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Research Article

On monotonic and logarithmic concavity properties of generalized k-Bessel function

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Abstract

In this study, our main objective is to determine some monotonic and log-concavity properties of generalized k-Bessel function by using its Hadamard product representation and some earlier results on power series. In addition, by using the relationships between Bessel-type special functions and some basic functions, we present some specific examples related to the monotonic and log-concavity properties of some trigonometric and hyperbolic functions.

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1. Introduction and preliminaries

In the recent years many geometric and monotonic properties of some special functions like Bessel, Struve, Lommel, Mittag-Leffler, Wright and their generalizations were investigated by many authors. Comprehensive information about these investigations can be found in [1–8, 10, 14] and references therein. Especially, some inequalities and monotonic properties of the above mentioned functions are usefull in engineering, physics, probability and statistics, and economics. It is known that log-concavity and log-convexity properties have a crucial role in economics. Comprehensive information about the log-concavity and the log-convexity properties can be found in [13] and its references. In this study, motivated by the some earlier results which are given in [14, 15], our main aim is to present some monotonic and log-concavity properties of generalized k-Bessel functions. Moreover, we give some specific examples regarding our obtained result by using the relationships between Bessel-type functions and elementary trigonometric and hyperbolic functions.

It is known that, most of special functions can be defined with the help of Euler's gamma function. Therefore, we would like to remind the definitions of gamma function and its k-generalization. The Euler's gamma function Γ is defined by the following improper integral, for $x > 0$:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

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Also, the k-gamma function is defined by (see [12])

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{\frac{-t^k}{k}} dt$$

for $k > 0$. We know that the k-gamma function Γ_k reduces to the classical gamma function Γ when $k \to 1$. In addition, Pochammer k-symbol is defined by

$$(\lambda)_{n,k} = \lambda(\lambda + k)(\lambda + 2k) \ldots ((\lambda + (n-1)k))$$

for $\lambda \in \mathbb{C}, k \in \mathbb{R}$ and $n \in \mathbb{N}^+$. Other properties of Pochammer k-symbol and k-gamma function can be found in [12].

In this paper, we are considering the generalized k-Bessel function defined by the following series representation (see [14]):

$$W^k_{\nu,c}(x) = \sum_{n=0}^\infty \frac{(-c)^n}{n!\Gamma_k(nk + \nu + k)} \left(\frac{x}{2}\right)^{2n+\frac{\nu}{k}} \tag{1.1}$$

for $k > 0, \nu > -1$ and $c \in \mathbb{R}$. It is clear that the generalized k-Bessel function reduces to classical Bessel and modified Bessel functions for appropriate values of the parameters k and c, respectively. More precisely, taking $k = c = 1$ and $k = -c = 1$ in (1.1), we have that

$$W^1_{\nu,1}(x) = \sum_{n=0}^\infty \frac{(-1)^n}{n!\Gamma(n + \nu + 1)} \left(\frac{x}{2}\right)^{2n+\nu} = J_\nu(x) \tag{1.2}$$

and

$$W^1_{\nu,-1}(x) = \sum_{n=0}^\infty \frac{1}{n!\Gamma(n + \nu + 1)} \left(\frac{x}{2}\right)^{2n+\nu} = I_\nu(x), \tag{1.3}$$

where $J_\nu(x)$ and $I_\nu(x)$ denote classical Bessel and modified Bessel functions of the first kind, respectively. In [15], the author studied some geomertric properties such as radii of starlikeness and convexity of generalized k-Bessel function. Also, the author gave an infinite product representation of generalized k-Bessel function by using Hadamard's theorem as follow (see [15, Lemma 1.1]):

$$W^k_{\nu,c}(x) = \frac{(\frac{x}{2})^{\frac{\nu}{k}}}{\Gamma_k(\nu + k)} \prod_{n\geq 1} \left(1 - \frac{x^2}{{}_kw^2_{\nu,c,n}}\right), \tag{1.4}$$

where ${}_kw_{\nu,c,n}$ denotes nth positive zero of generalized k-Bessel function $W^k_{\nu,c}(x)$.

Now, we would like to give the definition of logarithmic concavity of a function.

Definition 1.1 ([13])**.** A function f is said to be log-concave on interval (a, b) if the function $\log f$ is a concave function on (a, b).

To show log-concavity of a function f on the interval (a, b), it is sufficient to show one of the following two conditions:

i. $\frac{f'}{f}$ monotone decreasing on (a, b).

ii. $\log f'' < 0$.

Also the following lemma due to Biernacki and Krzyż (see [11]) will be used in order to prove some monotonic properties of the mentioned functions.

Lemma 1.2. *Consider the power series* $f(x) = \sum_{n\geq 0} a_n x^n$ *and* $g(x) = \sum_{n\geq 0} b_n x^n$*, where* $a_n \in \mathbb{R}$ *and* $b_n > 0$ *for all* $n \in \{0, 1, \ldots\}$*, and suppose that both converge on* $(-r, r), r > 0$*. If the sequence* $\{\frac{a_n}{b_n}\}_{n\geq 0}$ *is increasing(decreasing), then the function* $x \mapsto \left(\frac{f(x)}{g(x)}\right)$ *is also increasing(decreasing) on* $(0, r)$*.*

It is important to note that the above result remains true for the even or odd functions.

The outcomes of our paper is as follow: In Section 2, we give our main results and their consequences, while the Section 3 is devoted for some applications of our main results.

2. Main results

In this section, we present our main results and their consequences.

Theorem 2.1. *Let $k > 0, k + \nu > 0, c \in \mathbb{R}$ and ${}_k w_{\nu,c,n}$ denote the nth positive zero of the generalized k-Bessel function $W^k_{\nu,c}(x)$. Further, consider the following sets:*

$$\delta_1 = \bigcup_{n\geq 1} \left({}_k w_{\nu,c,2n-1}, {}_k w_{\nu,c,2n}\right), \delta_2 = \bigcup_{n\geq 1} \left({}_k w_{\nu,c,2n}, {}_k w_{\nu,c,2n+1}\right) \text{ and } \delta_3 = [0, {}_k w_{\nu,c,1}) \cup \delta_2.$$

The generalized k-Bessel function

$$\Theta^k_{\nu,c}(x) = \Gamma_k(\nu + k) 2^{\frac{\nu}{k}} x^{-\frac{\nu}{k}} W^k_{\nu,c}(x) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n! \, (\nu + k)_{n,k}} \left(\frac{x}{2}\right)^{2n} \tag{2.1}$$

has the following properties:

a. *the function $x \mapsto \Theta^k_{\nu,c}(x)$ is negative on δ_1 and it is positive on δ_3,*
b. *the function $x \mapsto \Theta^k_{\nu,c}(x)$ is a decreasing function on $[0, {}_k w_{\nu,c,1})$,*
c. *the function $x \mapsto \Theta^k_{\nu,c}(x)$ is strictly log-concave on δ_3.*

Proof. **a.** If we consider the infinite product representation of generalized k-Bessel function $W^k_{\nu,c}(x)$ which is given by (1.4), then it can be easily seen that the function $\Theta^k_{\nu,c}(x)$ can be written by the following product representation:

$$\Theta^k_{\nu,c}(x) = \prod_{n\geq 1} \left(1 - \frac{x^2}{{}_k w^2_{\nu,c,n}}\right). \tag{2.2}$$

In order to investigate the sign of the function $x \mapsto \Theta^k_{\nu,c}(x)$ on the mentioned sets, we rewrite the function $x \mapsto \Theta^k_{\nu,c}(x)$ as

$$\Theta^k_{\nu,c}(x) = U_n V_n,$$

where

$$U_n = \prod_{n\geq 1} \frac{{}_k w_{\nu,c,n} + x}{{}_k w^2_{\nu,c,n}} \text{ and } V_n = \prod_{n\geq 1} \left({}_k w_{\nu,c,n} - x\right).$$

It is clear that $U_n > 0$ for all $x \in \mathbb{R}^+ \cup \{0\}$. On the other hand, since

$$0 < {}_k w_{\nu,c,1} < {}_k w_{\nu,c,2} < \cdots < {}_k w_{\nu,c,n} < \cdots,$$

we can say that, if $x \in ({}_k w_{\nu,c,2n-1}, {}_k w_{\nu,c,2n})$, then the first $(2n-1)$ terms of V_n are strictly negative and remained terms are strictly positive. Also, if $x \in ({}_k w_{\nu,c,2n}, {}_k w_{\nu,c,2n+1})$, then the first $2n$ terms of V_n are strictly negative and the rest is strictly positive. In addition, all the terms of V_n are strictly positive for $x \in [0, {}_k w_{\nu,c,1})$. As a consequence, the function $x \mapsto \Theta^k_{\nu,c}(x)$ is negative on δ_1 and it is positive on δ_3.

b. We know from part ***a.*** that the function $x \mapsto \Theta^k_{\nu,c}(x)$ is positive on the interval $[0, {}_k w_{\nu,c,1})$. The logarithmic differentation of (2.2) implies that

$$\frac{\left(\Theta^k_{\nu,c}(x)\right)'}{\Theta^k_{\nu,c}(x)} = \sum_{n=1}^{\infty} \frac{2x}{x^2 - {}_k w^2_{\nu,c,n}}.$$

Thus, we get

$$\left(\Theta^k_{\nu,c}(x)\right)' = \Theta^k_{\nu,c}(x) \sum_{n=1}^{\infty} \frac{2x}{x^2 - {}_k w^2_{\nu,c,n}} < 0$$

for all $x \in [0, {}_k w_{\nu,c,1})$. As a result, the function $x \mapsto \Theta^k_{\nu,c}(x)$ is a decreasing function on $[0, {}_k w_{\nu,c,1})$.

c. In order to prove log-concavity of the function $x \mapsto \Theta^k_{\nu,c}(x)$, we need to show that

$$\frac{d^2}{dx^2}\left[\log \Theta^k_{\nu,c}(x)\right] < 0$$

for all $x \in \delta_3$. Now, by using the infinite product representation of the function $\Theta^k_{\nu,c}(x)$ which is given by (2.2) we infer that

$$\begin{aligned}
\frac{d^2}{dx^2}\left[\log \Theta^k_{\nu,c}(x)\right] &= \frac{d^2}{dx^2}\left[\log \prod_{n\geq 1}\left(1-\frac{x^2}{{}_kw^2_{\nu,c,n}}\right)\right] \\
&= \frac{d}{dx}\left[\frac{d}{dx}\sum_{n=1}^{\infty}\log\left(1-\frac{x^2}{{}_kw^2_{\nu,c,n}}\right)\right] \\
&= \frac{d}{dx}\sum_{n=1}^{\infty}\frac{-2x}{{}_kw^2_{\nu,c,n}-x^2} \\
&= -2\sum_{n=1}^{\infty}\frac{{}_kw^2_{\nu,c,n}+x^2}{\left({}_kw^2_{\nu,c,n}-x^2\right)^2} \\
&< 0
\end{aligned}$$

for $x \in \delta_3$. Thus, the proof is completed. □

By setting $k = c = 1$ and $k = 1, c = -1$ in the Theorem 2.1 we have the following properties for the classical Bessel and modified Bessel functions, respectively.

Corollary 2.2. *Let $\nu > -1$ and $j_{\nu,n}$ denote the nth positive zero of the classical Bessel function $J_\nu(x)$. Further, consider the next sets:*

$$A_1 = \bigcup_{n\geq 1}\left(j_{\nu,2n-1}, j_{\nu,2n}\right), A_2 = \bigcup_{n\geq 1}\left(j_{\nu,2n}, j_{\nu,2n+1}\right) \text{ and } A_3 = [0, j_{\nu,1}) \cup A_2.$$

The following assertions are true:

- ***a.*** *the function $\Theta^1_{\nu,1}(x) = \Gamma(\nu+1)2^\nu x^{-\nu} J_\nu(x)$ is negative on A_1 and it is positive on A_3,*
- ***b.*** *the function $\Theta^1_{\nu,1}(x) = \Gamma(\nu+1)2^\nu x^{-\nu} J_\nu(x)$ is a decreasing function on $[0, j_{\nu,1})$,*
- ***c.*** *the function $\Theta^1_{\nu,1}(x) = \Gamma(\nu+1)2^\nu x^{-\nu} J_\nu(x)$ is strictly log-concave on A_3.*

Corollary 2.3. *Let $\nu > -1$ and $\epsilon_{\nu,n}$ denote the nth positive zero of the modified Bessel function $I_\nu(x)$. Further, consider the next sets:*

$$B_1 = \bigcup_{n\geq 1}\left(\epsilon_{\nu,2n-1}, \epsilon_{\nu,2n}\right), B_2 = \bigcup_{n\geq 1}\left(\epsilon_{\nu,2n}, \epsilon_{\nu,2n+1}\right) \text{ and } B_3 = [0, \epsilon_{\nu,1}) \cup B_2.$$

The following assertions are true:

- ***a.*** *the function $\Theta^1_{\nu,-1}(x) = \Gamma(\nu+1)2^\nu x^{-\nu} I_\nu(x)$ is negative on B_1 and it is positive on B_3,*
- ***b.*** *the function $\Theta^1_{\nu,-1}(x) = \Gamma(\nu+1)2^\nu x^{-\nu} I_\nu(x)$ is a decreasing function on $[0, \epsilon_{\nu,1})$,*
- ***c.*** *the function $\Theta^1_{\nu,-1}(x) = \Gamma(\nu+1)2^\nu x^{-\nu} I_\nu(x)$ is strictly log-concave on B_3.*

Theorem 2.4. *Let $k > 0, \nu > 0, c \in \mathbb{R}$ and ${}_kw_{\nu,c,n}$ denote the nth positive zero of the generalized k-Bessel function $W^k_{\nu,c}(x)$. Then, the function $x \mapsto W^k_{\nu,c}(x)$ is strictly log-concave on $(0, {}_kw_{\nu,c,1}) \cup \delta_2$.*

Proof. It is known that the product of two strictly log-concave function is also strictly log-concave. By using this fact it is possible to prove the log-concavity of the generalized

k-Bessel function $W^k_{\nu,c}(x)$ on δ_3. Hence, we rewrite the function $W^k_{\nu,c}(x)$ as follow:

$$W^k_{\nu,c}(x) = \frac{\left(\frac{x}{2}\right)^{\frac{\nu}{k}}}{\Gamma_k(\nu+k)}\Theta^k_{\nu,c}(x).$$

Since

$$\frac{d^2}{dx^2}\left[\log\left(\frac{x}{2}\right)^{\frac{\nu}{k}}\right] = -\frac{\nu}{kx^2} < 0$$

for $\nu > 0, k > 0$ and $x \in \mathbb{R}^+$, the function $x \mapsto \left(\frac{x}{2}\right)^{\frac{\nu}{k}}$ is strictly log-concave on $\mathbb{R}^+$. In addition, it is known from part ***c.*** of Theorem 2.1 that the function $\Theta^k_{\nu,c}(x)$ is strictly log-concave on δ_3. As a result, the function $W^k_{\nu,c}(x)$ is strictly log-concave on $(0, {}_kw_{\nu,c,1}) \cup \delta_2$ as a product of two strictly log-concave functions. □

Now, by taking $k = c = 1$ and $k = 1, c = -1$ in Theorem 2.4, we deduce the following properties for the classical Bessel and modified Bessel functions, respectively.

Corollary 2.5. *The function $x \mapsto J_\nu(x)$ is strictly log-concave on $(0, j_{\nu,1}) \cup A_2$, while the function $x \mapsto I_\nu(x)$ is strictly log-concave on $(0, \epsilon_{\nu,1}) \cup B_2$.*

Our last main result is the following theorem.

Theorem 2.6. *The function $\Phi^k_{\nu,-1}(x) = \frac{x\left(\Theta^k_{\nu,-1}(x)\right)'}{\Theta^k_{\nu,-1}(x)}$ is increasing on $(0,\infty)$ for $v > -1$ and $\nu + k > 0$.*

Proof. If we put $c = -1$ in definition of the function $\Theta^k_{\nu,c}(x)$, then we get the following infinite series representation for the function $\Theta^k_{\nu,-1}(x)$, that is,

$$\Theta^k_{\nu,-1}(x) = \sum_{n=0}^{\infty} \mathcal{P}_{n,\nu,k}x^{2n}, \tag{2.3}$$

where $\mathcal{P}_{n,\nu,k} = \frac{1}{n!4^n(\nu+k)_{n,k}}$. Differentiating both sides of the equality (2.3) and by multiplying by x obtained equality, we get that

$$x\left(\Theta^k_{\nu,-1}(x)\right)' = \sum_{n=0}^{\infty} \mathcal{R}_{n,\nu,k}x^{2n},$$

where $\mathcal{R}_{n,\nu,k} = \frac{2n}{n!4^n(\nu+k)_{n,k}}$. According to Cauchy-Hadamard theorem for power series, it can be easily shown that both power series $\sum_{n=0}^{\infty} \mathcal{P}_{n,\nu,k}x^{2n}$ and $\sum_{n=0}^{\infty} \mathcal{R}_{n,\nu,k}x^{2n}$ are convergent on $(-\infty,\infty)$, since

$$\lim_{n\to\infty}\left|\frac{\mathcal{P}_{n,\nu,k}}{\mathcal{P}_{n+1,\nu,k}}\right| = \lim_{n\to\infty}\left|\frac{\mathcal{R}_{n,\nu,k}}{\mathcal{R}_{n+1,\nu,k}}\right| = \infty.$$

Here we used the equality $(\nu+k)_{n+1,k} = (\nu+k+nk)(\nu+k)_{n,k}$ for the Pochammer k-symbol. On the other hand, it can be easily seen that $\mathcal{R}_{n,\nu,k} \in \mathbb{R}$ and $\mathcal{P}_{n,\nu,k} > 0$ for all $n \in \{0,1,\dots\}, \nu > -1$ and $\nu + k > 0$. Now, if we consider the sequence

$$U_n = \frac{\mathcal{R}_{n,\nu,k}}{\mathcal{P}_{n,\nu,k}} = 2n,$$

then we have

$$\frac{U_{n+1}}{U_n} = \frac{n+1}{n} > 1.$$

So the sequence $\{U_n\}_{n\geq 0}$ is increasing. The proof is completed by applying Lemma 1.2. □

3. Applications

In this section, we want to give some applications of our main results. Therefore, we consider the relationships among of the functions $x \mapsto \Theta^k_{\nu,c}(x)$, $x \mapsto J_\nu(x)$ and $x \mapsto I_\nu(x)$. We know from (1.2) and (1.3) that, the following equalities

$$W^1_{\nu,1}(x) = J_\nu(x) \text{ and } W^1_{\nu,-1}(x) = I_\nu(x)$$

hold true for $k = c = 1$ and $k = 1, c = -1$, respectively. On the other hand, we know from [9] that some basic trigonometric and hyperbolic functions can be written in terms of Bessel and modified Bessel functions for some special values of ν. Especially, for $\nu = -\frac{1}{2}, \nu = \frac{1}{2}$ and $\nu = \frac{3}{2}$ we have the following basic trigonometric and hyperbolic functions:

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x, \quad J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x, \quad J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

and

$$I_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cosh x, \quad I_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sinh x, \quad I_{\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\sinh x}{x} - \cosh x\right).$$

By using above relationships, we have the followings:

$$\Theta^1_{-\frac{1}{2},1}(x) = \cos x, \quad \Theta^1_{\frac{1}{2},1}(x) = \frac{\sin x}{x}, \quad \Theta^1_{\frac{3}{2},1}(x) = 3\left(\frac{\sin x - x\cos x}{x^3}\right)$$

and

$$\Theta^1_{-\frac{1}{2},-1}(x) = \cosh x, \quad \Theta^1_{\frac{1}{2},-1}(x) = \frac{\sinh x}{x}, \quad \Theta^1_{\frac{3}{2},-1}(x) = 3\left(\frac{x\cosh x - \sinh x}{x^3}\right)$$

respectively.

Now, by using the above relationships in Corollary 2.2, Corollary 2.3, Corollary 2.5 and Theorem 2.6, respectively, we can give the following some interesting examples.

Example 3.1. The following assertions hold true.

- **i.** The function $x \mapsto \Theta^1_{-\frac{1}{2},1}(x) = \cos x$ is strictly log-concave on $\left[0, j_{-\frac{1}{2},1}\right) \cup T_1$, where $T_1 = \bigcup_{n\geq 1}\left(j_{-\frac{1}{2},2n}, j_{-\frac{1}{2},2n+1}\right)$ and $j_{-\frac{1}{2},n}$ denotes the nth positive zero of the equation $\cos x = 0$.
- **ii.** The function $x \mapsto \Theta^1_{\frac{1}{2},1}(x) = \frac{\sin x}{x}$ is strictly log-concave on $\left[0, j_{\frac{1}{2},1}\right) \cup T_2$, where $T_2 = \bigcup_{n\geq 1}\left(j_{\frac{1}{2},2n}, j_{\frac{1}{2},2n+1}\right)$ and $j_{\frac{1}{2},n}$ denotes the nth positive zero of the equation $\sin x = 0$.
- **iii.** The function $x \mapsto \Theta^1_{\frac{3}{2},1}(x) = 3\left(\frac{\sin x - x\cos x}{x^3}\right)$ is strictly log-concave on $\left[0, j_{\frac{3}{2},1}\right) \cup T_3$, where $T_3 = \bigcup_{n\geq 1}\left(j_{\frac{3}{2},2n}, j_{\frac{3}{2},2n+1}\right)$ and $j_{\frac{3}{2},n}$ denotes the nth positive zero of the equation $\tan x = x$.

Example 3.2. The following statements are valid.

- **i.** The function $x \mapsto \Theta^1_{-\frac{1}{2},-1}(x) = \cosh x$ is strictly log-concave on $\left[0, \epsilon_{-\frac{1}{2},1}\right) \cup S_1$, where $S_1 = \bigcup_{n\geq 1}\left(\epsilon_{-\frac{1}{2},2n}, \epsilon_{-\frac{1}{2},2n+1}\right)$ and $\epsilon_{-\frac{1}{2},n}$ denotes the nth positive zero of the equation $\cosh x = 0$.
- **ii.** The function $x \mapsto \Theta^1_{\frac{1}{2},-1}(x) = \frac{\sinh x}{x}$ is strictly log-concave on $\left[0, \epsilon_{\frac{1}{2},1}\right) \cup S_2$, where $S_2 = \bigcup_{n\geq 1}\left(\epsilon_{\frac{1}{2},2n}, \epsilon_{\frac{1}{2},2n+1}\right)$ and $\epsilon_{\frac{1}{2},n}$ denotes the nth positive zero of the equation $\sinh x = 0$.

iii. The function $x \mapsto \Theta^1_{\frac{3}{2},-1}(x) = 3\left(\frac{\sinh x - x\cosh x}{x^3}\right)$ is strictly log-concave on $\left[0, \epsilon_{\frac{3}{2},1}\right) \cup S_3$, where $S_3 = \bigcup_{n\geq 1}\left(\epsilon_{\frac{3}{2},2n}, \epsilon_{\frac{3}{2},2n+1}\right)$ and $\epsilon_{\frac{3}{2},n}$ denotes the nth positive zero of the equation $\tanh x = x$.

Example 3.3. The following assertions hold true.

i. The function $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$ is strictly log-concave on $\left[0, j_{-\frac{1}{2},1}\right) \cup T_1$.

ii. The function $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$ is strictly log-concave on $\left[0, j_{\frac{1}{2},1}\right) \cup T_2$.

iii. The function $J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$ is strictly log-concave on $\left[0, j_{\frac{3}{2},1}\right) \cup T_3$.

iv. The function $I_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cosh x$ is strictly log-concave on $\left[0, \epsilon_{-\frac{1}{2},1}\right) \cup S_1$.

v. The function $I_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sinh x$ is strictly log-concave on $\left[0, \epsilon_{\frac{1}{2},1}\right) \cup S_2$.

vi. The function $I_{\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\sinh x}{x} - \cosh x\right)$ is strictly log-concave on $\left[0, \epsilon_{\frac{3}{2},1}\right) \cup S_3$.

Example 3.4. The following functions

$$\Phi^1_{-\frac{1}{2},-1}(x) = x\tanh x, \quad \Phi^1_{\frac{1}{2},-1}(x) = x\coth x - 1$$

and

$$\Phi^1_{\frac{3}{2},-1}(x) = \frac{(x^2+3)\sinh x - 3x\cosh x}{x\cosh x - \sinh x}$$

are increasing functions on $(0, \infty)$.

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