


## NEW QUADRATIC FUNCTIONAL EQUATION AND ITS (HURS)

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### Abstract

The primary subject in the stability of differential equations is to answer the question of when is it real that a mapping which roundly satisfies a differential equation must be close to an exact solution of the equation. For this reason, the Hyers-Ulam and Hyers-Ulam Rassias stability of differential equations is fundemantal. Currently, researchers have used various methods (open mapping, direct method, integral factor, fixed point method) to research that the Hyers-Ulam Rassias and Hyers-Ulam stability of differential equations. The direct method has been succesfully apllied for investigate of the Hyers-Ulam Rassias stability of many different functional differential equations. But it does not enough for some important cases. The second most popular method is the fixed point method. In this study, we make an attemp to establish the Hyers-Ulam Rassias stability (HURS) of a new quadratic type functional equation (QFE)

$$g(\chi + \gamma + \nu + \varsigma) + g(\chi - \gamma - \nu - \varsigma) = 4g(\chi) + g(\gamma + \varsigma) + g(\gamma + \varsigma + 2\nu) - g(\chi - \nu) - g(\chi + \nu)$$

by direct method and fixed point method. We consider that this research will contribute to the related literature and it may be useful for authors studying on the Hyers-Ulam Stability of the quadratic functional differential equations.

**Keywords:** (HURS), (QFE), fixed point method.

## YENİ QUADRATIC FONKSİYONEL DENKLEM VE BU DENKLEMİN HYERS ULAM RASSIAS KARARLILIĞI

### Özet

Diferansiyel denklemlerin kararlılığında asıl mesele bir diferansiyel denklemi yaklaşık olarak sağlayan bir dönüşümün denklemin tam çözümüne yaklaşması ne zaman gerçek olur sorusuna cevap verilmesidir. Bu nedenle diferansiyel denklemlerin Hyers-Ulam ve Hyers Ulam Rassias kararlılığı esastır. Bu günlerde araştırmacılar diferansiyel denklemlerin Hyers-Ulam ve Hyers-Ulam Rassias kararlılığını araştırmak için çeşitli metotlar (açık dönüşüm, direkt metot, integral çarpanı, sabit nokta metodu) kullanmaktadır. Direkt metot birçok farklı fonksiyonel diferansiyel denklemlerin Hyers-Ulam Rassias kararlılığını araştırmak için başarılı bir şekilde uygulanmaktadır. Fakat bu metot bazı önemli durumlar için yeterli değildir. İkinci en popüler metot sabit nokta metodudur.

Bu çalışmada direkt metot ve sabit nokta metodunu kullanarak

$$g(\chi + \gamma + \nu + \varsigma) + g(\chi - \gamma - \nu - \varsigma) = 4g(\chi) + g(\gamma + \varsigma) + g(\gamma + \varsigma + 2\nu) - g(\chi - \nu) - g(\chi + \nu)$$

şeklindeki yeni bir quadratic tipten fonksiyonel denklemin Hyers-Ulam Rassias kararlılığını belirlemek için girişimde bulunduk. Bu araştırmanın quadratic fonksiyonel denklemlerin Hyers Ulam kararlılığı üzerine çalışan yazarlara fayda sağlayabileceğini ve ilgili literatüre katkı sağlayacağını düşünüyoruz.

**Anahtar Kelimeler:** Hyers-Ulam Rassias kararlılık, quadratic fonksiyonel denklem, sabit nokta metodu.

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### 1. Introduction

In 1940, the concept of Hyers-Ulam stability (HUS) came to light as a result of the problem introduced by Ulam. In 1941, this problem was partially brightened, by Hyers. After then many mathematician have searched this topic.

The following equation

$$g(\chi + \gamma) + g(\chi - \gamma) = 2g(\chi) + 2g(\gamma) \quad (1)$$

is named a quadratic functional equation (QFE). Every solution of the (QFE) (1) is said to be a quadratic function.

F. Skof [1] seems to first author who investigated the (HUS) of a (QFE) for functions between  $X_1$  and  $X_2$ , in which  $X_1$  is a normed space and  $X_2$  is a Banach space.

Cholewa [2] proved that, if  $X_1$  is an Abelian Grup, then the result of Skof is also true. Czerwik [3] proved the (HURS) of the (QFE) in [1].

Jung [4] studied the (HURS) problems of a (QFE) of the form

$$f(\chi - \gamma - z) + f(\chi) + f(\gamma) + f(z) \\ = f(\chi - \gamma) + f(\gamma + z) + f(z - \chi).$$

In the literature, it can be seen that (HUS) and (HURS) of (QFEs) can attract the intensive attentions of the researchers. Actually, recently, particularly, the (HUS) and (HURS) of (QFEs) are investigated by many researchers.

In particular, for more results on the (HURS) of (QFE) see the papers [5], [6], [7], [8] and on the (HURS) stability of the ordinary or functional differential equations see the papers [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21] and the references therein.

In this study, we deal with a new functional equation:

$$g(\chi + \gamma + \nu + \varsigma) + g(\chi - \gamma - \nu - \varsigma) \\ = 4g(\chi) + g(\gamma + \varsigma) \\ + g(\gamma + \varsigma + 2\nu) - g(\chi - \nu) - g(\chi + \nu). \quad (2)$$

We prove (HURS) of above (QFE) by using direct method and fixed point method. During the process we use in the following theorem.

**Theorem 1.1. ([8])** Let  $(X, d)$  be a complete generalized metric space and  $P : X \rightarrow X$  be a strictly contractive mapping with Lipschitz constant  $L < 1$ . Then, for each given  $\chi \in X$ , either

$$d(P^n \chi, P^{n+1} \chi) = \infty$$

for all  $n \geq 0$  or there exists a natural number  $n_0$  such that

- i.  $d(P^n \chi, P^{n+1} \chi) < \infty$  for all  $n \geq n_0$ ,
- ii. The sequence  $\{P^n \chi\}$  converges to a fixed point  $\gamma^*$  of  $P$ ,
- iii.  $\gamma^*$  is the unique fixed point of  $P$  in the set  $Y = \{\gamma \in X : d(P^{n_0} \chi, \gamma) < \infty\}$ ,
- iv.  $d(\gamma, \gamma^*) \leq \frac{1}{1-L} d(\gamma, P\gamma)$  for all  $\gamma \in Y$ .

## 2. (HURS) of (QFE)

Firstly, we will give a lemma to use in proof of main theorem.

**Lemma 2.1.** Let  $X_1$  and  $X_2$  be real vector spaces.  $g : X_1 \rightarrow X_2$  is quadratic if and only if  $g$  satisfies equation (2).

**Proof.** ( $\Rightarrow$ ) Assume that  $g$  is quadratic. Letting  $\chi = 0 = \gamma$  in (1) we obtain  $g(0) = 0$ . Taking  $\chi = 0$  in (1) gives

$$g(\gamma) + g(-\gamma) = 2g(0) + 2g(\gamma)$$

and  $g(\gamma) = g(-\gamma)$ . Putting  $\gamma = \chi$  in (1), we obtain that  $g(2\chi) = 4g(\chi)$ .

We substitute  $\gamma = \chi + \gamma$  in (1) and then  $\gamma = \chi - \gamma$  in (1) to obtain that

$$g(2\chi + \gamma) + g(\gamma) = 2g(\chi) + 2g(\chi + \gamma) \quad (3)$$

and

$$g(2\chi - \gamma) + g(\gamma) = 2g(\chi) + 2g(\chi - \gamma). \quad (4)$$

Adding (3) and (4), we get

$$g(2\chi + \gamma) + g(2\chi - \gamma) = 8g(\chi) + 2g(\gamma). \quad (5)$$

Substituting  $\gamma = \gamma + \nu$ , in equalities of

$$g(2\chi + \gamma) = g(\chi + \gamma + \chi),$$

and

$$g(2\chi - \gamma) = g(\chi - \gamma + \chi),$$

from (1), we obtain

$$g(2\chi + \gamma) = g(\chi + \gamma + \chi + \nu) \\ = 2g(\chi + \gamma) + 2g(\chi + \nu) - g(\gamma - \nu) \quad (6)$$

and

$$g(2\chi - \gamma) = g(\chi - \gamma + \chi - \nu) \\ = 2g(\chi - \gamma) + 2g(\chi - \nu) - g(\nu - \gamma). \quad (7)$$

Adding (6) and (7) and using (5), we arrive at

$$g(\chi + \gamma) + g(\chi + \nu) + g(\chi - \gamma) \\ + g(\chi - \nu) - g(\gamma - \nu) \\ = 4g(\chi) + g(\gamma + \nu). \quad (8)$$

Putting  $\gamma = \gamma + \nu + \varsigma$ , in (8), we obtain

$$g(\chi + \gamma + \nu + \varsigma) + g(\chi - \gamma - \nu - \varsigma) \\ = 4g(\chi) + g(\gamma + \varsigma) + g(\gamma + \varsigma + 2\nu) - g(\chi - \nu) - g(\chi + \nu).$$

( $\Leftarrow$ ) Substituting  $\nu = 0 = \varsigma$  in (2) gives the identity

$$g(\chi + \gamma) + g(\chi - \gamma) = 2g(\chi) + 2g(\gamma)$$

for all  $\chi, \gamma \in X_1$ , which implies that  $g$  is quadratic. The proof is completed.

Given a mapping  $g : X_1 \rightarrow X_2$ , we define

$$Dg(\chi, \gamma, \nu, \varsigma) = g(\chi + \gamma + \nu + \varsigma) \\ + g(\chi - \gamma - \nu - \varsigma) + g(\chi - \nu) \\ + g(\chi + \nu) - g(\gamma + \varsigma + 2\nu) \\ - g(\gamma + \varsigma) - 4g(\chi) \quad (9)$$

for all  $\chi, \gamma, \nu, \varsigma \in X_1$ , where  $X_1$  is a real vector space and  $X_2$  is a real Banach space.

We investigate the (HURS) of the (QFE)  $Dg(\chi, \gamma, \nu, \varsigma) = 0$ .

Let  $\phi : X_1^4 \rightarrow [0, \infty)$  be a function such that

$$\lim_{n \rightarrow \infty} \alpha_p^{2n} \phi\left(\frac{\chi}{\alpha_p^n}, \frac{\gamma}{\alpha_p^n}, \frac{\nu}{\alpha_p^n}, \frac{\varsigma}{\alpha_p^n}\right) = 0 \quad (10)$$

for all  $\chi, \gamma, \nu, \varsigma \in X_1$ , where  $\alpha_p = \frac{1}{2}$  for  $p=0$  and  $\alpha_p = 2$  for  $p=1$ .

**Theorem 2.2.** For  $g(0) = 0$  let  $g : X_1 \rightarrow X_2$  be a mapping in which there exists a function  $\phi : X_1^4 \rightarrow [0, \infty)$  such that

$$\|Dg(\chi, \gamma, \nu, \varsigma)\| \leq \phi(\chi, \gamma, \nu, \varsigma) \quad (11)$$

for all  $\chi, \gamma, \nu, \varsigma \in X_1$ . If there exists  $L(p) < 1$  such that

$$\varphi(\nu) \leq \frac{1}{\alpha_p^2} L \varphi(\alpha_p \nu) \quad (12)$$

for all  $\nu \in X_1$ , in which  $\varphi(\nu) = \phi(0, 0, \frac{\nu}{2}, 0)$ . Then there exists a unique quadratic mapping  $Q : X_1 \rightarrow X_2$  satisfying (2) and

$$\|g(\nu) - Q(\nu)\| \leq \frac{L^{1-p}}{1-L} \varphi(\nu) \quad (13)$$

for all  $\nu \in X_1$ .

**Proof.** Let  $M = \{h : X_1 \rightarrow X_2, h(0) = 0\}$  be a set and define the generalized metric on  $M$  :

$$d(h, f) = \inf \{K > 0 : \|h(\nu) - f(\nu)\| \leq K \phi(0, 0, \frac{\nu}{2}, 0), \forall \nu \in X_1\}.$$

Then  $(M, d)$  is complete.

Let  $\Lambda : M \rightarrow M$  be linear mapping such that

$$\alpha_p^2 g\left(\frac{\nu}{\alpha_p}\right) = \Lambda g(\nu)$$

for all  $\nu \in X_1$ . Then for all  $h, f \in M$ , we obtain

$$\|h(\nu) - f(\nu)\| \leq K \varphi(\nu), \nu \in X_1,$$

$$\left\| \alpha_p^2 h\left(\frac{\nu}{\alpha_p}\right) - \alpha_p^2 f\left(\frac{\nu}{\alpha_p}\right) \right\| \leq \alpha_p^2 K \varphi\left(\frac{\nu}{\alpha_p}\right),$$

$$\left\| \alpha_p^2 h\left(\frac{\nu}{\alpha_p}\right) - \alpha_p^2 f\left(\frac{\nu}{\alpha_p}\right) \right\| \leq L K \varphi(\nu).$$

Then, we obtain

$$d(\Lambda h, \Lambda f) \leq L d(h, f)$$

for all  $h, f \in M$ . So,  $\Lambda$  is a strictly self mapping on  $M$ .

Putting  $\chi = \gamma = \varsigma = 0$  in (11), we obtain

$$\|g(2\nu) - 4g(\nu)\| \leq \phi(0, 0, \nu, 0) \quad (14)$$

for all  $\nu \in X_1$ . For the case  $p=0$ , from (12), we get

$$\left\| g(\nu) - \frac{1}{4} g(2\nu) \right\| \leq \frac{1}{2^2} \varphi(2\nu) \leq L \varphi(\nu)$$

for all  $\nu \in X_1$ . That is,

$$d(g, \Lambda g) \leq L.$$

Letting  $\nu = \frac{\nu}{2}$  in (14), we get,

$$\left\| g(\nu) - 2^2 g\left(\frac{\nu}{2}\right) \right\| \leq \varphi(\nu)$$

for all  $\nu \in X_1$ . Using (12) with the case  $p=1$ , we get

$$d(g, \Lambda g) \leq L^p = 1.$$

For both situations, by Theorem 1.1, there exists a fixed point  $Q$  of  $\Lambda$  in  $M$  and from Theorem 1.1,  $d(\Lambda^n g, P) \rightarrow 0$  as  $n \rightarrow \infty$ . This implies that,

$$\lim_{n \rightarrow \infty} \alpha_p^{2n} g\left(\frac{\nu}{\alpha_p^n}\right) = Q(\nu) \quad (15)$$

holds for all  $\nu \in X_1$ .

Replacing  $\chi, \gamma, \nu, \varsigma$  by  $\frac{\chi}{\alpha_p^n}, \frac{\gamma}{\alpha_p^n}, \frac{\nu}{\alpha_p^n}, \frac{\varsigma}{\alpha_p^n}$  in (11)

respectively and multiplying by  $\alpha_p^{2n}$ , then from (10) and (15) we get

$$\begin{aligned} \|DQ(\chi, \gamma, \nu, \varsigma)\| &= \lim_{n \rightarrow \infty} \alpha_p^{2n} \left\| Dg\left(\frac{\chi}{\alpha_p^n}, \frac{\gamma}{\alpha_p^n}, \frac{\nu}{\alpha_p^n}, \frac{\varsigma}{\alpha_p^n}\right) \right\| \\ &\leq \lim_{n \rightarrow \infty} \alpha_p^{2n} \phi\left(\frac{\chi}{\alpha_p^n}, \frac{\gamma}{\alpha_p^n}, \frac{\nu}{\alpha_p^n}, \frac{\varsigma}{\alpha_p^n}\right) = 0 \end{aligned}$$

for all  $\chi, \gamma, \nu, \varsigma \in X_1$ . Hence  $Q$  satisfies the equation (2). So,  $Q$  is quadratic.

Because of  $Q$  is the unique fixed point of  $\Lambda$  in  $S = \{u \in M : d(g, u) < \infty\}$ ,  $Q$  satisfies in the following inequality

$$\|g(\nu) - Q(\nu)\| \leq K \varphi(\nu)$$

for all  $\nu \in X_1$  and some  $K > 0$ . From Theorem 1.1, we can write

$$d(g, Q) \leq \frac{1}{1-L} d(g, \Lambda g),$$

and we get

$$d(g, Q) \leq \frac{L^{1-p}}{1-L}.$$

The proof is completed.

**Corollary.** Let  $X_1$  and  $X_2$  be normed spaces. Let  $r \geq 0$  be given. Suppose that  $\delta > 0$  is fixed. Let  $g : X_1 \rightarrow X_2$  be a mapping such that

$$\|Dg(\chi, \gamma, \nu, \varsigma)\| \leq \delta (\|\chi\|^r + \|\gamma\|^r + \|\nu\|^r + \|\varsigma\|^r) \quad (16)$$

for all  $\chi, \gamma, \nu, \varsigma \in X_1$ . Then the inequality

$$\|g(\nu) - Q(\nu)\| \leq \frac{\delta}{2^r - 4} \|\nu\|^r$$

holds for all  $\nu \in X_1$ , in which  $r > 2$ , or

$$\|g(\nu) - Q(\nu)\| \leq \frac{\delta}{4 - 2^r} \|\nu\|^r$$

holds for all  $v \in X_1$ , where  $r < 2$ .

**Proof.** Let  $\phi(\chi, \gamma, \nu, \varsigma) = \delta(\|\chi\|^r + \|\gamma\|^r + \|\nu\|^r + \|\varsigma\|^r)$

for all  $\chi, \gamma, \nu, \varsigma \in X_1$ . Then  $L = 2^{2-r}$  for  $p = 1$  or  $L = \frac{1}{2^{2-r}}$  for  $p = 0$ , and we get desired results.

### 3. (HURS) of equation (2) with direct method

Here, we investigate (HURS) of equation (2) by using the direct method.

Let  $Dg(\chi, \gamma, \nu, \varsigma)$  defined as in (9).

**Theorem 3.1.** Let  $\phi: X_1^4 \rightarrow [0, \infty)$  be a mapping such that

$$\Psi(\chi, \gamma, \nu, \varsigma) = \sum_{j=0}^{\infty} \alpha_p^{2j} \phi\left(\frac{\chi}{\alpha_p^j}, \frac{\gamma}{\alpha_p^j}, \frac{\nu}{\alpha_p^j}, \frac{\varsigma}{\alpha_p^j}\right) < \infty \quad (17)$$

for all  $\chi, \gamma, \nu, \varsigma \in X_1$ . Assume  $g: X_1 \rightarrow X_2$  satisfies inequality (11) for all  $\chi, \gamma, \nu, \varsigma \in X_1$ . Then

$$\lim_{n \rightarrow \infty} \alpha_p^{2n} g\left(\frac{v}{\alpha_p^n}\right) = Q(v)$$

exists for each  $v \in X_1$  and describes unique quadratic mapping  $Q: X_1 \rightarrow X_2$  such that

$$\|g(v) - Q(v)\| \leq \alpha_p^{2(1-p)} \Psi(0, 0, \frac{v}{\alpha_p}, 0) \quad (18)$$

for all  $v \in X_1$ .

**Proof.** Since  $\Psi(0, 0, 0, 0) = \sum_{j=0}^{\infty} \alpha_p^{2j} \phi(0, 0, 0, 0) < \infty$ , setting  $\chi = \gamma = \varsigma = 0$  in (11) we have

$$\|4g(v) - g(2v)\| \leq \phi(0, 0, v, 0) \quad (19)$$

for all  $v \in X_1$ . Interchanging  $v$  with  $\frac{v}{2}$  in (19), we arrive at

$$\left\|4g\left(\frac{v}{2}\right) - g(v)\right\| \leq \phi\left(0, 0, \frac{v}{2}, 0\right). \quad (20)$$

Replacing  $v$  by  $\frac{v}{\alpha_p^{n-1}}$  and multiplying both sides by  $\alpha_p^{2(n-1)}$ , in formerly inequality, we get

$$\left\|4\alpha_p^{2(n-1)} g\left(\frac{v}{2\alpha_p^{n-1}}\right) - \alpha_p^{2(n-1)} g\left(\frac{v}{\alpha_p^{n-1}}\right)\right\| \leq \alpha_p^{2(n-1)} \phi\left(0, 0, \frac{v}{2\alpha_p^{n-1}}, 0\right)$$

(21)

for all  $v \in X_1$  and  $n \in N$ . Then from (20) and (21), we deduce that for  $p = 1$ ,

$$\left\|\alpha_p^{2n} g\left(\frac{v}{\alpha_p^n}\right) - g(v)\right\| \leq \sum_{k=0}^{n-1} \alpha_p^{2k} \phi\left(0, 0, \frac{v}{\alpha_p^{k+1}}, 0\right) \quad (22)$$

and

$$\left\|4g\left(\frac{v}{2}\right) - \alpha_p^{2(n-1)} g\left(\frac{v}{\alpha_p^{n-1}}\right)\right\| \leq \sum_{k=1}^n \alpha_p^{2k} \phi\left(0, 0, \frac{v}{\alpha_p^{k-1}}, 0\right)$$

(23)

for  $p = 0$ . Replacing  $v$  by  $2v$  in (23), we obtain

$$\left\|g(v) - \alpha_p^{2n} g\left(\frac{v}{\alpha_p^n}\right)\right\| \leq \frac{1}{4} \sum_{k=0}^{n-1} \alpha_p^{2k} \phi\left(0, 0, \frac{v}{\alpha_p^k}, 0\right) \quad (24)$$

for all  $v \in X_1$  and any positive integer  $n$ .

Now, we show that the sequence  $\{\alpha_p^{2n} g(\frac{v}{\alpha_p^n})\}$  is a Cauchy sequence. For any  $m \in N$ , by (22) we have

$$\begin{aligned} \left\|\alpha_p^{2(n+m)} g\left(\frac{v}{\alpha_p^{n+m}}\right) - \alpha_p^{2m} g\left(\frac{v}{\alpha_p^m}\right)\right\| &= \alpha_p^{2m} \left\|\alpha_p^{2n} g\left(\frac{v}{\alpha_p^{n+m}}\right) - g\left(\frac{v}{\alpha_p^m}\right)\right\| \\ &\leq \alpha_p^{2m} \sum_{k=0}^{n-1} \alpha_p^{2k} \phi\left(0, 0, \frac{v}{\alpha_p^{k+m+1}}, 0\right) \\ &\leq \frac{1}{4} \sum_{k=0}^{n-1} \alpha_p^{2(k+m+1)} \phi\left(0, 0, \frac{v}{\alpha_p^{k+m+1}}, 0\right). \end{aligned}$$

By condition (17), we obtain

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \alpha_p^{2(k+m+1)} \phi\left(0, 0, \frac{v}{\alpha_p^{k+m+1}}, 0\right) = 0,$$

for all  $v \in X_1$ . Consequently,  $\{\alpha_p^{2n} g(\frac{v}{\alpha_p^n})\}$  is a Cauchy sequence.

Similarly, we can also show that  $\{\alpha_p^{2n} g(\frac{v}{\alpha_p^n})\}$  is a Cauchy

sequence by considering of inequality (24).

Thus, we can set

$$Q(v) = \alpha_p^{2n} g\left(\frac{v}{\alpha_p^n}\right)$$

for all  $v \in X_1$ .

Now, we show that,  $Q$  is a solution of equation (2).

Replacing  $\chi, \gamma, \nu, \varsigma$  by  $\frac{\chi}{\alpha_p^n}, \frac{\gamma}{\alpha_p^n}, \frac{v}{\alpha_p^n}, \frac{\varsigma}{\alpha_p^n}$  in (11),

respectively.

If we multiply both sides of obtained equation by  $\alpha_p^{2n}$ , we get

$$\alpha_p^{2n} \left\| Dg\left(\frac{\chi}{\alpha_p^n}, \frac{\gamma}{\alpha_p^n}, \frac{\nu}{\alpha_p^n}, \frac{\varsigma}{\alpha_p^n}\right) \right\|$$

$$\leq \alpha_p^{2n} \phi\left(\frac{\chi}{\alpha_p^n}, \frac{\gamma}{\alpha_p^n}, \frac{\nu}{\alpha_p^n}, \frac{\varsigma}{\alpha_p^n}\right).$$

Since  $\lim_{n \rightarrow \infty} \alpha_p^{2n} \phi\left(\frac{\chi}{\alpha_p^n}, \frac{\gamma}{\alpha_p^n}, \frac{\nu}{\alpha_p^n}, \frac{\varsigma}{\alpha_p^n}\right) = 0$ , the mapping

$Q$  satisfies (2). From (22) we obtain

$$\lim_{n \rightarrow \infty} \left\| \alpha_p^{2n} g\left(\frac{\nu}{\alpha_p^n}\right) - g(\nu) \right\| \leq \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \alpha_p^{2k} \phi\left(0, 0, \frac{\nu}{\alpha_p^{k+1}}, 0\right).$$

We can also obtain above inequality using (24). So, above inequality holds both for  $p=1$  and for  $p=0$ .

Consequently, we get

$$\|g(\nu) - Q(\nu)\| \leq \alpha_p^{2(1-p)} \Psi\left(0, 0, \frac{\nu}{\alpha_p^p}, 0\right)$$

for all  $\nu \in X_1$ .

We should show that the uniqueness of  $Q$ . Suppose that there exists  $Q' : X_1 \rightarrow X_2$  such that  $Q(\nu)$  not equivalent to  $Q'(\nu)$ . Then

$$\|Q(\nu) - Q'(\nu)\| = \alpha_p^{2n} \left\| Q\left(\frac{\nu}{\alpha_p^n}\right) - Q'\left(\frac{\nu}{\alpha_p^n}\right) \right\|$$

$$\leq \alpha_p^{2n} \left( \left\| Q\left(\frac{\nu}{\alpha_p^n}\right) - f\left(\frac{\nu}{\alpha_p^n}\right) \right\| \right.$$

$$\left. + \left\| f\left(\frac{\nu}{\alpha_p^n}\right) - Q'\left(\frac{\nu}{\alpha_p^n}\right) \right\| \right)$$

$$\leq 2\alpha_p^{2(n+1-p)} \Psi\left(0, 0, \frac{\nu}{\alpha_p^{n+p}}, 0\right)$$

$$\leq \frac{2}{\alpha_p^{4p-2}} \alpha_p^{2(n+p)} \Psi\left(0, 0, \frac{\nu}{\alpha_p^{n+p}}, 0\right).$$

Since  $\lim_{n \rightarrow \infty} \alpha_p^{2(n+p)} \Psi\left(0, 0, \frac{\nu}{\alpha_p^{n+p}}, 0\right) = 0$ , we get  $Q(\nu) = Q'(\nu)$ ,

for all  $\nu \in X_1$ .

#### 4. Conclusion

A new quadratic functional equation was considered. The Hyers-Ulam Rassias stability of this quadratic equation was investigated. We benefited from fixed point method and direct method.

#### 5. References

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