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Research Article

A note on a transform to self-inverse sequences

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Abstract

The sequences which are fixed by the binomial transform are called self-inverse sequences. In this paper, an identity satisfied by Fibonacci numbers is modified to provide a transform which maps a specific subset of sequences to self-inverse sequences bijectively. The image of some classes of sequences under this transform are explicitly found which provides a new formulation and a class of examples of self-inverse sequences. A criterion for the solutions of some difference equations to be self-inverse is also given.

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1. Introduction

Let $\mathcal{A}$ denote the set of sequences $\mathbf{d} = \{d_n\}_{n\geq 0}$ over a field of characteristic zero. The binomial transform, $\mathcal{B}$ of a sequence $\mathbf{d} \in \mathcal{A}$ is defined as

$$\mathcal{B} : \mathcal{A} \to \mathcal{A},\ \mathcal{B}(\mathbf{d})_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k d_k,\ n \geq 0.$$

The ordinary generating function of a sequence $\mathbf{d} = \{d_n\}_{n\geq 0}$ is the formal power series $f(T) = \sum_{n=0}^{\infty} d_n T^n$. In this case we simple write

$$\mathbf{d} \sim f(T).$$

The generating function of a sequence $\mathbf{d}$ and of its binomial transform are related to each other; for any sequence $\mathbf{d}$,

$$\mathbf{d} \sim f(T) \implies \mathcal{B}(\mathbf{d}) \sim \frac{1}{1-T} f\left(\frac{-T}{1-T}\right)$$

(See [3] or [4]). It is easy to see that $\mathcal{B}^2(\mathbf{d}) = \mathbf{d}$, so that the eigenvalues of $\mathcal{B}$ are ± 1. The eigenspaces of $\mathcal{B}$, i.e. sequences for which $\mathcal{B}(\mathbf{d}) = \pm\,\mathbf{d}$ are of common interest. A sequence $\mathbf{d} \in \mathcal{A}$ is said to be self-inverse (or invariant under $\mathcal{B}$) if $\mathcal{B}(\mathbf{d}) = \mathbf{d}$. We denote the subset of self-inverse sequences by $\mathcal{A}^+$. Note that $\mathcal{A}^+$ is a subspace of $\mathcal{A}$. Typical examples of self-inverse sequences are

$$\{1/2^n\}_{n\geq 0},\ \{nF_{n-1}\}_{n\geq 0},\ \{(-1)^n B_n\}_{n\geq 0},$$

where F_n and B_n denote the n-th Fibonacci and Bernoulli numbers respectively with $F_0 = 0$. An extensive list of examples of both eigenspaces of $\mathcal{B}$ with related recurrence

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relations are proved in [4]. Also some congruences modulo prime powers involving both eigenspaces are given in [5]. Another interesting result is that $\mathcal{A}$ is equal to the direct sum of eigenspaces of $\mathcal{B}$ which can be proved by using the matrix representation of $\mathcal{B}$, [6]. In this paper, we shall restrict to eigenspace of $\mathcal{B}$ corresponding to the eigenvalue 1.

There is another convention for the binomial transform where the term $(-1)^k$ in the summation in $\mathcal{B}$ is omitted, but we will adopt the above convention. The reader is referred to [1] for the alternative convention and also a general exposition on transforms of integer sequences.

Let $\mathcal{A}_0$ denote the subspace of $\mathcal{A}$ consisting of sequences $\mathbf{d} = \{d_n\}_{n\geq 0}$ for which $d_0 = 0$. We denote the subspace of self-inverse sequences in $\mathcal{A}_0$ by $\mathcal{A}_0^+$. Note that for any $\mathbf{d} \in \mathcal{A}_0^+$, we necessarily have $d_0 = d_1 = 0$. Let $\mathcal{H}$ be the transform on $\mathcal{A}_0$ defined as

$$\begin{aligned} &\mathcal{H} : \mathcal{A}_0 \to \mathcal{A}_0, \\ &\mathcal{H}(\mathbf{c})_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k-1}{k-1} (-1)^k c_k, \ n \geq 2, \end{aligned}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Note that the binomial term corresponding to $k = 0$ is $\binom{n-1}{-1}$ which may be defined in different ways. But we don't need to care about this ambiguity since we apply $\mathcal{H}$ on sequences for which $c_0 = 0$. So we actually have that

$$\mathcal{H}(\mathbf{c})_0 = \mathcal{H}(\mathbf{c})_1 = 0, \ \mathcal{H}(\mathbf{c})_n = \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n-k-1}{k-1} (-1)^k c_k, \ n \geq 2, \ \mathbf{c} \in \mathcal{A}_0 \,. \tag{1.1}$$

The main result of this paper is the following theorem.

Theorem 1.1. *For any $\mathbf{c} \in \mathcal{A}_0$, the sequence $\{n\,\mathcal{H}(\mathbf{c})_n\}_{n\geq 0}$ is a self-inverse sequence. Moreover the map*

$$\mathcal{A}_0 \to \mathcal{A}_0^+, \ \mathbf{c} \mapsto \{n\,\mathcal{H}(\mathbf{c})_n\}_{n\geq 0}$$

is a bijection.

This result is a characterization of self-inverse sequences in $\mathcal{A}_0^+$. For example the sequence $\mathbf{c}$ for which $c_0 = 0$, $c_k = (-1)^k$ for $k \geq 1$ maps to $\{nF_{n-1}\}_{n\geq 0}$ where $\{F_n\}$ is the Fibonacci sequence (We take $F_{-1} = F_0 = 0$). It is possible to recover $\mathcal{A}^+$ from $\mathcal{A}_0^+$ since we have the following;

$$(0, 0, d_2, d_3, \ldots) \in \mathcal{A}_0^+ \ \text{ if and only if } \ (d_2, d_3 - d_2, d_4 - d_3, \ldots.) \in \mathcal{A}^+, \tag{1.2}$$

This follows by [4, Corollary 3.1-3.2].

The organization of the paper is as follows. In Section 2 we study the ordinary generating functions and as a result prove above theorem. The idea of the proof is expressing the difference operator as a conjugation by a differential operator applied on ordinary generating functions.

Section 3 is devoted to characterization of specific types of self-inverse sequences. We shall focus on two types of sequences. We find a basis for self-inverse sequences of *polynomial type* in $\mathcal{A}_0^+$. We also study sequences which satisfy specific recurrence relations.

2. Main theorem

We define the formal differential operator D on the ring of formal power series as $D = T\dfrac{d}{dT}$, explicitly if $f(T) = \sum_{n=0}^{\infty} d_n T^n$ then

$$Df(T) = \sum_{n=1}^{\infty} n d_n T^n.$$

By abuse of notation we may also use D to denote the corresponding transform on $\mathcal{A}$ and write $D(\mathbf{d}) = \{nd_n\}_{n\geq 0}$ for any $\mathbf{d} \in \mathcal{A}$. Clearly D does not have an inverse. But if we restrict D to $\mathcal{A}_0$, then D is invertible with inverse D^{-1} where

$$\begin{aligned} &D^{-1} : \mathcal{A}_0 \to \mathcal{A}_0, \\ &D^{-1}(\mathbf{d})_0 = 0,\ D^{-1}(\mathbf{d})_n = d_n/n \ \text{ for } \ n \geq 1 \end{aligned}$$

In terms of generating functions we have

$$f(T) = \sum_{n=1}^{\infty} d_n T^n \implies D^{-1} f(T) = \sum_{n=1}^{\infty} \frac{d_n}{n} T^n.$$

We define another transform $\mathcal{B}'$ on $\mathcal{A}_0$ as $\mathcal{B}' = D^{-1}\,\mathcal{B}\,D$, explicitly for any $\mathbf{d} \in \mathcal{A}_0$

$$\mathcal{B}'(\mathbf{d})_0 = 0,\ \mathcal{B}'(\mathbf{d})_n = \sum_{k=1}^{n} \binom{n-1}{k-1} d_k (-1)^k \ \text{ for } \ n \geq 1.$$

Equivalently, we can define $\mathcal{B}'$ by $\mathcal{B}'(\mathbf{d})_n = \mathcal{B}(\mathbf{d})_n - \mathcal{B}(\mathbf{d})_{n-1}$ for $n \geq 1$. This difference operator in relation to self-inverse sequences allows one to obtain formulas involving harmonic numbers and Stirling numbers, [2].

But we emphasize the relation $\mathcal{B}' = D^{-1}\,\mathcal{B}\,D$ on $\mathcal{A}_0$ which simply implies that

$$\mathcal{B}'(\mathbf{d}) = \mathbf{d} \ \text{ if and only if } \ \mathcal{B}(D(\mathbf{d})) = D(\mathbf{d}) \tag{2.1}$$

for $\mathbf{d} \in \mathcal{A}_0$. We say that $\mathbf{d}$ is invariant under $\mathcal{B}'$ if $\mathcal{B}'(\mathbf{d}) = \mathbf{d}$. So we shall study the invariant sequences under $\mathcal{B}'$ and then relate to $\mathcal{B}$. Note that if $\mathbf{d} \in \mathcal{A}_0$ is invariant under $\mathcal{B}$ (or $\mathcal{B}'$) then necessarily $d_0 = d_1 = 0$. We have a simple description of $\mathcal{B}'$ in terms of the ordinary generating functions.

Lemma 2.1. *Let $\mathbf{d} \in \mathcal{A}_0$ and $\mathbf{d} \sim f(T)$. Then*

$$\mathcal{B}'(\mathbf{d}) \sim f\left(\frac{-T}{1-T}\right),$$

where $\dfrac{1}{1-T} = \sum_{n=0}^{\infty} T^n$.

Proof. Recall that for any $\mathbf{a} \in \mathcal{A}$, if $\mathbf{a} \sim f(T)$ then

$$\mathcal{B}(\mathbf{a}) \sim \frac{1}{1-T} f\left(\frac{-T}{1-T}\right).$$

Since $D(\mathbf{d}) \sim Tf'(T)$ we have that

$$\mathcal{B}\,D(\mathbf{d}) \sim \frac{-T}{(1-T)^2} f'\left(\frac{-T}{1-T}\right).$$

But by chain rule we also see that

$$D\left[f\left(\frac{-T}{1-T}\right)\right] = \frac{-T}{(1-T)^2} f'\left(\frac{-T}{1-T}\right).$$

Applying D^{-1} we obtain

$$\mathcal{B}'(\mathbf{d}) = D^{-1}\,\mathcal{B}\,D(\mathbf{d}) \sim f\left(\frac{-T}{1-T}\right).$$

□

In particular, for any sequence $\mathbf{d} \sim f(T)$ we have that

$$\mathcal{B}'(\mathbf{d}) = (\mathbf{d}) \iff f(T) = f\left(\frac{-T}{1-T}\right).$$

So we shall investigate the formal power series which are *symmetric* in T and $-T/(1-T)$. The simplest nonzero power series symmetric in T and $-T/(1-T)$ is

$$T + \frac{-T}{1-T} = T\frac{-T}{1-T} = \frac{-T^2}{1-T}.$$

As a consequence any sequence in $\mathcal{A}_0$ with an ordinary generating function of the form $f\left(-T^2/(1-T)\right)$ for some $f(T)$ is invariant under $\mathcal{B}'$.

Example 2.2. Let $\mathbf{d} \sim f(T) = \ln^2(1-T)/2$ where $\ln(1-T) = -\sum_{n=1}^{\infty} \frac{T^n}{n}$, so that $\mathbf{d} \in \mathcal{A}_0$. We may easily verify that

$$f(T) = f\left(-T/(1-T)\right),$$

which implies that $\mathcal{B}'(\mathbf{d}) = \mathbf{d}$. But by computing the square of $\ln(1-T)$ we can also see that

$$f(T) = \sum_{k=2}^{\infty} \frac{H_{k-1}}{k} T^k,$$

where $H_{k-1} = 1 + 1/2 + 1/3 + ... + 1/(k-1)$ are the harmonic numbers. Then we have that $D(\mathbf{d}) = \{0, 0, H_1, H_2, ...\}$. By (2.1), $D(\mathbf{d})$ is invariant under $\mathcal{B}$, i.e. we have the recurrence

$$\sum_{k=2}^{n} \binom{n}{k} (-1)^k H_{k-1} = H_{n-1},\ n \geq 2.$$

Note that this fact can also be derived by applying (1.2) to the self-inverse sequence $\{1, 1/2, 1/3, ...\}$.

Theorem 2.3. *For any $\mathbf{c} \in \mathcal{A}_0$, the sequence $\{n\,\mathcal{H}(\mathbf{c})_n\}_{n\geq 0}$ is a self-inverse sequence. Moreover the map*

$$\mathcal{A}_0 \to \mathcal{A}_0^+,\ \mathbf{c} \mapsto \{n\,\mathcal{H}(\mathbf{c})_n\}_{n\geq 0}$$

is a bijection.

Proof. Let $\mathbf{c} \in \mathcal{A}_0$ and $\mathbf{c} \sim g(T)$. For the first part of the theorem by (2.1) it is enough to prove that $\mathcal{H}(\mathbf{c})$ is invariant under $\mathcal{B}'$. Let $\mathbf{d}$ be the sequence for which $\mathbf{d} \sim g(-T^2/(1-T))$. By Lemma 2.1 and the conclusion following it, $\mathbf{d}$ is invariant under $\mathcal{B}'$. Now we will show that indeed $\mathbf{d} = \mathcal{H}(\mathbf{c})$. Let $g(T) = \sum_{k=1}^{\infty} c_k T^k$. Then

$$\begin{aligned} g\left(\frac{-T^2}{1-T}\right) &= \sum_{k=1}^{\infty} c_k \left(\frac{-T^2}{1-T}\right)^k = \sum_{k=1}^{\infty} c_k (-1)^k T^{2k} \frac{1}{(1-T)^k} \\ &= \sum_{k=1}^{\infty} c_k (-1)^k \sum_{l=k}^{\infty} \binom{l-1}{k-1} T^{l+k}. \end{aligned}$$

In the last equality we use that $1/(1-T)^k = \sum_{l=k}^{\infty} \binom{l-1}{k-1} T^{l-k}$ for $k \geq 1$ which can be proven inductively. We set $n = l + k$ and obtain that

$$g\left(\frac{-T^2}{1-T}\right) = \sum_{k=1}^{\infty} c_k (-1)^k \sum_{n=2k}^{\infty} \binom{n-k-1}{k-1} T^n.$$

We interchange the indices k and n and deduce that

$$g\left(\frac{-T^2}{1-T}\right) = \sum_{n=2}^{\infty} \left(\sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n-k-1}{k-1} c_k (-1)^k\right) T^n,$$

which completes the proof of the first part of the theorem.

For the second part, first we note that the map $\mathbf{c} \mapsto \{n\,\mathcal{H}(c)_n\}_{n\geq 0}$ on $\mathcal{A}_0$ is linear. If $\mathcal{H}(\mathbf{c})(n) = 0$ for all $n \geq 2$ then by induction it follows that $c_k = 0$ for $k \geq 1$. So the map $\mathbf{c} \mapsto \{n\,\mathcal{H}(\mathbf{c})_n\}_{n\geq 0}$ on $\mathcal{A}_0$ is injective.

Now we prove the surjectivity. Again by (2.1) it is enough to show that for any sequence $\mathbf{d} \in \mathcal{A}_0$ which is invariant under $\mathcal{B}'$ there exists $\mathbf{c} \in \mathcal{A}_0$ such that $\mathcal{H}(\mathbf{c}) = \mathbf{d}$. Let $\mathbf{d}$ be invariant under $\mathcal{B}'$. Note that we necessarily have $d_0 = d_1 = 0$. Consider the subsequence $\{d_{2m}\}_{m\geq 1}$. We can uniquely determine $\{c_k\}_{k\geq 1}$ such that

$$d_{2m} = \sum_{k=1}^{m} \binom{2m-k-1}{k-1} c_k(-1)^k$$

recursively. Now we set $\mathbf{d}' = \mathcal{H}(\mathbf{c})$, so that $\mathbf{d}'$ is invariant under $\mathcal{B}'$ and $d_{2m} = d'_{2m}$ for $m \geq 0$. Recall that $\mathcal{B}' = D^{-1}\,\mathcal{B}\,D$, and that D and $\mathcal{B}$ are linear. So $\mathcal{B}'$ is also linear which implies that $\mathbf{d}'' = \mathbf{d}' - \mathbf{d}$ is also invariant under $\mathcal{B}'$. Explicitly

$$d''_n = \sum_{k=2}^{n} \binom{n-1}{k-1} d''_k(-1)^k, n \geq 2$$

where $d''_0 = d''_1 = d''_{2m} = 0$ for $m \geq 1$. Inductively it follows that $d''_n = 0$ also for all odd $n \geq 0$, and so

$$\mathbf{d} = \mathbf{d}' = \mathcal{H}(\mathbf{c}).$$

This completes the proof of surjectivity. □

3. Examples and applications

The well-known examples of invariant sequences under $\mathcal{B}$ have been given in Section 1. A convenient reference for other examples is [4]. Here we will focus on two specific type of self-inverse sequences using results of Section 2.

First we need to recall a difference operator required in this section. We define the (forward) difference operator Δ as

$$(\Delta\,\mathbf{d})(n) = d_{n+1} - d_n, \quad n \geq 0$$

for any sequence $\mathbf{d} = \{d_n\}_{n\geq 0}$. If f is a function on $\mathbb{N}$, we may write (Δf) for $(\Delta\,\mathbf{d})$ where $\mathbf{d} = \{f(n)\}_{n\geq 0}$, i.e. $(\Delta f)(n) = f(n+1) - f(n)$ for $n \geq 0$.

We say that a sequence $\mathbf{d} \in \mathcal{A}_0$ is of *polynomial type* if there exists a polynomial $f(x)$ and $N \in \mathbb{N}$ such that $d_n = f(n)$ for all $n \geq N$. In this case we say that $\mathbf{d}$ is *associated* to $f(x)$. The existence of invariant sequences of polynomial type is already known. Indeed, if $f(x)$ is a polynomial of degree r then the sequence $\{d_n\}_{n\geq 0}$ defined as

$$d_n = (-1)^n(\Delta^n f)(0) + f(n),\ n \geq 0$$

is an invariant sequence of polynomial type, [6, Theorem 3.2]. Note that in this case $d_n = f(n)$ for $n > r$. In particular, for any $r \geq 1$ the sequence

$$d_n = (-1)^n n! S(r,n) + n^r,\ n \geq 0 \tag{3.1}$$

is in $\mathcal{A}_0^+$ for which $d_n = n^r$ if $n > r$ where $S(r,n)$ denotes the Stirling numbers of the second kind, [6, Example 3.3]. We say that a sequence $\mathbf{d} = \{d_0, d_1, ...\}$ is finitely supported if there exists $n \in \mathbb{N}$ such that $d_n = 0$ for $n \geq N$.

Lemma 3.1. *The only finitely supported sequence in $\mathcal{A}^+$ is the zero sequence.*

Proof. For an arbitrary sequence $\mathbf{d}$, if $\mathbf{d} \sim f(T)$ then

$$\mathcal{B}(\mathbf{d}) \sim \frac{1}{1-T}\, f\left(\frac{-T}{1-T}\right).$$

If $\mathbf{d}$ is finitely supported then $f(T)$ is a polynomial. Additionally if $\mathbf{d} \in \mathcal{A}^+$ then $f(T) = \dfrac{1}{1-T} f\left(\dfrac{-T}{1-T}\right)$. But by considering the degree of $f(T)$ we see that this is possible if and only if $f(T) = 0$. □

Let $\mathbf{d}$ and $\mathbf{d}'$ be of polynomial type, with $d_n = d'_n = f(n)$ for $n \geq N$ for some polynomial f of degree r satisfying $f(0) = 0$. Then $\{d_n - d'_n\}_{n\geq 0}$ is also of polynomial type. But it is also a finitely supported sequence in $\mathcal{A}_0^+$, so it must be the zero sequence by Lemma 3.1. Hence there is a unique self-inverse sequence associated to any polynomial. Since $\{x^r\}_{r\geq 1}$ is a basis for the space of polynomials with zero constant term we have the following result.

Corollary 3.2. *The sequences in (3.1) for $r \geq 1$ form a basis for the subspace of self-inverse sequences of polynomial type in $\mathcal{A}_0^+$.*

Now we give another basis using the transform $\mathcal{H}$.

Theorem 3.3. *Let $l \in \mathbb{N}$ with $l \geq 1$ and f_l be the function on $\mathbb{N}$ defined as*

$$f_l(n) = \begin{cases} 0, & n < 2l \\ \binom{n-l-1}{l-1}, & n \geq 2l \end{cases}$$

Then $\{nf_l(n)\}_{n\geq 0} \in \mathcal{A}_0^+$. Conversely, let $\mathbf{d} = \{d_n\}_{n\geq 0} \in \mathcal{A}_0^+$ be of polynomial type. Suppose that $d_n = f(n)$ for $n \geq N$ where f is a polynomial of degree r satisfying $f(0) = 0$. Then there exist unique $\alpha_1, \alpha_2, ..., \alpha_r$ such that

$$\mathbf{d} = \left\{ n \sum_{l=1}^{r} \alpha_l f_l(n) \right\}_{n\geq 0}$$

Proof. Note that for a fixed $l \geq 1$, $\{nf_l(n)\}_{n\geq 0}$ is of polynomial type. Let $\mathbf{c}^{(l)} \in \mathcal{A}_0$ be the sequence for which

$$\mathbf{c}_k^{(l)} = \begin{cases} 0, & k \neq l \\ (-1)^l, & k = l \end{cases}$$

Then we have $\{f_l(n)\}_{n\geq 0} = \mathcal{H}(\mathbf{c}^{(l)})$. So the first part follows by Theorem 2.3.

To prove the converse, take any $\mathbf{d}$ and f satisfying the hypothesis of the theorem. Without loss of generality we may assume that $N \geq 2r$. For $l \geq 1$, the degree of the polynomial $x\binom{x-l-1}{l-1}$ is l, so in particular, the set

$$\left\{ x\binom{x-l-1}{l-1} \right\}_{l\geq 1}$$

forms a basis for the space of polynomials with no constant term. Since $f(0) = 0$ there exist unique $\alpha_1, \alpha_2, ..., \alpha_r$ such that

$$f(x) = x \sum_{l=1}^{r} \alpha_l \binom{x-l-1}{l-1}.$$

Now for $n \geq N$, we have $d_n = f(n) = \sum_{l=1}^{r} \alpha_l \, n f_l(n)$. Also both of the sequences

$$\{d_n\}_{\geq 0} \text{ and } \left\{ n \sum_{l=1}^{r} \alpha_l \, f_l(n) \right\}_{n\geq 0}$$

are invariant under $\mathcal{B}$. So the difference

$$\left\{ d_n - n \sum_{l=1}^{r} \alpha_l \, f_l(n) \right\}_{n\geq 0}$$

is a finitely supported sequence in $\mathcal{A}_0^+$ which is necessarily the zero sequence by Lemma 3.1. □

So the set

$$\{\{nf_l(n)\}_{n\geq 0}\,|\,l\geq 1\}$$

is also a basis for the self-inverse sequences of polynomial type in $\mathcal{A}_0$. We also see that $\mathcal{H}$ maps the finitely supported sequence in $\mathcal{A}_0$ to self-inverse sequences of polynomial type in $\mathcal{A}_0$ bijectively.

Example 3.4. $l = 1$ produces the sequence $d_0 = d_1 = 0$, $d_n = n$ for $n \geq 2$. This elementary case is already known, [4]. Similarly the invariant sequence corresponding to $l = 2$ is $d_0 = d_1 = d_2 = d_3 = 0$ and $d_n = n(n-3)$ for $n \geq 4$.

Now we consider another type of self-inverse sequences which generalizes the Fibonacci sequence. First we need to introduce an operator V on $\mathcal{A}_0$. For any $\mathbf{c} \in \mathcal{A}_0$ we define $V(\mathbf{c})$ as

$$V(\mathbf{c})_0 = 0,\ V(\mathbf{c})_n = -c_{n+1} \text{ for } n \geq 1$$

i.e. if $\mathbf{c} = (0, c_1, c_2, ...)$ then $V(c) = (0, -c_2, -c_3, ...)$. So the L-th power of V is

$$V^L(\mathbf{c}) = (0, (-1)^L c_{1+L}, (-1)^L c_{2+L}, ...)$$

The images of the sequences $\mathbf{c}$ and $V(\mathbf{c})$ under $\mathcal{H}$ are related to each other.

Proposition 3.5. *For any $\mathbf{c} \in \mathcal{A}_0$ and $n \geq 1$ we have that*

$$\mathcal{H}(V^L(\mathbf{c}))_n = (\Delta^L\,\mathcal{H}(\mathbf{c}))(L+n)$$

Proof. We use induction on L. We set $\mathbf{d} = \mathcal{H}(\mathbf{c})$ and $\mathbf{d}^{(L)} = \mathcal{H}(V^L(\mathbf{c}))$ to simplify the notation. For $L = 1$ the problem reduces to showing that $d_n^{(1)} = d_{n+2} - d_{n+1}$ for $n \geq 1$. This holds for $n = 1$ as $\mathbf{d}$ is self-inverse and $d_0 = d_1 = 0$.

For even n we set $n = 2m$ for $m \geq 2$. Then

$$\begin{aligned}
d_{2m} - d_{2m-1} &= \sum_{k=1}^{\lfloor 2m/2\rfloor}\binom{2m-k-1}{k-1}(-1)^k c_k - \sum_{k=1}^{\lfloor (2m-1)/2\rfloor}\binom{2m-k-2}{k-1}(-1)^k c_k\\
&= (-1)^m c_m + \sum_{k=2}^{m-1}\binom{2m-k-2}{k-2}(-1)^k c_k\\
&= (-1)^{m-1}(-c_m) + \sum_{k=1}^{m-2}\binom{2m-k-3}{k-1}(-1)^k(-c_{k+1})\\
&= \sum_{k=1}^{\lfloor (2(m-1)/2\rfloor}\binom{2(m-1)-k-1}{k-1}(-1)^k(-c_{k+1}) = d_{2m-2}^{(1)}
\end{aligned}$$

In a similar way it also follows that $d_{2m+1} - d_{2m} = d_{2m-1}^{(1)}$ for $m \geq 2$ which completes the proof of the claim for $L = 1$.

Now assume that $d_n^{(L-1)} = (\Delta^{L-1}\,\mathbf{d})(L-1+n)$. Since $\mathbf{d}^{(L)} = \mathcal{H}(V^L(\mathbf{c})) = \mathcal{H}(V(V^{L-1}(\mathbf{c})))$ by using the case $L = 1$ and the induction hypothesis we obtain that

$$\begin{aligned}
d_n^{(L)} &= \mathcal{H}(V^{L-1}(\mathbf{c}))_{n+2} - \mathcal{H}(V^{L-1}(\mathbf{c}))_{n+1} = d_{n+2}^{(L-1)} - d_{n+1}^{(L-1)}\\
&= (\Delta^{L-1}\,\mathbf{d})(L+1+n) - (\Delta^{L-1}\,\mathbf{d})(L+n) = (\Delta^L\,\mathbf{d})(L+n).
\end{aligned}$$

□

Corollary 3.6. *Let $\mathbf{d} \in \mathcal{A}_0$ be invariant under $\mathcal{B}'$. Then the sequence $\mathbf{d}' = (d_n)_{n\geq 0}$ defined as*

$$d'_n = \begin{cases} 0, & n = 0\\ (\Delta^L\,\mathbf{d})(L+n), & n > 0\end{cases}$$

is also invariant under $\mathcal{B}'$. Equivalently, if $\{nd_n\}_{n\geq 0}$ is a self-inverse sequence in $\mathcal{A}_0$ then so is $\{n(\Delta^L\,\mathbf{d})(L+n)\}_{n\geq 0}$

Proof. By Theorem 2.3, there exists $\mathbf{c} \in \mathcal{A}_0$ such that $\mathbf{d} = \mathcal{H}(\mathbf{c})$. Then the result follows by Proposition 3.5. $\square$

We may compare this result with [6]; if we set $a_n = nd_n$ in the notation of the above corollary, it follows by Theorem 3.4 of [6] that $\{(\Delta^L \mathbf{a})(L+n)\}_{n\geq 0}$ is also self-inverse. But $\{(\Delta^L \mathbf{a})(L+n)\}_{n\geq 0}$ and $\{n(\Delta^L \mathbf{d})(L+n)\}_{n\geq 0}$ are obviously distinct sequences.

A related fact is that a sequence $\{nd_{n-1}\}$ for which

$$d_0 = 0, d_1 = 1 \text{ and } (\Delta^1 \mathbf{d})(1+n) = td_n, n \geq 1$$

where t is an indeterminate is self-inverse, [4, Example 5 of Section 2]. In other words by (2.1) the sequence $(0, 0, d_1, d_2, ...)$ is invariant under $\mathcal{B}'$. It is worthwhile to extend this fact to higher order difference equations $(\Delta^L \mathbf{d})(L+n) = d_n$ for $L \geq 2$. Consider a sequence $\mathbf{d} \in \mathcal{A}_0$ which satisfies the difference equation

$$(\Delta^L \mathbf{d})(L+n) = d_n,\ n \geq 1.$$

Then $\mathbf{d}$ is uniquely determined by the initial conditions $d_1, d_2, ..., d_{2L}$. It turns out that if the initial conditions on $d_0, d_1, d_2, ..., d_{2L}$ obey the invariance under $\mathcal{B}'$ then $\mathbf{d}$ is itself invariant under $\mathcal{B}'$.

Theorem 3.7. *Let $\mathbf{d} \in \mathcal{A}_0$, $L \in \mathbb{Z}$ with $L \geq 1$ and t be an indeterminate. Suppose that*

$$(\Delta^L \mathbf{d})(L+n) = td_n, n \geq 1.$$

If $\mathcal{B}'(d)_n = d_n$ for $0 \leq n \leq 2L$ then $\mathcal{B}'(\mathbf{d}) = \mathbf{d}$.

Proof. Note that by assumption $d_0 = d_1 = 0$. By using all even integers n between 1 and $2L$ we can recursively find $c_1, c_2, ..., c_L$ such that

$$d_n = \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n-k-1}{k-1} (-1)^k c_k,\ 1 \leq n \leq 2L,\ n: \text{ even}$$

By hypothesis $\mathcal{B}'(\mathbf{d})_n = d_n$ for all n between 0 and $2L$. So it follows that for any extension of $(0, c_1, c_2, ...c_L)$ to an infinite sequence $\mathbf{c} = (0, c_1, c_2, ...c_L, ...)$ we have that $\mathcal{H}(\mathbf{c})_n = d_n$ for all odd $n \in [0, 2L]$. In particular, we can choose $\mathbf{c}$ so that $V^L(\mathbf{c}) = t\,\mathbf{c}$. Hence by Proposition 3.5 and linearity we see that

$$t\,\mathcal{H}(\mathbf{c})_n = \mathcal{H}(t\,\mathbf{c})_n = \mathcal{H}(V^L(\mathbf{c}))_n = (\Delta^L \mathcal{H}(\mathbf{c}))(L+n),\ 1 \leq n \leq 2L$$

Now let $\mathbf{d}' = \mathcal{H}(\mathbf{c}) - \mathbf{d}$. Since the mapping $\mathbf{a} \mapsto \{(\Delta^L \mathbf{a})(L+n)\}$ is linear, we see that $\mathbf{d}'$ satisfies the difference equation

$$(\Delta^L \mathbf{d}')(L+n) = td'_n, n \geq 1$$

with the initial conditions $d'_0 = d'_1 = ... = d'_{2L} = 0$. So $\mathbf{d}'$ must be the zero sequence. $\square$

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