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Determination of energy spectra by using proper quantization rule of woods-saxon potential

Woods-Saxon potansiyeline ait uygun kuantumlamış çözüm metodu ile enerji spektrumlarının belirlenmesi

Yazar(lar) (Author(s)): *Rezvan REZAEIZADEH[1], Niloufer ZOGHI FOUMANI[2], Abbas GHASEMIZAD[3], Aybaba HANÇERLİOĞULLARI[4]*

ORCID[1]: 0000-0001-6219-6174

ORCID[2]: 0000-0002-2508-3303

ORCID[3]: 0000-0001-6219-6174

ORCID[4]: 0000-0000-1700-8480

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Determination of Energy Spectra By Using Proper Quantization Rule Of Woods-Saxon Potential

Highlights

- *The PQR method is described and the related mechanism is presented in detail.*
- *Then the energy spectrum is obtained for the WS potential.*
- *The numerical calculations for four various light nuclei are presented*
- *The E-V_0-a diagrams are plotted to optimize and provide the appropriate coefficients*

Graphical Abstract

In this study, the energy spectra of Schrodinger equation for non-zero l values considering Woods Saxon potential (WSP) is calculated using proper quantization rulFor the energy, potential, surface thickness for those nuclei (^{7}Li, ^{9}Be ,^{11}B and ^{15}N) shown in figure-1 .

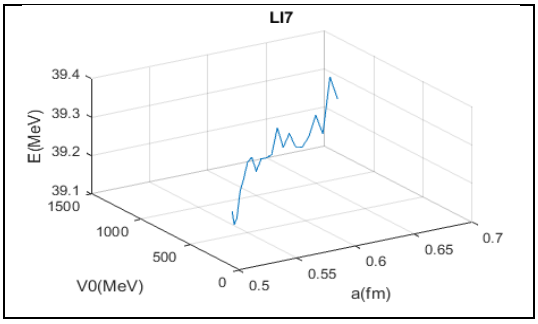

Figure . ^{7}Li nuclei, BE as function of V_0 and a

Aim

This study of aim was to the energy spectra of Schrodinger equation for non-zero l values considering Woods-Saxon (WS) potential is calculated using Proper Quantization Rule (PQR).

Design & Methodology

The investigation was done numerically using Matlab simulation program and Pearson correlation coefficient has been shown to be related to various nuclear properties of the nuclei.

Originality

The most important feature and originality of our study was that until this time the studies required numerical and complex and high mathematics.

Findings

Results have shown thatthe initial state of $\varphi_0(x)$ achieved from the Riccati equation, we were able to achieve binding energy of the nucleus at the presence of WS potential with the Schrodinger equation via PQR method.

Conclusion

In this study, the PQR method of the related mechanism is described and then the energy spectrum is obtained for the WS potential .the numerical calculations for four various light nuclei are presented and the results are compared with experimental values. correlation.

Declaration of Ethical Standards

The author(s) of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

Determination of Energy Spectra By Using Proper Quantization Rule of Woods-Saxon Potential

Araştırma Makalesi / Research Article

Rezvan REZAEIZADEH[1], Niloufer ZOGHI FOUMANI[2], Abbas GHASEMIZAD[1*], Aybaba HANÇERLİOĞULLARI[3]

[1]Department of Physics, University Campus 2, University of Guilan, Rasht, Iran

[2]Department of Physics, Faculty of Science,University of Guilan, P.o Box 1914, Rasht, Iran

[3]Department of Physics, Science & Arts Faculty, Kastamonu University, 37150, Kastamonu, Turkey

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ABSTRACT

In this study, the energy spectra of Schrodinger equation for non-zero l values considering Woods Saxon potential (WSP) is calculated using proper quantization rule, then the binding energies (BE) of random light nuclei is obtained and the optimized potential parameters such as potential depth (V0) and surface thickness (a) are found. In order to calculate the energy levels of the nuclei with WSP, the PQR method was used, which has not been considered before. In quantum mechanics, the exact solution of energy systems, momentum, and quantum states can be found using the proper quantization rule(PQR) method.Using the Matlab calculation program, we have achieved numerical values of the energy spectrum for random light nuclei and compared the result with the experimental Nuclear Data Center (NDC) values. In addition, we found potential depth and surface thickness for four light nuclei. Correlations between the light nuclei show the facts about the nuclear structure characteristics, origin, and energies of these nuclei. Pearson's correlation coefficient is accepted as the most common correlation coefficient. According to the values of Pearson correlation coefficients, it is observed that there is a significant positive correlation between the nucleons examined. Finally, we plot the E-V0-a diagrams for those values to optimize and provide the appropriate coefficients. It is shown that there is a good agreement between the results of this work and experimental values.

keywords: Schrodinger equation, woods saxon potential, proper quantization rule, binding energy.

Woods-Saxon Potansiyeline Ait Uygun Kuantumlamış Çözüm Metodu İle Enerji Spektrumlarının Belirlenmesi

ÖZ

Bu çalışmada, Woods -Saxon potansiyeli (WSP) göz önünde bulundurularak sıfır olmayan L değerleri için Schrodinger denkleminin enerji spektrumu uygun tam çözüm metodu kuralı kullanılarak hesaplanmıştır. Çalışmamızda,rastgele hafif nükleer çekirdeklerinin bağlanma enerjileri (BE),optimize edilmiş potansiyel derinlik (V0) ve yüzey kalınlıkları (a) hesaplanmıştır. Çekirdeklerin WSP ile enerji seviyelerini hesaplamak için, daha önce dikkate alınmamış olan PQR yöntemi kullanılmıştır.Kuantum mekaniğinde, enerji sistemlerinin, momentumun ve kuantum durumlarının kesin çözümü, uygun kuantumlanmış çözüm (PQR) yöntemi kullanılarak bulunabilir. MatLab simülasyon programını kullanarak, bu hafif nükleer çekirdekleri için enerji spektrumunun sayısal değerlerini elde ederek, sonuçları, dört hafif nükleer çekirdeği için potansiyel derinlik ve yüzey kalınlığı deneysel verilerle nükleer veri merkezi (NDC) MeV karşılaştırılmıştır. Bu çekirdekler arasındaki korelasyon ilişkileri için istatistiksel analizler yapılarak, çekirdeklerin nükleer yapı özellikleri ve enerjileri seviyeleri arasındaki ilikliler detaylandırılmıştır. Hafif nükleer çekirdekler için Pearson'un korelasyon katsayısı en yaygın korelasyon katsayısı olarak kabul edilerek, incelenen nükleonlar arasında anlamlı bir pozitif korelasyon olduğu gösterilmiştir. Son olarak, uygun katsayıları optimize etmek için bu çekirdeklere ait (E-V_0-a) ilişkili olarak gerekli grafiksel diyagramlarını çizilmiştir. Çalışmanın sonuçları ile deneysel değerler arasında iyi bir uyum olduğu gösterilmiştir.

Anahtar Kelimeler: Schrodinger denklemli,woods saxon potansiyeli, tam çözüm metodu, bağlanma enerjisi.

1.INTRODUCTION

Since the investigation of quantum systems is accompanied by computational challenges and some great complexities, physicists are usually trying to make calculations as simple as possible. Examining precisely solvable systems is very important in quantum

**Sorumlu yazar (Corresponding author)*
e-posta: ghasemi@guilan.ac.ir

mechanics. The Schrodinger equation for all quantum numbers n andl can only be solved for a small number

of potentials such as Hulthen, Harmonic Oscillator, and Hydrogen atom[1,2]. The WS potential is important in describing the interaction between a light nucleus and a heavy nucleus as a solvable potential. However, the Schrodinger equation with $l \neq 0$ in the presence of WS potential does not have an analytical solution[3]. Recently, several methods have been presented for solving quantum systems. They are the supersymmetric quantum mechanics (SUSYQM) approach[4], the supersymmetric Wentzel–Kramer–Brillouin (SWKB) method[5], the Nikiforov–Uvarov (NU) method[6,7], the factorization formalism[8] and exact quantization rule method(EQR)[9–11], the exact quantization rule with the generalization of the Bohr-Sommerfeld quantization rule[12] and the Wentzel–Kramer–Brillouin(WKB) $method$. Apart from these approaches, the quasilinearization method (QLM) is also applied to investigate random physical potentials[13-21]. The EQR method is an effective instrument for obtaining the eigenvalues of all solvable quantum potentials[22 − 27]. Since the complex integral calculations of quantum correction are considered a problem, The EQR method is developed to improve the quantum correction term. Furthermore, in quantum mechanics, the solution of energy systems and quantum states can be obtained using the proper quantization rule (PQR) method. This method has been developed by converting the exact quantization rule (EQR) into simpler basic integrals [1,2], especially for the calculation of complex energy spectra with nuclear potentials. As a matter of fact, the PQR method is achieved with the aim of creating more symmetry [29 − 33]. The PQR method is applied for some exactly solvable quantum systems such as the finite square well, Morse, hyperbolic- Rosen–Morse, Poschl–Teller, Hulthen, harmonic oscillator, and the hydrogen atom, WSP with Pekeris approximation, WS potential, the Kratzer, modified harmonic oscillator, trigonometric Rosen–Morse potential and others [2].

The aim of this study is to investigate the energy spectra of nuclei using the Schrodinger equation for the WS potential by the PQR method which has not been considered for nuclei before. The first benefit of PQR method over EQR method is that by finding the solution of the complex quantum correction term in EQR method, we must find the energy spectra and wave function of the initial state of the nucleus at the same time, but in order to find the energy spectra of a quantum system such as the nucleus via the PQR method, we only need to know the ground state energy. Another benefit of the PQR method is that finding the solution of one of two integrals is enough to examine the system. Thus, for quantum systems such as nuclei and exactly solvable potentials, the PQR method can be useful and simpler. So, In order to calculate the energy levels of the nuclei with WSP, the PQR method is utilized. Using the Matlab computational program, the numerical values of the binding energy for random light nuclei have been achieved, the potential depth and surface thickness for random light nuclei have been found and the results have been compared with experimental values. correlations between the light nuclei show the facts about the nuclear structure characteristics, origin, and energies of these nuclei. Since Pearson's correlation coefficient is accepted as the most common correlation coefficient, it is shown that there is a good agreement between the results of this work and experimental values.

2.PROPER QUANTIZATION RULE

The Schrodinger equation is stated as

$$\frac{d^2}{dx^2}\psi(x) = -\frac{2M}{h^2}[E - V(x)]\psi(x) \tag{1}$$

which is equal to the Riccati equation

$$\frac{d}{dx}\varphi(x) = -\frac{2M}{h^2}[E - V(x)] - \varphi(x)^2 \tag{2}$$

where $\varphi(x) = \Psi(x)^{-1}\, d\Psi(x)/dx$ is the logarithmic derivative of wave function ψ(x). This exact quantization rule was displayed[5, 6] for the Schrodinger equation as

$$\int_{x_A}^{x_B} k(x)dx = N\pi + \int_{x_A}^{x_B} k'(x)\frac{\varphi(x)}{\varphi'(x)}dx \tag{3}$$

where $k(x) = \sqrt{2M[E - V(x)]}/\hbar$ and $x_A$and $x_B$are two turning points determined by E = V (x). N = n + 1 is the number of nodes of φ(x) in the E ≥ V (x). The quantum correction term, which is the second integral of the equation (3), is derived from the ground state[1],

$$Q = Q_0 = \int_{x_A}^{x_B} k_0'(x)\varphi_0(x)/\varphi_0'(x)dx \tag{4}$$

The momentum k(x) in equation (3) is relevant to the energy spectrum. Finally, we can use this rule to achieve the energy spectrum of nuclei in presence of WSP:

$$\begin{aligned} &\int_{r_A}^{r_B} k(r)dr = N\pi + \int_{r_A}^{r_B} k_0'(r)\frac{\varphi_0(r)}{\varphi_0'(r)}dr \\ &k(r) = \sqrt{2M[E - V(r)]}/\hbar \quad E \geq V_{eff}(r) \end{aligned} \tag{5}$$

The solution of two integrals of quantum correction in equation (3) and (4), for some physical potentials, may be difficult[9]. Hence, with the development of the method mentioned below, the PQR method is achieved. Finally, considering N = 1, i.e., n = 0 in equation (3) gives

$$\begin{aligned} &\int_{x_{0A}}^{x_{0B}} k_0(x)dx = \pi + \int_{x_{0A}}^{x_{0B}} k_0'(x)\frac{\varphi_0(x)}{\varphi_0'(x)}dx \\ &k_0(x) = \sqrt{2M[E_0 - V(x)]}/\hbar \end{aligned} \tag{6}$$

Which we can get

$$\int_{x_{0A}}^{x_{0B}} k_0'(x)\frac{\varphi_0(x)}{\varphi_0'(x)}dx = \int_{x_{0A}}^{x_{0B}} k_0(x)dx - \pi \tag{7}$$

After replacing equation (7) into equation (3), we achieve

$$\int_{x_A}^{x_B} k(x)dx - \int_{x_{0A}}^{x_{0B}} k_0(x)dx = (N-1)\pi = n\pi \tag{8}$$

likewise, equation (4) can also be written in the same form

$$\int_{x_A}^{x_B} k(x)dx - \int_{x_{0A}}^{x_{0B}} k_0(x)dx = n\pi \tag{9}$$

Equations (8) and (9) are introduced as the PQR method.

3.EIGENVALUES OF WOODS-SAXON POTENTIAL USING PQR METHOD

The WS potential in N- dimensions is characterized by

$$V_{eff} = V(r) + V_l(r) = -\frac{V_0 e^{-\frac{(r-R_0)}{a}}}{1+e^{-\frac{(r-R_0)}{a}}} + \frac{(\eta^2-1)\hbar^2}{8\mu r^2}\ ,\ \eta = 2l+N-2 \tag{10}$$

where $R_0 = r_0 A^{1/3}$ is the nuclear radius with $r_0 = 1.25$ fm, V_0 is the potential depth and $a \approx 0.5-0.6$ fm is the surface thickness. In addition, $1/r^{-2}$ is the orbital coupling term. describing

$$b = \left(\frac{L}{R_0}\right)^2,\ L^2 = \frac{\hbar^2}{2\mu}\left(l+\frac{N-1}{2}\right)\left(l+\frac{N-3}{2}\right) \tag{11}$$

and introducing the new variable $z(r) = \frac{e^{-\gamma x}}{1+e^{-\gamma x}}$, solving the equation (12)

$$V_{eff}(z) = bc_2 z^2 + (bc_1 - V_0)z + bc_0 = E_{n,l} \tag{12}$$

The turning points z_A and z_B will be determined

$$z_A = \frac{V_0}{2bc_2} - \frac{c_1}{2c_2} - \frac{1}{2bc_2}\sqrt{(V_0-bc_1)^2 + 4dc_2(E_{n,l}-bc_0)}$$
$$z_B = \frac{V_0}{2bc_2} - \frac{c_1}{2c_2} - \frac{1}{2bc_2}\sqrt{(V_0-bc_1)^2 + 4bc_2(E_{n,l}-bc_0)} \tag{13}$$

The momentum k(z) is given as

$$k(z) = \frac{\sqrt{2\mu}}{\hbar}\sqrt{bc_2}\sqrt{(z_B-z)(z-z_A)} \tag{14}$$

The Riccati equation (2) becomes

$$-\frac{\gamma}{R_0}z(1-z)\frac{d\varphi_0(z)}{dz} = -\frac{2\mu}{\hbar^2}\left[E_0 - bc_2 z^2 + (V_0 - bc_1)z - bc_0\right] - \varphi_0 (\tag{15}$$

Based on the Sturm–Liouville theorem[33], after getting $\varphi_0(r) = c_1 z + c_2 (c_1 > 0)$ and replacing $\varphi_0(r)$ into equation (15) the Energy equation of ground state can be achieved

$$E_0 = -\frac{\hbar^2}{2\mu}\left[\frac{\gamma}{2R_0} - \frac{2\mu}{\hbar^2}\frac{(V_0-bc_1)}{2m\gamma}R_0\right]^2 \tag{16}$$

After solving the first integral in equation (8), we have

$$\int_{r_A}^{r_B} k(r)dr = \pi\frac{\sqrt{2\mu}}{\hbar}\frac{R_0}{\gamma}\left(\sqrt{bc_2} + \sqrt{bc_0 - E_{n,l}} - \sqrt{b(c_0+c_1+c_2) - V_0 - E_{n,l}}\right) \tag{17}$$

substituting E_n in equation (17) with E_0 given in equation (16) and considering equation (8), we achieve the following result for the energy levels:

$$E_{n,l}^{(N)} = \frac{L^2}{R_0^2}c_0 - \frac{\hbar^2 a^2}{2\mu}\left[\frac{\left(2n+1-\sqrt{1+\frac{8\mu}{\hbar^2}\frac{L^2}{\gamma^2}Ac_2}\right)}{4a^2} - \frac{\frac{2\mu}{\hbar^2}\frac{L^2}{R_0^2}(c_1+c_2) - \frac{2\mu}{\hbar^2}V_0}{\left(2n+1-\sqrt{1+\frac{8\mu}{\hbar^2}\frac{L^2}{\gamma^2}c_2}\right)}\right]^2 \tag{18}$$

4. DISCUSSION AND RESULTS

In this study, the binding energy for four random light nuclei including, ^{7}Li, ^{9}Be ,^{11}B and ^{15}N is obtained using the final energy Equation-18 and the results are compared with the experimental values. In addition, we found the optimized potential depth and thickness of the surface for these nuclei. The results are presented in Table- 1 of https://www-nds.iaea.org/ NDC and PQR. Thus, it seems that the method can obtain the Binding Energy (BE) of light nuclei in a good agreement with the experimental values. An increase in the number of neutrons in the nuclei leads to the instability of the nuclei and their activity. Therefore, the shell model for heavy nuclei is not well justified. For each of the nuclei, at the approximate eligible potential depth and surface thickness listed in Table- 1, the expected approximate amount of energy is obtained. It can be found that as the surface thickness of these nuclei increases, their energy gradually decreases. Besides, the energy gradually increases with the increment of the potential depth.Moreover, as the nuclear radii of the nuclei increase, the potential depth and binding energies of the nuclei increase. In this study, the Pearson's correlation

method was used that analyzes the relationship between variables.

The direction of the relationship and correlation coefficient indicates its degree. The degree of this relationship can be determined by the correlation of significant correlation at $p \leq 0.01$ .The correlation analysis degree determines the r coefficient. This value is between (-1 and +1). If it is close to r = -1, there will be an inverse negative relationship[34-36]. It shows a positive relationship if r = +1 and if r = 0 there will be no correlation connection between two variables. Table 2 shows the Pearson's correlation matrix between the analyzed nuclei ^{7}Li, ^{9}Be,^{11}B, and ^{15}N. In Table-2, Pearson's correlation coefficient has been shown in relation to various nuclear properties of the nuclei. Table 2 shows that the relation between ^{15}N and ^{7}Li (0,98) is stronger in terms of correlation than^{15}N and ^{11}B (0,80). In addition, ^{7}Li is significantly correlated with ^{9}Be (0,99). According to the statistical analysis of the nuclei in Table -1, The average of nuclear radii 0.56 fm corresponds to the radius of ^{11}B nucleus in the median.Furthermore, the average binding energy of the nuclei calculated with the help of the experiment and Theory is compatible.

The experimental average binding energy of the cores is 95.13 MeV and the calculated value is 95.21 MeV. In terms of energy, the standard deviation is 0.06, which is 95.13 ± 0.06 and 95.21 ± 0.06. According to the calculated PQR method, as shown in Table-1, 7Li, 9Be, 11B and 15N nuclei have values of the standard deviation than the experimental binding energies as 39.24 ±0; 58.16 ±0.01; 76.19 ±0,27; 115.49 ± 0.12; 186.56 ±0.11, respectively.

According to the values obtained through the Matlab computational program, diagrams of the energy, potential, surface thickness for selective nuclei (7Li, 9Be ,11B and 15N) are plotted as shown in Figures 1(a,b,c), 2(a,b,c), 3(a,b,c) and 4(a,b,c) and the optimized values are found by using these diagrams.

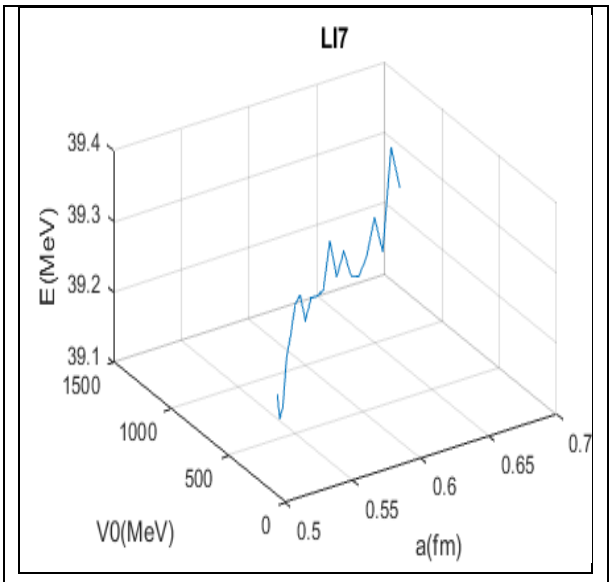
Table .1 BE of ^{7}Li, ^{9}Be ,^{11}B and ^{15}N nuclei

Nuclei	a (fm)	V_0 (Mev)	BE(Exper.) (NDC) MeV	BE(Calcul.) (PQR) MeV
Li^{7}	0.5	69.00	39.24	39.24
Be^{9}	0.50	89.30	58.16	58.18
B^{11}	0.55	152.00	76.19	76.57
N^{15}	0.59	114.00	115.49	115.66
Min	0.50	69.00	39.24	39.24

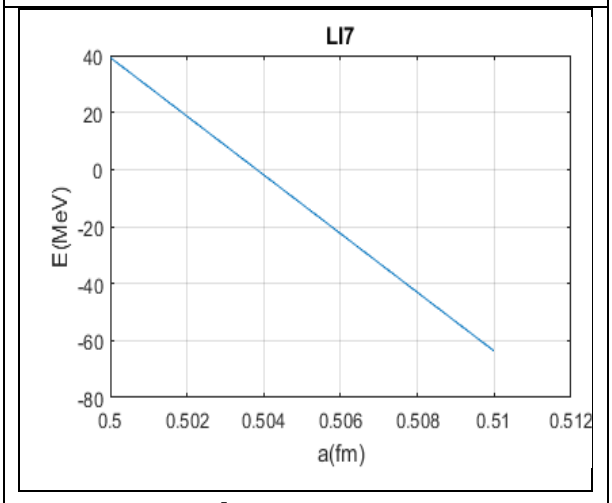
Table.2 Pearson's correlation matrix between the analyzed nuclei

Nuclei	^{7}Li	^{9}Be	^{11}B	^{15}N
^{7}Li	1			
^{9}Be	0,993298	1		
^{11}B	0,995818*	0,97859	1	
^{15}N	0,988766	0,913015	0,809514	1

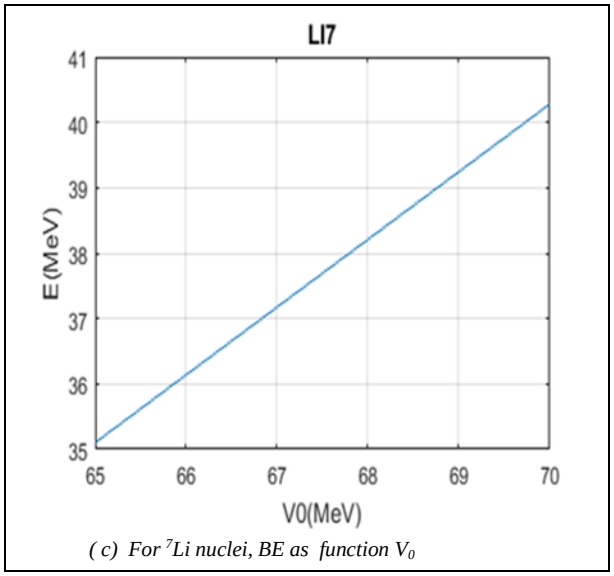
*****Bold value indicates significant correlation at $p \leq 0.01$*



(a)For ^{7}Li nuclei, BE as function of V_0 and a



(b) For ^{7}Li nuclei, BE as function a



(c) For ^{7}Li nuclei, BE as function V_0

Figure1. BE as a function of V_0 and a, BE as a function of a, BE as a function of V_0 for ^{7}Li

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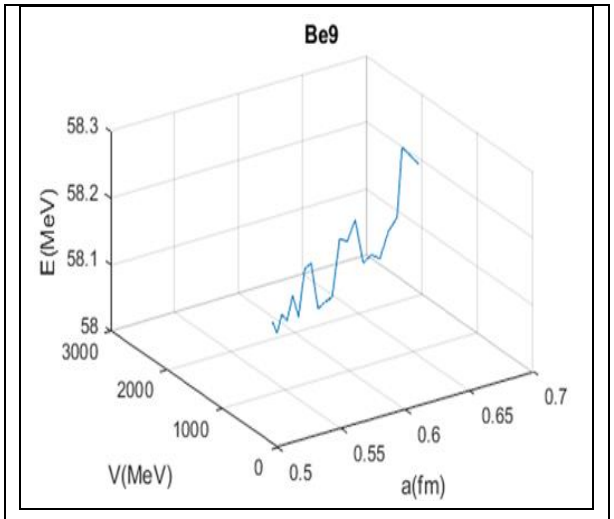
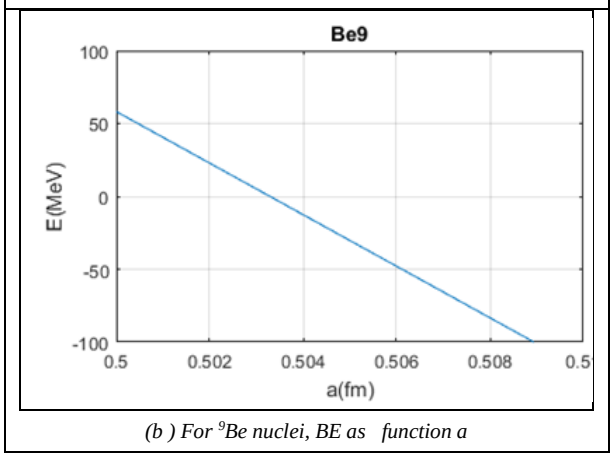
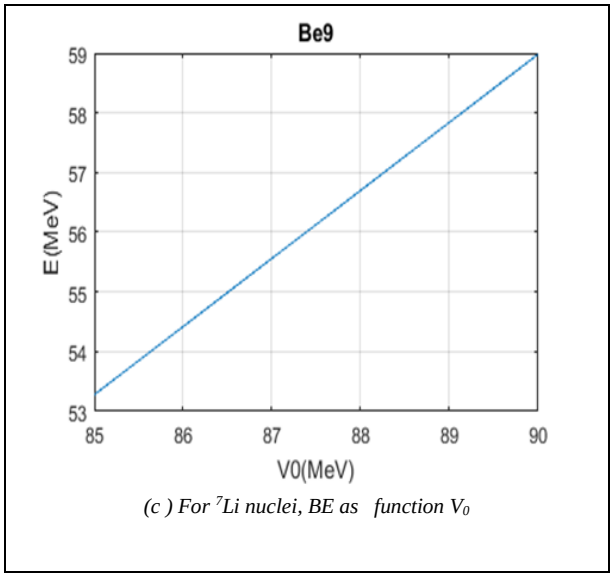


(a)For ^{9}Be nuclei, BE as function of V_0 and a

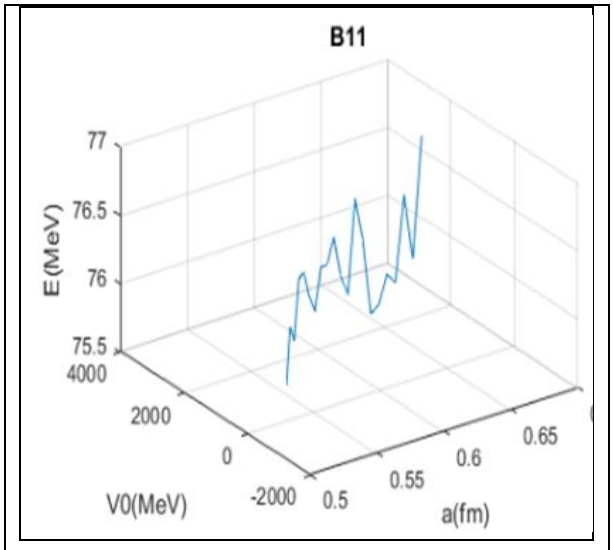


(b) For ^{9}Be nuclei, BE as function a

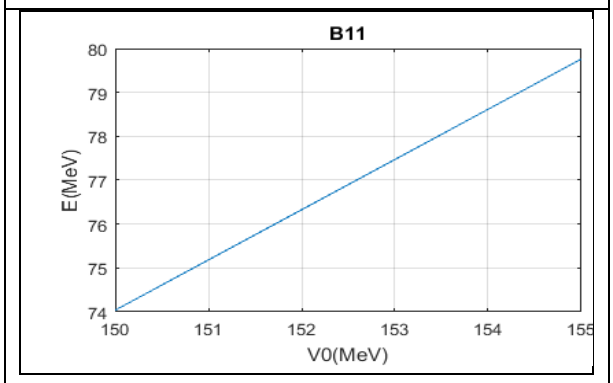


(c) For ^{7}Li nuclei, BE as function V_0

Figure 2. BE as a function of V_0 and a, BE as a function of a, BE as a function of V_0 for ^{9}Be

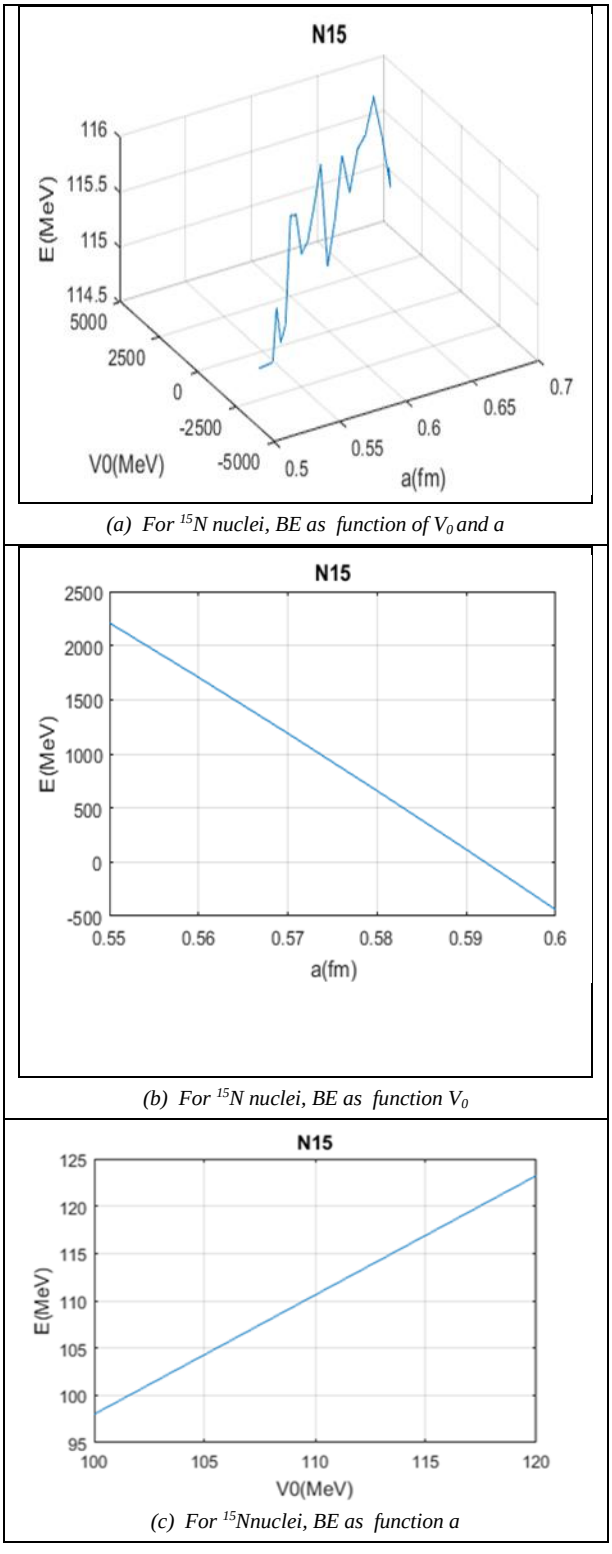


(a) For ^{11}B nuclei, BE as function of V_0 and a



(b) For ^{11}B nuclei, BE as function V_0

Figure 3. BE as a function of V_0 and a, BE as a function of a, BE as a function of V_0 for ^{11}B.



(a) For ^{15}N nuclei, BE as function of V_0 and a

(b) For ^{15}N nuclei, BE as function V_0

(c) For ^{15}Nnuclei, BE as function a

Figure 4. BE as a function of V_0 and a, BE as a function of a, BE as a function of V0 for .

5. CONCLUSIONS

In this study, the PQR method of the related mechanism is described and then the energy spectrum is obtained for the WS potential .the numerical calculations for four various light nuclei are presented and the results are compared with experimental values. The most important feature and originality of our study were that until this time the studies required numerical and complex and high mathematics. But, PQR method developed by converting the exact quantization rule (EQR) into simpler basic integrations, especially for the calculation of complex energy spectra with nuclear potential. According to the initial state of $\varphi_0(x)$ achieved from the Riccati equation, we were able to calculate the binding energy of the nucleus at the presence of WS potential with the Schrodinger equation via PQR method. This is considered as an important feature that the symmetry of the PQR method is greater than that of the EQR method. In fact, the momentum integral $\int_{x_A}^{x_B} k(x)\,dx$ increases by one when the number of the nodes of the wave function $\varphi(x)$ increases by one. The Pearson correlation coefficient matrix between the various nuclear structures between samples within the WS potential frame is given in Table-2. The relationship between binding energy, nucleon radius, and nuclear potential is multiple correlations and gives information about the Pearson correlation.

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This study was conducted as aproduct of the doctoral thesis prepared .

DECLARATION OF ETHICAL STANDARDS

The author(s) of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

AUTHORS' CONTRIBUTIONS

Rezvan REZAEIZADEH : She made the necessary simulation calculations by writing the article.

Niloufer ZOGHI FOUMANI: Performed the experiments and analyse the results.

Abbas GHASEMIZAD: Performed the experiments and analyse the results.

Aybaba HANÇERLİOĞULLARI: Performed the experiments and analyse the results.

CONFLICT OF INTEREST

There is no conflict of interest in this study.

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