$

Theorem 2.2. *If $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ and $z = (z_1, z_2, z_3)$ in $\mathbb{R}^3_1$, then,*[17]
(1) $x \times y = -y \times x$
(2) $x \times (y \times z) = \langle x, y\rangle z - \langle z, x\rangle y$.

Theorem 2.3. *Let x, y in $\mathbb{R}^3_1$. We have,* [18]
(1) *If x and y are spacelike vectors, $x \times y$ is a timelike vector.*
(2) *If x and y are timelike vectors, $x \times y$ is a spacelike vector.*
(3) *If x is a spacelike vector and y is a timelike vector, $x \times y$ is a spacelike vector.*

Let $\alpha(s) : I \subset \mathbb{R} \to \mathbb{R}^3_1$, be a smooth regular curve in $\mathbb{R}^3_1$. For any $s \in I$, the curve is said to be a spacelike, timelike or null if the velocity vector $\alpha'(s)$ is a spacelike, timelike or null vector, respectively. Denote by $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ the moving Frenet frame along the curve $\alpha(s)$ in Lorentz 3-space. For an arbitrary spacelike curve $\alpha(s)$, then the following Frenet formulae are given,

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\varepsilon\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix},$$

where $\langle \mathbf{T}, \mathbf{T}\rangle = 1, \langle \mathbf{N}, \mathbf{N}\rangle = \varepsilon = \pm, \langle \mathbf{B}, \mathbf{B}\rangle = -\varepsilon, \langle \mathbf{T}, \mathbf{N}\rangle = \langle \mathbf{T}, \mathbf{B}\rangle = \langle \mathbf{N}, \mathbf{B}\rangle = 0$ and κ and τ are curvature and torsion of the spacelike curve, respectively. The Darboux vector of this motion is $\mathbf{D} = \tau\mathbf{T} - \kappa\mathbf{B}$, where $\mathbf{T} \times \mathbf{N} = \mathbf{B}, \mathbf{N} \times \mathbf{B} = -\varepsilon\mathbf{T}$ and $\mathbf{T} \times \mathbf{B} = \mathbf{N}$, [19].

Furthermore, for any timelike curve $\alpha(s)$, then the following Frenet formulae are given,

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix},$$

where $\langle \mathbf{T}, \mathbf{T}\rangle = -1, \langle \mathbf{N}, \mathbf{N}\rangle = \langle \mathbf{B}, \mathbf{B}\rangle = 1, \langle \mathbf{T}, \mathbf{N}\rangle = \langle \mathbf{T}, \mathbf{B}\rangle = \langle \mathbf{N}, \mathbf{B}\rangle = 0$ and κ and τ are curvature and torsion of the spacelike curve, respectively. The Darboux vector of this motion is $\mathbf{D} = \tau\mathbf{T} + \kappa\mathbf{B}$, where $\mathbf{T} \times \mathbf{N} = \mathbf{B}, \mathbf{N} \times \mathbf{B} = -\mathbf{T}$ and $\mathbf{T} \times \mathbf{B} = -\mathbf{N}$, [19].

Definition 2.1. ***i) The timelike angle between timelike vectors:*** *Let x and y be positive (negative) timelike vectors in $\mathbb{R}^3_1$. Then there is a unique real number $\theta \geq 0$ such that $\langle x, y\rangle = |x| |y| \cosh\theta$.*
ii) The timelike angle between spacelike vectors: *Let x and y be spacelike vectors in $\mathbb{R}^3_1$ that span a timelike vector subspace. Then there is a unique real number $\theta \geq 0$ such that $\langle x, y\rangle = |x| |y| \cosh\theta$.*
iii) The spacelike angle between spacelike vectors: *Let x and y be spacelike vectors in $\mathbb{R}^3_1$ that span a spacelike vector subspace. Then there is a unique real number $\theta \geq 0$ such that $\langle x, y\rangle = |x| |y| \cos\theta$.*
iv) The angle between spacelike and timelike vectors: *Let x be a spacelike vector and y be a timelike vector in $\mathbb{R}^3_1$. Then there is a unique real number $\theta \geq 0$ such that $\langle x, y\rangle = |x| |y| \sinh\theta$,* [17].

A surface in Lorentz 3-space is called a timelike or spacelike if the normal on surface is a spacelike or timelike vector, respectively, [16, 17]. Let M be an oriented surface in Lorentz 3-space and $\alpha(s)$ be a non-null curve lying on M. Since $\alpha(s)$ is also a space curve, there exists the moving Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ along the curve. $\mathbf{T}$ is a unit tangent vector, $\mathbf{N}$ is a principal normal vector and $\mathbf{B}$ is a binormal vector.Due to the curve $\alpha(s)$ that lies on the surface there exists the Darboux Frame and it is denoted by $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$. In Darboux Frame $\mathbf{T}$ is the unit tangent vector of the curve like the Frenet Frame. $\mathbf{n}$ is the unit normal vector of the surface and $\mathbf{g}$ is the unit vector which is defined by $\mathbf{g} = \mathbf{T} \times \mathbf{n}$. Due to the unit tangent vector **T** is common Frenet Frame and Darboux Frame, the vectors **N** , **B**, **g**, **n** lie on the same plane. Then, if the surface M is an oriented timelike surface, the relations between these frames can be given as follows

i) If the curve $\alpha(s)$ is timelike,

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{g} \\ \mathbf{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}.$$

ii) If the curve $\alpha(s)$ is spacelike,

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{g} \\ \mathbf{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\varphi & \sinh\varphi \\ 0 & \sinh\varphi & \cosh\varphi \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}.$$

If the surface M is an oriented spacelike surface, then the curve $\alpha(s)$ is spacelike. So, the relations between these frames can be given as follows

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{g} \\ \mathbf{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\varphi & \sinh\varphi \\ 0 & \sinh\varphi & \cosh\varphi \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}.$$

In all cases, φ is the angle between the vectors **g** and **N**. The derivative formulae of the Darboux frame can be changed as follows:

i) If the surface M is a timelike surface, then the curve $\alpha(s)$ can be a spacelike or a timelike curve. Thus, the derivative formulae of the Darboux frame of $\alpha(s)$ is given by

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{g}' \\ \mathbf{n}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & -\eta\kappa_n \\ \kappa_g & 0 & \eta\tau_g \\ \kappa_n & \tau_g & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{g} \\ \mathbf{n} \end{bmatrix}, \langle \mathbf{T}, \mathbf{T} \rangle = \eta = \pm, \langle \mathbf{g}, \mathbf{g} \rangle = -\eta, \langle \mathbf{n}, \mathbf{n} \rangle = 1.$$

ii) If the surface M is a spacelike surface, then the curve $\alpha(s)$ can be a spacelike curve. Thus, the derivative formulae of the Darboux frame of $\alpha(s)$ is given by

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{g}' \\ \mathbf{n}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ \kappa_n & \tau_g & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{g} \\ \mathbf{n} \end{bmatrix}, \langle \mathbf{T}, \mathbf{T} \rangle = \langle \mathbf{g}, \mathbf{g} \rangle = 1, \langle \mathbf{n}, \mathbf{n} \rangle = -1.$$

In these formulae κ_g is the geodesic curvature, κ_n is the normal curvature and τ_g is the geodesic torsion of $\alpha(s)$, [19, 20, 21]. In this article, we prefer using -prime- to denote the derivative with respect to the arc length parameter of a curve.

In addition, the geodesic curvature κ_g and geodesic torsion τ_g of the curve $\alpha(s)$ can be calculated as follows:

$$\kappa_g = < \frac{d\alpha}{ds}, \frac{d^2\alpha}{ds^2} \times \mathbf{n} >, \tau_g = < \frac{d\alpha}{ds}, \mathbf{n} \times \frac{d\mathbf{n}}{ds} > .$$

Corollary 2.1. *There are not closed curves in $\mathbb{R}^3_1$ that are timelike or lightlike,* [22].

2.2. **Ruled non-degenerate Surfaces.** A ruled surface is obtained by a straight line moving along a curve. A ruled surface in $\mathbb{R}^3_1$ is given by the parametrization

$$X : I \times \mathbb{R} \to \mathbb{R}^3_1, \; X(s,v) = \alpha(s) + v\mathbf{e}(s), \tag{1}$$

where the curve $\alpha : I \to \mathbb{R}^3_1$ is called the base curve and $\mathbf{e} : \mathbb{R} \to \mathbb{R}^3_1$ is called the ruling. If the normal vector on ruled surface $\mathbf{n} = \frac{X_u \times X_v}{\|X_u \times X_v\|}$ is a spacelike or timelike vector, the surface is called timelike or spacelike, respectively.

Let M be a ruled surface with the eq. 1. An orthonormal basis of $\chi(M)$, $\{\mathbf{T}, \mathbf{e}\}$ can be chosen, where **T** is the unit tangent vector of α. Thus $\mathbf{n} = \mathbf{T} \times \mathbf{e}$ is a normal vector of M. In this case, spacelike ruled surface in $\mathbb{R}_1^3$ is obtained by a spacelike straight line moving along a spacelike curve, [2]. Similarly, a timelike ruled surface in $\mathbb{R}_1^3$ is obtained by a spacelike straight line moving along a timelike curve or by a timelike straight line moving along a spacelike curve, [3].

The striction point on ruled surface is the foot of the common perpendicular line of the successive rulings on the main ruling.The set of striction points of the ruled surface generates its striction curve. The striction curve of any spacelike and timelike ruled surface is given by , [2, 3]

$$c(s) = \alpha(s) - \frac{< \alpha_s, \mathbf{e}_s >}{< \mathbf{e}_s, \mathbf{e}_s >}\mathbf{e}(s).$$

The distribution parameter of any spacelike and timelike ruled surface is given by, [2, 3]

$$P_e = \frac{\langle \alpha_s \times \mathbf{e}, \mathbf{e}_s \rangle}{< \mathbf{e}_s, \mathbf{e}_s >} = -\frac{det(\alpha_s, \mathbf{e}, \mathbf{e}_s)}{< \mathbf{e}_s, \mathbf{e}_s >}.$$

Theorem 2.4. *The ruled surface is developable if and only if the distribution parameter P_e is zero,* [1].

A curve which intersects perpendicularly each one of rulings is called an orthogonal trajectory of the ruled surface. It is calculated by

$$< \mathbf{e}, d\varphi >= 0.$$

Let $\alpha : I \to \mathbb{E}_1^3$ be a differentiable closed curve and $H = sp\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be a moving space along the curve α, where $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is Frenet frame of α and $H' = \mathbb{E}_1^3$ be a fixed space. In this case, a closed space motion can be defined on H' of H along the curve α. We denote by H/H' the closed space motion. The pitch of closed ruled surface is defined by

$$L_e = -\oint_\alpha < \mathbf{e}, d\alpha >= \oint_\alpha dv =< \mathbf{V}, \mathbf{e} >,$$

where $\mathbf{V} = \oint_\alpha d\alpha$ is Steiner translation vector of the motion. The angle of pitch of closed ruled surface is defined by

$$\lambda_e =< \mathbf{D}, \mathbf{e} >,$$

where $\mathbf{D} = \lambda_T \mathbf{T} - \varepsilon\lambda_B \mathbf{B}$ is Steiner rotation vector of the motion.

3. On ruled non-degenerate surfaces with Darboux Frame in Minkowski 3-space

3.1. **Spacelike ruled surfaces.** Let M be a ruled surface which obtain by a spacelike straight line moving along a spacelike curve in $\mathbb{R}_1^3$. In the study [2], $\mathbf{T}$ and $\mathbf{e}$ are orthogonal spacelike vectors. Because of this special choice, $\mathbf{n}$ is obtained as a timelike, so M is a spacelike ruled surface. Unlike this study, we take that $\mathbf{T}$ and $\mathbf{e}$ are not orthogonal vectors. In this case, the normal of the surface $\mathbf{n}$ can be spacelike or timelike. We assumed that the normal vector $\mathbf{n}$ is a timelike. We take the relation between the Darboux frame $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$ and the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ of base curve. As the tangent vector of base curve appears in both frames, then that relation is indeed, a relation between the last two vectors of both frames and a certain angles between the vectors. Then, we can write $\mathbf{e}$ in linear combination of both frames.

A unit direction vector of a spacelike ruling $\mathbf{e}$ is spanned by the system $\{\mathbf{T}, \mathbf{g}\}$. So $\mathbf{e}$ can be written as:

$$\mathbf{e} = \mathbf{T}\cos\phi + \mathbf{g}\sin\phi$$

where ϕ is the angle between the spacelike vectors $\mathbf{T}$ and $\mathbf{e}$.

$\mathbf{e}_s$ is also provided the equation,

$$\mathbf{e}_s = -\mathbf{T}(\phi' + \kappa_g)\sin\phi + \mathbf{g}(\phi' + \kappa_g)\cos\phi + \mathbf{n}(\kappa_n \cos\phi + \tau_g \sin\phi).$$

Holding v constant, we obtain a curve $\beta(s) = \alpha(s) + v\mathbf{e}(s)$ on a ruled surface whose vector field is

$$\mathbf{T}^* = \mathbf{T}(1 - v(\phi' + \kappa_g)\sin\phi) + \mathbf{g}(\phi' + \kappa_g)v\cos\phi + \mathbf{n}(\kappa_n \cos\phi + \tau_g \sin\phi)v.$$

The relation between the vectors $\mathbf{e}$ and $\mathbf{T}^*$ is

$$< \mathbf{T}^*, \mathbf{e} >= \cos\phi.$$

The distribution parameter of the spacelike RSDF is

$$P_e = -\frac{sin\phi(\kappa_n \cos\phi + \tau_g \sin\phi)}{(\phi' + \kappa_g)^2 - (\kappa_n \cos\phi + \tau_g \sin\phi)^2}.$$

The orthogonal trajectories of the spacelike RSDF is

$$\cos\phi ds = -dv.$$

The striction curve of the spacelike RSDF is

$$c(s) = \alpha(s) + \frac{sin\phi(\phi' + \kappa_g)}{(\phi' + \kappa_g)^2 - (\kappa_n \cos\phi + \tau_g \sin\phi)^2}\mathbf{e}(s).$$

Theorem 3.1. *Let M be a spacelike RSDF, which is given by $X(s,v) = \alpha(s) + v\mathbf{e}(s)$. In this case, the shortest distance between the spacelike rulings of the surface along the orthogonal trajectories is*

$$v = \frac{sin\phi(\phi' + \kappa_g)}{(\phi' + \kappa_g)^2 - (\kappa_n \cos\phi + \tau_g \sin\phi)^2}$$

along the curve $X_v : I \to M$.

Proof. Supposing that the two spacelike rulings pass through the points α_{s_1} and α_{s_2} where $s_1 < s_2$, the distance between these rulings along an orthogonal trajectory is given:

$$J(v) = \int_{s_1}^{s_2} \|T^*\| \mathrm{d}s$$

where $\mathbf{T}^* = \mathbf{T}(1 - v(\phi' + \kappa_g)\sin\phi) + \mathbf{g}(\phi' + \kappa_g)v\cos\phi + \mathbf{n}(\kappa_n \cos\phi + \tau_g \sin\phi)v$. From there we obtain

$$J(v) = \int_{s_1}^{s_2} \sqrt{1 - 2v\sin\phi(\phi' + \kappa_g) + v^2(\phi' + \kappa_g)^2 - v^2(\kappa_n \cos\phi + \tau_g \sin\phi)^2}\mathrm{d}s.$$

To find value of v which minimizes $J(v)$, we have to use

$$J'(v) = \int_{s_1}^{s_2} \frac{-2\sin\phi(\phi' + \kappa_g) + 2v(\phi' + \kappa_g)^2 - 2v(\kappa_n \cos\phi + \tau_g \sin\phi)^2}{\sqrt{1 - 2v\sin\phi(\phi' + \kappa_g) + v^2(\phi' + \kappa_g)^2 - v^2(\kappa_n \cos\phi + \tau_g \sin\phi)^2}} ds = 0$$

which satisfies

$$v = \frac{sin\phi(\phi' + \kappa_g)}{(\phi' + \kappa_g)^2 - (\kappa_n \cos\phi + \tau_g \sin\phi)^2}.$$

□

Theorem 3.2. *Let M be a spacelike RSDF. Moreover, the point $X(s, v_0)$, $v_0 \in \mathbb{R}$, on the main spacelike ruling which passes the point $\alpha(s)$, is a striction point if and only if $\mathbf{e}_s$ is the unit normal vector field of tangent plane in the point $X(s, v_0)$.*

Proof. While suggesting that the point $X(s, v_0)$ on the main spacelike ruling which passes through the point $\alpha(s)$ is a striction point, we have to show that $< \mathbf{e}_s, \mathbf{e} >=< \mathbf{e}_s, \mathbf{T}^* >= 0$. We know that $< \mathbf{e}, \mathbf{e} >= 1$ so if we take differential this equation with respect to s, we obtain $< \mathbf{e}_s, \mathbf{e} >= 0$. Also if we calculate the value of $< \mathbf{e}_s, \mathbf{T}^* >$, we get

$$< \mathbf{e}_s, \mathbf{T}^* >= -\sin\phi(\phi' + \kappa_g) + v(\phi' + \kappa_g)^2 - v(\kappa_n \cos\phi + \tau_g \sin\phi)^2. \tag{2}$$

From $X(s, v_0)$, we can write the striction point as

$$v_0 = \frac{sin\phi(\phi' + \kappa_g)}{(\phi' + \kappa_g)^2 - (\kappa_n \cos\phi + \tau_g \sin\phi)^2}.$$

If we calculate the value v_0 into the eq. 2, then we get $< \mathbf{e}_s, \mathbf{T}^* >= 0$. So, $\mathbf{e}_s$ is normal to $\mathbf{e}$ and the vector field $\mathbf{T}^*$.

Conversely, it can easily be obtained. □

Let

$$\mathbf{e} = \mathbf{T}\cos\phi - \mathbf{N}\sin\phi\sinh\varphi + \mathbf{B}\sin\phi\cosh\varphi, \langle \mathbf{e}, \mathbf{e} \rangle = 1 \tag{3}$$

where φ is the angle between spacelike $\mathbf{g}$ and timelike $\mathbf{N}$ and where ϕ is the angle between the spacelike vectors $\mathbf{T}$ and $\mathbf{e}$, be a unit vector line in Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ on the RSDF drawn by a line $\mathbf{e}$. (Similarly, we can take $\mathbf{B}$ is a timelike vector. Then, all below theorems can be shown using it.)

Let $\alpha : I \to \mathbb{E}^3_1$ be a differentiable closed spacelike curve and $H = sp\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be a moving space along the curve α, where $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is Frenet frame of α and $H' = \mathbb{E}^3_1$ be a fixed space. In this case, a closed space motion can be defined on H' of H along the curve α. We denote by H/H' the closed space motion.

Theorem 3.3. *The angle of pitch of the spacelike RSDF, which is drawn by a fixed line in* $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ *during the motion* H/H' *in fixed space* H', *is*

$$\lambda_e = \lambda_T \cos\phi + \lambda_B \sin\phi\cosh\varphi, \tag{4}$$

where λ_T *and* λ_B *are the angle of pitches of the ruled surfaces which are drawn by the vectors* $\mathbf{T}$ *and* $\mathbf{B}$, *respectively.*

Theorem 3.4. *The pitch of the spacelike RSDF, which is drawn by a fixed line in* $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ *during the motion* H/H' *in fixed space* H', *is*

$$L_e = \cos\phi L_T,$$

where L_T *is the pitch of the ruled surface which is drawn by the vector* $\mathbf{T}$.

Proof. For the pitch of the spacelike RSDF, which is drawn by the fixed line $\mathbf{e}$, we get

$$L_e = \oint_\alpha < d\alpha, e >$$

$$L_e = \oint_\alpha < \mathbf{T}ds, \mathbf{T}\cos\phi - \mathbf{N}\sin\phi\sinh\varphi + \mathbf{B}\sin\phi\cosh\varphi >$$

or

$$L_e = \cos\phi L_T. \tag{5}$$

□

Theorem 3.5. *If the spacelike RSDF, which is drawn by a fixed spacelike line* $\mathbf{e}$ *in* $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ *during the motion* H/H', *is developable then the harmonic curvature is calculated as follows:*

$$h = \frac{\kappa}{\tau} = -\frac{\sin^2\phi}{\cos\phi\sin\phi\cosh\varphi} = -\frac{(L_T^2 - L_e^2)\lambda_B}{(\lambda_e L_T - L_e\lambda_T)L_e}$$

of the base curve of the ruled surface, is constant.

Proof. Let $\mathbf{e}$ draws a developable spacelike RSDF. In this case, the distribution parameter of the ruled surface is zero. Hence,

$$\frac{de}{ds} = -\mathbf{T}\left(\kappa \sin\phi \sinh\varphi\right) + \mathbf{N}\left(\kappa\cos\phi + \tau\sin\phi\cosh\varphi\right) - \mathbf{B}\left(\tau\sin\phi\sinh\varphi\right) \tag{6}$$

and

$$\frac{d\alpha}{ds} \times \mathbf{e} = \mathbf{T} \times \mathbf{e} = -\mathbf{N}\sin\phi\cosh\varphi + \mathbf{B}\sin\phi\sinh\varphi$$

so

$$P_e = \kappa\cos\phi\sin\phi\cosh\varphi + \tau \sin^2\phi = 0 \tag{7}$$

Using this last equation, we get the following

$$\frac{\kappa}{\tau} = -\frac{\sin^2\phi}{\cos\phi\sin\phi\cosh\varphi} \tag{8}$$

Solving $\cos\phi$ and $\sin\phi\cosh\varphi$ from the eq. 5 and the eq. 4, we get the following equations

$$\cos\phi = \frac{L_e}{L_T} \tag{9}$$

and

$$\sin\phi\cosh\varphi = \frac{(\lambda_e L_T - L_e\lambda_T)}{L_T\lambda_B}. \tag{10}$$

Then, substituting the eq. 9 and the eq. 10 into the eq. 8 gives

$$h = \frac{\kappa}{\tau} = -\frac{\sin^2\phi}{\cos\phi\sin\phi\cosh\varphi} = -\frac{(L_T^2 - L_e^2)\lambda_B}{(\lambda_e L_T - L_e\lambda_T)L_e}. \tag{11}$$

Besides, $\mathbf{e}$ is a fixed spacelike line in $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$. Hence, the components of e in $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ are fixed. So that from the eq. 11, h is constant. So the harmonic curvature of the spacelike RSDF is constant. □

Theorem 3.6. *The spacelike RSDF, which is drawn by a fixed spacelike line* $\mathbf{e}$ *in a normal plane in* $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ *during the motion* H/H'*, is developable if and only if* (α) *is a plane curve.*

Proof. If $\mathbf{e}$ is a line in a normal plane then from the eq. 3,

$$\cos\phi = 0 \tag{12}$$

can be obtained. Since the spacelike ruled surface is developable from the eq. 7,

$$\sin^2\phi\tau = 0 \tag{13}$$

can be obtained. When we use the eq. 12 and the eq. 13, τ is zero, so (α) is a plane spacelike curve. □

Theorem 3.7. *The spacelike RSDF, which is drawn by a fixed spacelike line* $\mathbf{e}$ *in an osculator plane in* $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ *during the motion* H/H'*, is always developable.*

Proof. If $\mathbf{e}$ is a line in an osculator plane from, form the eq. 3 we get,

$$\sin\phi\cosh\varphi = 0$$

Since $\cosh\varphi \neq 0$, we get $\sin\phi = 0$. So, the ruled surface is always developable from the eq. 7. □

Theorem 3.8. *The spacelike RSDF, which is drawn by a fixed spacelike line* $\mathbf{e}$ *in a rectifian plane in* $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ *during the motion* H/H'*, is developable if and only if* $\frac{\kappa}{\tau} = -\tan\phi$ *or* $\phi = 0$.

Proof. If $\mathbf{e}$ is a line in a rectifian plane from the eq. 3, then

$$\sin\phi\sinh\varphi = 0$$

Hence $\sin\phi = 0$ or $\sinh\varphi = 0$. Moreover, the ruled surface is developable from the eq. 7, so

$$\kappa\cos\phi\sin\phi\cosh\varphi + \tau\sin\phi^2 = 0.$$

Then,

i) If $\sin\phi = 0$ when $\sinh\varphi \neq 0$, then $\phi = 0$.
ii) If $\sinh\varphi = 0$ when $\sin\phi \neq 0$, then $\frac{\kappa}{\tau} = -\tan\phi$..
iii) Both of $\sin\phi$ and $\sinh\varphi$ are both zero, then $\phi = 0$. □

Theorem 3.9. *The spacelike ruling of the spacelike RSDF, which is drawn by a fixed line* $\mathbf{e}$ *in* $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ *during the motion* H/H' *in fixed space, is always in the rectifian plane of the striction curve.*

Proof. Since the base curve of the ruled surface is the striction curve,

$$< \alpha_s, \mathbf{e}_s >= 0. \tag{14}$$

Hence if we substitute $\mathbf{T}$ and the eq. 6 into the eq. 14,

$$\kappa\sin\phi\sinh\varphi = 0$$

can be obtained. Since $\kappa \neq 0$, $\sin\phi\sinh\varphi$ is zero. So $\mathbf{e}$ is always in the rectifian plane of the striction curve.

3.2. **Timelike ruled surfaces.**

3.2.1. *Timelike ruled surfaces with timelike rulings.* Let M be a ruled surface which obtain by a timelike straight line moving along a spacelike curve in $\mathbb{R}^3_1$. In the paper [3], $\mathbf{T}$ and $\mathbf{e}$ are orthogonal vectors. Because of this special choice, $\mathbf{n}$ is obtained as a spacelike, so M is a timelike ruled surface. Unlike this study, we take that $\mathbf{T}$ and $\mathbf{e}$ are not orthogonal vectors. In this case, we assumed that the normal vector $\mathbf{n}$ is a spacelike. We take the relation between the Darboux frame $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$ and the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ of base curve. As the tangent vector of base curve appears in both frames, then that relation is indeed, a relation between the last two vectors of both frames and a certain angles between the vectors. Then, we can write $\mathbf{e}$ in linear combination of both frames.
A unit direction vector of a timelike straight ruling $\mathbf{e}$ is spanned by the system $\{\mathbf{T}, \mathbf{g}\}$. So $\mathbf{e}$ can be written as:

$$\mathbf{e} = \mathbf{T}\sinh\phi - \mathbf{g}\cosh\phi$$

where ϕ is the angle between the vectors $\mathbf{T}$ and $\mathbf{e}$.
$\mathbf{e}_s$ is also provided the equation,

$$\mathbf{e}_s = \mathbf{T}(\phi' - \kappa_g)\cosh\phi - \mathbf{g}(\phi' - \kappa_g)\sinh\phi - \mathbf{n}(\kappa_n\sinh\phi + \tau_g\cosh\phi).$$

Holding v constant, we obtain a curve $\beta(s) = \alpha(s) + v\mathbf{e}(s)$ on a ruled surface whose vector field is

$$\mathbf{T}^* = \mathbf{T}(1 + v(\phi' - \kappa_g)\cosh\phi) - \mathbf{g}(\phi' - \kappa_g)v\sinh\phi - \mathbf{n}(\kappa_n\sinh\phi + \tau_g\cosh\phi)v.$$

The relation between the vectors $\mathbf{e}$ and $\mathbf{T}^*$ is

$$< \mathbf{T}^*, \mathbf{e} >= \sinh\phi.$$

The distribution parameter of the timelike RSDF is

$$P_e = \frac{\cosh\phi(\kappa_n \sinh\phi + \tau_g \cosh\phi)}{(\phi' - \kappa_g)^2 + (\kappa_n \sinh\phi + \tau_g \cosh\phi)^2}.$$

The orthogonal trajectories of the timelike RSDF is

$$\sinh\phi ds = dv.$$

The striction curve of the timelike RSDF with is

$$c(s) = \alpha(s) - \frac{\cosh\phi(\phi' - \kappa_g)}{(\phi' - \kappa_g)^2 + (\kappa_n \sinh\phi + \tau_g \cosh\phi)^2}\mathbf{e}(s).$$

Theorem 3.10. *Let M be a timelike RSDF which is given by $X(s, v) = \alpha(s) + v\mathbf{e}(s)$. In this case, the shortest distance between the timelike rulings of the surface along the orthogonal trajectories is*

$$v = -\frac{\cosh\phi(\phi' - \kappa_g)}{(\phi' - \kappa_g)^2 + (\kappa_n \sinh\phi + \tau_g \cosh\phi)^2}$$

along the curve $X_v : I \to M$.

Theorem 3.11. *Let M be a timelike RSDF. Moreover, the point $X(s, v_0)$, $v_0 \in \mathbb{R}$, on the main timelike ruling which passes the point $\alpha(s)$, is a striction point if and only if $\mathbf{e}_s$ is the unit normal vector field of tangent plane in the point $X(s, v_0)$.*

Let

$$\mathbf{e} = \mathbf{T}\sinh\phi + \mathbf{N}\cosh\phi\cosh\varphi - \mathbf{B}\cosh\phi\sinh\varphi, \langle\mathbf{e}, \mathbf{e}\rangle = -1 \qquad (15)$$

where φ is the angle between timelike vectors $\mathbf{g}$ and $\mathbf{N}$ and where ϕ is the angle between the spacelike $\mathbf{T}$ and timelike $\mathbf{e}$, be a unit vector line in $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ Frenet frame on the RSDF drawn by a line $\mathbf{e}$. (Similarly, we can take $\mathbf{B}$ is a timelike vector. Then, all below theorems can be shown using it.)

Let $\alpha : I \to \mathbb{E}^3_1$ be a differentiable closed spacelike curve and $H = sp\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be a moving space along the curve α, where $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is Frenet frame of α and $H' = \mathbb{E}^3_1$ be a fixed space. In this case, a closed space motion can be defined on H' of H along the curve α. We denote by H/H' the closed space motion.

Theorem 3.12. *The angle of pitch of the timelike RSDF, which is drawn by a fixed line in $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ during the motion H/H' in fixed space H', is*

$$\lambda_e = \lambda_T \sinh\phi - \lambda_B \cosh\phi \sinh\varphi, \qquad (16)$$

where λ_T and λ_B are the angle of pitches of the ruled surfaces which are drawn by the vectors $\mathbf{T}$ and $\mathbf{B}$, respectively.

Theorem 3.13. *The pitch of the timelike RSDF, which is drawn by a fixed line in $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ during the motion H/H' in fixed space H', is*

$$L_e = \sinh\phi L_T,$$

where L_T is the pitch of the ruled surfaces which is drawn by the vector $\mathbf{T}$.

Theorem 3.14. *If the timelike RSDF, which is drawn by a fixed timelike line $\mathbf{e}$ in $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ during the motion H/H', is developable, then the harmonic curvature is calculated as follows:*

$$h = \frac{\kappa}{\tau} = -\frac{\cosh^2\phi}{\cosh\phi\sinh\phi\sinh\varphi} = \frac{(L_T^2 + L_e^2)\lambda_B}{(\lambda_e L_T - L_e\lambda_T)L_e}$$

of the base curve of the ruled surface, is constant.

Theorem 3.15. *The timelike RSDF, which is drawn by a fixed timelike line* $\mathbf{e}$ *in a normal plane in* $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ *during the motion* H/H'*, is developable if and only if* (α) *is a plane curve.*

Theorem 3.16. *The timelike RSDF, which is drawn by a fixed timelike line* $\mathbf{e}$ *in an osculator plane in* $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ *during the motion* H/H'*, is developable if and only if* (α) *is a plane curve.*

3.2.2. *Timelike ruled surfaces with spacelike rulings.* Let M be a ruled surface which obtain by a spacelike straight line moving along a timelike curve in $\mathbb{R}^3_1$. In the paper [3], $\mathbf{T}$ and $\mathbf{e}$ are orthogonal vectors. Because of this special choice, $\mathbf{n}$ is a spacelike vector and M is a timelike ruled surface. Unlike this study, we take that $\mathbf{T}$ and $\mathbf{e}$ are not orthogonal vectors. In this case, the normal of M can be spacelike or timelike. We assumed that the normal vector $\mathbf{n}$ is a spacelike. Then, we take the relation between the Darboux frame $\{\mathbf{T},\mathbf{g},\mathbf{n}\}$ and the Frenet frame $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ of base curve. As the tangent vector of base curve appears in both frames, then that relation is indeed, a relation between the last two vectors of both frames and a certain angles between the vectors. Then, we can write $\mathbf{e}$ in linear combination of both frames.

A unit direction vector of a spacelike ruling $\mathbf{e}$ is spanned by the system $\{\mathbf{T},\mathbf{g}\}$. So $\mathbf{e}$ can be written as:

$$\mathbf{e} = -\mathbf{T}\sinh\phi + \mathbf{g}\cosh\phi$$

where ϕ is the angle between the vectors $\mathbf{T}$ and $\mathbf{e}$.

$\mathbf{e}_s$ is also provided the equation,

$$\mathbf{e}_s = -\mathbf{T}(\phi' - \kappa_g)\cosh\phi + \mathbf{g}(\phi' - \kappa_g)\sinh\phi - \mathbf{n}(\kappa_n\sinh\phi + \tau_g\cosh\phi).$$

Holding v constant, we obtain a curve $\beta(s) = \alpha(s) + v\mathbf{e}(s)$ on a ruled surface whose vector field is

$$\mathbf{T}^* = \mathbf{T}(1 - v(\phi' - \kappa_g)\cosh\phi) + \mathbf{g}(\phi' - \kappa_g)v\sinh\phi - \mathbf{n}(\kappa_n\sinh\phi + \tau_g\cosh\phi)v.$$

The relation between the vectors $\mathbf{e}$ and $\mathbf{T}^*$ is:

$$< \mathbf{T}^*, \mathbf{e} >= -\sinh\phi.$$

The distribution parameter of the timelike RSDF is

$$P_e = \frac{\cosh\phi(\kappa_n\sinh\phi + \tau_g\cosh\phi)}{-(\phi' - \kappa_g)^2 + (\kappa_n\sinh\phi + \tau_g\cosh\phi)^2}.$$

The orthogonal trajectories of the timelike RSDF is

$$\sinh\phi ds = -dv.$$

The striction curve of the timelike RSDF is

$$c(s) = \alpha(s) - \frac{\cosh\phi(\phi' - \kappa_g)}{-(\phi' - \kappa_g)^2 + (\kappa_n\sinh\phi + \tau_g\cosh\phi)^2}\mathbf{e}(s).$$

Theorem 3.17. *Let* M *be a timelike RSDF, which is given by* $X(s,v) = \alpha(s) + v\mathbf{e}(s)$*. In this case, the shortest distance between the spacelike rulings of the ruled surface along the orthogonal trajectories is*

$$v = -\frac{\cosh\phi(\phi' - \kappa_g)}{-(\phi' - \kappa_g)^2 + (\kappa_n\sinh\phi + \tau_g\cosh\phi)^2}$$

along the curve $X_v : I \to M$*.*

Theorem 3.18. *Let M be a timelike RSDF. Moreover, the point $X(s, v_0)$, $v_0 \in \mathbb{R}$, on the main spacelike ruling which passes the point $\alpha(s)$, is a striction point if and only if $\mathbf{e}_s$ is the unit normal vector field of tangent plane in the point $X(s, v_0)$.*

Theorem 3.19. *The harmonic curvature of the closed spacelike curve $\alpha(s)$ of ruled non-degenerate ruled surface with Darboux frame, during the motion H/H', is calculated as follows:*

$$(\frac{\kappa}{\tau})^2 = -\epsilon \frac{P_B}{P_N} + 1 \tag{17}$$

where P_B and P_N are the distribution parameters of ruled non-degenerate surfaces which are drawn by $\mathbf{B}$ and $\mathbf{N}$.

4. Conclusion

We gave the characteristic properties and integral invariants, which are depended to the geodesic torsion, the normal curvature and the geodesic curvature, of ruled non-degenerate surfaces with respect to the Darboux frame in Minkowski 3-space.

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