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Research Article

The Hermite-Hadamard inequalities for p-convex functions

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Abstract

In this paper, the Hermite-Hadamard inequality for $p-$convex function is provided. Some integral inequalities for them are also presented. Also, based on the integral and double integral of $p-$convex sets, the new functions are defined and under certain conditions, $p-$convexity of these functions are shown. Some inequalities for these functions are expressed.

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1. Introduction

The inequalities are very useful tools in almost all branches of mathematics. They have been sometimes a control mechanism or filtration for mathematical facts to be proved, sometimes they impose constraints, and sometimes the sets such as the feasible sets in constrained optimization problems are expressed by inequalities. The inequalities have been indispensable tools to natural sciences as well as all branches of mathematics to handle various problems. In many engineering sciences, they are especially used in estimating error bounds or in limiting physical variables to suitable boundaries in algorithms [21] the Chebyshev and the Markov inequalities in stochastic analysis and probability [12, 18], the Hadamard inequality in matrix theory [17], the Lyapunov inequality in control theory, which is inevitable for current artificial intelligent based technology [26], variational inequalities, which is frequently used at handling equilibrium issues in economics [16] can be given as the most well-known examples.

The existence of many inequalities are connected with the convexity. The convexity of a set is defined as follows:

Let X be a set in a vector space V over real number field $\mathbb{R}$. X is said to be a convex set, if

$$\lambda x + \mu y \in X$$

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for all $x, y \in X$ and $\lambda, \mu \in \mathbb{R}$ with $\lambda+\mu=1$, which states that each line segment connecting for any elements of X is contained by the set X.

At the beginning, the geometric aspect of convex bodies (compact nonempty convex sets) caused to obtain many inequalities such as the Brunn-Minkowski and Blashcke Santalo inequalities related to the volumes of sum and product of two convex bodies respectively. Later, convexity has been extended for different mathematical structures equipped with different operations such as posets, lattices, metric spaces etc [4,13,20].

Later, the convexity of a function is defined as follows:

Let X be a convex set and $f : X \to \mathbb{R}$. f is said to be a real valued convex set If

$$f(\lambda x + \mu y) \leq \lambda f(x) + \mu f(y)$$

for all $x, y \in X$ and $\lambda, \mu \in \mathbb{R}$ with $\lambda + \mu = 1$.

Also functional convexity gave birth to a large number of inequalities such as Jensen's inequality and the Hermite-Hadamard inequalities, which are the most leading two inequalities based on convexity.

The theoretical and applied developments leads to the new kind of convexities such as $\mathbb{B}-$convexity, $\mathbb{B}^{-1}-$convexity, $p-$convexity etc [2,5,15,19]. Many types of inequalities obtained for classical convex functions are studied and extended for new convexity types. By far the most modified one for new classes is the Hermite-Hadamard inequalities. A great number of studies on it can be seen in [1,6–11,14,23–25] and references therein.

In this work, the Hermite-Hadamard type inequality for p-convex functions is expressed, then inequality relations involving the right and left part of it are given. Finally, new functions via integral and double integral of p-convex functions are defined. p-convexity of them and some related inequalities are given.

2. Preliminaries

Throughout the paper, $\mathbb{R}, \mathbb{R}_+, \mathbb{Z}_+, \mathbb{R}^n$ denote the set of real numbers, the set of nonnegative real numbers, the set of positive integers and $n-$dimensional Euclidean space, respectively.

Definition 2.1 ([19])**.** Let U be a subset of $\mathbb{R}^n$ and $0 < p \leq 1$. U is called $p-$convex set if

$$tx + sy \in U$$

for all $x, y \in U$ and $t, s \in [0,1]$ such that $t^p + s^p = 1$.

Since we study on $\mathbb{R}_+$, let us express a related proposition.

Proposition 2.2 ([3])**.** *For $a > 0$, $[0, a)$ is a p-convex set.*

Definition 2.3 ([19])**.** Let $U \subseteq \mathbb{R}^n$ be a $p-$convex set and $f : U \to \mathbb{R}$. If for all $x, y \in U$ and $t, s \in [0,1]$ such that $t^p + s^p = 1$,

$$f(tx + sy) \leq tf(x) + sf(y), \tag{2.1}$$

then f is said to be p-convex function.

Inequality (2.1) can be expressed in terms of one parameter in two ways we often use in results. First, since $t^p + s^p = 1$, hence, $s = (1 - t^p)^{\frac{1}{p}}$, we can write

$$f(tx + (1 - t^p)^{\frac{1}{p}} y) \leq tf(x) + (1 - t^p)^{\frac{1}{p}} f(y).$$

Second, replacing the condition $t^p + s^p = 1$ with $\left(t^{\frac{1}{p}}\right)^p + \left(s^{\frac{1}{p}}\right)^p = 1$ and solving for s in the definition, we have

$$f(t^{\frac{1}{p}} x + (1 - t)^{\frac{1}{p}} y) \leq t^{\frac{1}{p}} f(x) + (1 - t)^{\frac{1}{p}} f(y).$$

The Hermite-Hadamard inequality for convex functions is given as follows:

Let $f:[a,b]\to\mathbb{R}$ be an integrable convex function. Then

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}. \tag{2.2}$$

This theorem states that the average value of a function on closed interval can be squeezed between the image of the average of boundary points under the function and the average of the images of boundary points under function.

Two of the tools used in some of the results below are the gamma and beta functions and some relations. These functions are defined as follows, respectively:

$$\Gamma(\alpha)=\int_0^\infty e^{-x}x^{\alpha-1}dx \text{ for } \alpha>0 \quad , \quad B(\alpha,\beta)=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx \quad \text{for } \alpha,\beta>0.$$

Beta and gamma functions have the following properties:

For $x,y>0$ and $n\in\mathbb{Z}_+$,

$$B(x,y)=B(y,x),\;\; B(x+1,y)=\frac{x}{x+y}B(x,y)$$

and

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\quad \Gamma(n)=(n-1)!.$$

Now, let us give an inequality on gamma function:

Theorem 2.4 ([22]). *Let $x>0$ and $0<a<1$. Then*

$$\left(\frac{x}{x+a}\right)^{1-a}\leq\frac{\Gamma(x+a)}{x^a\Gamma(x+1)}\leq 1. \tag{2.3}$$

3. Main results

Theorem 3.1. *Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be an integrable $p-$convex function For $a,b\in\mathbb{R}_+$ with $a<b$, the following inequality holds*

$$2^{\frac{1}{p}-1}f(\frac{a+b}{2^{\frac{1}{p}}})(b-a)\leqslant\int_a^b f(x)dx\leqslant\frac{1}{2p}\left\{p\left[bf(b)+af(a)\right]+\left[bf(a)+af(b)\right]B(\frac{1}{p},\frac{1}{p})\right\}.$$

Proof. First we show the right part of the inequality. By changing variable $x=t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a$, we have

$$\int_a^b f(x)dx=\frac{1}{p}\int_0^1 f(t^{\frac{1}{p}}b+(1-t)^{\frac{1}{p}}a)(bt^{\frac{1}{p}-1}-a\,(1-t)^{\frac{1}{p}-1})dt.$$

From the $p-$convexity of f and the nonnegativity of a and b,

$$\begin{aligned}\int_a^b f(x)dx&\leq\frac{1}{p}\int_0^1\left\{\left[t^{\frac{1}{p}}f(b)+(1-t)^{\frac{1}{p}}f(a)\right]\left[bt^{\frac{1}{p}-1}+a\,(1-t)^{\frac{1}{p}-1}\right]\right\}dt\\&\leq\frac{1}{p}\left\{f(b)\int_0^1\left[bt^{\frac{2}{p}-1}+at^{\frac{1}{p}}\,(1-t)^{\frac{1}{p}-1}\right]dt+f(a)\int_0^1\left[b(1-t)^{\frac{1}{p}}t^{\frac{1}{p}-1}+a(1-t)^{\frac{2}{p}-1}\right]dt\right\}.\end{aligned}$$

Using the beta function and its properties, we obtain

$$\int_a^b f(x)dx \le \frac{1}{p}\left\{\frac{p}{2}bf(b) + af(b)B(\frac{p+1}{p},\frac{1}{p}) + bf(a)B(\frac{1}{p},\frac{p+1}{p}) + \frac{p}{2}af(a)\right\}$$
$$= \frac{1}{2}\left[bf(b)+af(a)\right] + \frac{1}{2p}\left[bf(a)+af(b)\right]B(\frac{1}{p},\frac{1}{p}).$$

For the first part of the inequality, from the $p-$convexity of f, for all $x, w > 0$, we have

$$f(\frac{x+w}{2^{\frac{1}{p}}}) \leqslant \frac{f(x)+f(w)}{2^{\frac{1}{p}}}.$$

Let $x = ta + (1-t)b$ and $w = tb + (1-t)a$ for $t \in (0,1]$. Then

$$f(\frac{a+b}{2^{\frac{1}{p}}}) \leqslant \frac{f(ta+(1-t)b) + f((1-t)a+tb)}{2^{\frac{1}{p}}}.$$

Integrating both side and using the fact that

$$\int_0^1 f\left(ta + (1-t)\,b\right)dt = \int_0^1 f\left(tb + (1-t)\,a\right)dt$$

one can have

$$2^{\frac{1}{p}-1} f(\frac{a+b}{2^{\frac{1}{p}}}) \leqslant \frac{1}{b-a}\int_a^b f(x)dx.$$

□

Theorem 3.2. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be an integrable $p-$convex function. For $a, b \in \mathbb{R}_+$ with $a < b$, the following inequality holds:*

$$\int_0^1 f\left((1-t)^{\frac{1}{p}}a + t^{\frac{1}{p}}b\right)\left[1 + (1-t^p)^{\frac{1}{p}-1}t^{\frac{1}{p}}\right]dt \le \left(1 + \frac{\Gamma^2(\frac{1}{p})}{p\Gamma(\frac{2}{p})}\right)\left[\frac{f(a)+f(b)}{2}\right]. \tag{3.1}$$

Proof. From the p-convexity of f, for $t \in [0,1]$ and $a, b \in \mathbb{R}_+$ with $a < b$, we can write

$$f\left(ta + (1-t^p)^{\frac{1}{p}}b\right) \le tf(a) + (1-t^p)^{\frac{1}{p}}f(b)$$

and

$$f\left((1-t^p)^{\frac{1}{p}}a + tb\right) \le (1-t^p)^{\frac{1}{p}}f(a) + tf(b).$$

By summing these inequalities side by side, it is derived that

$$f\left(ta + (1-t^p)^{\frac{1}{p}}b\right) + f\left((1-t^p)^{\frac{1}{p}}a + tb\right) \le \left[f(a)+f(b)\right]\left[(1-t^p)^{\frac{1}{p}} + t\right].$$

Using the definite integration on $[0,1]$ yields

$$\int_0^1 f\left(ta + (1-t^p)^{\frac{1}{p}}b\right)dt + \int_0^1 f\left((1-t^p)^{\frac{1}{p}}a + tb\right)dt \le \left[f(a)+f(b)\right]\int_0^1 \left[(1-t^p)^{\frac{1}{p}} + t\right]dt. \tag{3.2}$$

By changing variable $v = (1-t^p)^{\frac{1}{p}}$ at first integral of right hand side of the inequality above, $t^p = 1 - v^p$, $v \in [0,1]$ and $t^{p-1}dt = -v^{p-1}dv$, hence

$$\int_0^1 f\left(ta + (1-t^p)^{\frac{1}{p}}b\right)dt = -\int_1^0 f\left((1-v^p)^{\frac{1}{p}}a + vb\right)\left[\frac{v}{(1-v^p)^{\frac{1}{p}}}\right]^{p-1}dv$$
$$= \int_0^1 f\left((1-t^p)^{\frac{1}{p}}a + tb\right)(1-t^p)^{\frac{1}{p}-1}t^{p-1}dt.$$

Using this equality, we have

$$\int_0^1 f\left(ta+(1-t^p)^{\frac{1}{p}}\, b\right) dt + \int_0^1 f\left((1-t^p)^{\frac{1}{p}}\, a+tb\right) dt$$
$$= \int_0^1 f\left((1-t^p)^{\frac{1}{p}}\, a+tb\right)\left[1+(1-t^p)^{\frac{1}{p}-1}t^{p-1}\right] dt.$$

Since

$$\int_0^1 \left[(1-t^p)^{\frac{1}{p}}+t\right] dt = \frac{1}{2}\left(1+\frac{\Gamma^2(\frac{1}{p})}{p\Gamma(\frac{2}{p})}\right),$$

we have

$$\int_0^1 f\left((1-t)^{\frac{1}{p}}\, a+t^{\frac{1}{p}}b\right)\left[1+(1-t^p)^{\frac{1}{p}-1}t^{p-1}\right] dt \le \left(1+\frac{\Gamma^2(\frac{1}{p})}{p\Gamma(\frac{2}{p})}\right)\left[\frac{f(a)+f(b)}{2}\right].$$

□

The upper bound for the integral at the left side of (3.1) is given in terms gamma function. We can find the upper bound without gamma function for $\frac{1}{p} \notin \mathbb{Z}_+$.

Theorem 3.3. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be an integrable $p-$convex function and $\frac{1}{p} \notin \mathbb{Z}_+$. For $a, b \in \mathbb{R}_+$ with $a < b$, the following inequality holds*

$$\int_0^1 f\left((1-t)^{\frac{1}{p}}\, a+t^{\frac{1}{p}}b\right)\left[1+(1-t^p)^{\frac{1}{p}-1}t^{\frac{1}{p}}\right] dt \le \psi(p)\left[\frac{f(a)+f(b)}{2}\right]$$

where

$$\psi(p) = 1+\frac{\left\lfloor\frac{1}{p}\right\rfloor^{\frac{2}{p}-2\left\lfloor\frac{1}{p}\right\rfloor}\left(\left\lfloor\frac{1}{p}\right\rfloor!\right)^2\left(\frac{2}{p}\right)^{\frac{p-2}{p}+\left\lfloor\frac{2}{p}\right\rfloor}}{p\left\lfloor\frac{2}{p}\right\rfloor\left\lfloor\frac{2}{p}\right\rfloor!}$$

and $\lfloor . \rfloor$ denotes the greatest integer value.

Proof. By $x = \left\lfloor\frac{1}{p}\right\rfloor$ and $a = \frac{1}{p} - \left\lfloor\frac{1}{p}\right\rfloor$ at the right side of (2.3), we have

$$\Gamma(\frac{1}{p}) \le \left\lfloor\frac{1}{p}\right\rfloor^{\frac{1}{p}-\left\lfloor\frac{1}{p}\right\rfloor}\left(\left\lfloor\frac{1}{p}\right\rfloor\right)!\,.$$

By using (2.3) also for $x = \left\lfloor\frac{2}{p}\right\rfloor$ and $a = \frac{2}{p} - \left\lfloor\frac{2}{p}\right\rfloor$,

$$\frac{\left\lfloor\frac{2}{p}\right\rfloor\left(\left\lfloor\frac{2}{p}\right\rfloor\right)!}{\left(\frac{2}{p}\right)^{\frac{p-2}{p}+\left\lfloor\frac{2}{p}\right\rfloor}} \le \Gamma(\frac{2}{p}) \le \left\lfloor\frac{2}{p}\right\rfloor^{\frac{2}{p}-\left\lfloor\frac{2}{p}\right\rfloor}\left(\left\lfloor\frac{2}{p}\right\rfloor\right)!\,.$$

So

$$1+\frac{\left[\Gamma(\frac{1}{p})\right]^2}{p\left[\Gamma(\frac{2}{p})\right]} \le 1+\frac{\left\lfloor\frac{1}{p}\right\rfloor^{\frac{2}{p}-2\left\lfloor\frac{1}{p}\right\rfloor}\left(\left\lfloor\frac{1}{p}\right\rfloor!\right)^2\left(\frac{2}{p}\right)^{\frac{p-2}{p}+\left\lfloor\frac{2}{p}\right\rfloor}}{p\left\lfloor\frac{2}{p}\right\rfloor\left\lfloor\frac{2}{p}\right\rfloor!}.$$

Using this inequality in Theorem 3.2, we get desired inequality. □

Theorem 3.4. *Let $U \subseteq \mathbb{R}_+$ be a $p-$convex set and $f : U \to \mathbb{R}$ be a nondecreasing $p-$convex function. For $a, b \in U$ with $a < b$, the following inequality holds:*

$$f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right) \leq \int_0^1 f\left(\frac{a+b}{2^{\frac{1}{p}}}\left[t+(1-t^p)^{\frac{1}{p}}\right]\right) dt$$
$$\leq \frac{1}{2^{\frac{1}{p}}} \int_0^1 f\left((1-t^p)^{\frac{1}{p}} a + tb\right)\left[1+(1-t^p)^{\frac{1}{p}-1} t^{p-1}\right] dt. \tag{3.3}$$

Proof. Since $h(x) = x^{\frac{1}{p}}$ is convex function on $\mathbb{R}_+$ for $0 < p < 1$, we know that

$$h\left(\frac{x+y}{2}\right) \leq \frac{h(x)+h(y)}{2}.$$

Writing $x = t^p$ and $y = 1 - t^p$, we have

$$\left(\frac{t^p + 1 - t^p}{2}\right)^{\frac{1}{p}} = \frac{1}{2^{\frac{1}{p}}} \leq \frac{(t^p)^{\frac{1}{p}} + (1-t^p)^{\frac{1}{p}}}{2}$$

$$\frac{a+b}{2^{\frac{1}{p}}} \cdot \frac{1}{2^{\frac{1}{p}}} \leq \frac{a+b}{2^{\frac{1}{p}}} \cdot \frac{t + (1-t^p)^{\frac{1}{p}}}{2}$$

hence,

$$\frac{a+b}{2^{\frac{2}{p}-1}} \leq \frac{a+b}{2^{\frac{1}{p}}} \cdot \left[t+(1-t^p)^{\frac{1}{p}}\right].$$

Using the fact that f is monotonic nondecreasing on $(0, \infty)$, we obtain

$$f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right) \leq f\left(\frac{a+b}{2^{\frac{1}{p}}}\left[t+(1-t^p)^{\frac{1}{p}}\right]\right), \ \forall t \in [0,1].$$

Integration on $[0, 1]$ yields to the left inequality in (3.3). From f being $p-$convex, it is known that for all $x, y \in \mathbb{R}_+$,

$$f\left(\frac{x+y}{2^{\frac{1}{p}}}\right) \leq \frac{f(x)+f(y)}{2^{\frac{1}{p}}}. \tag{3.4}$$

Putting $x = ta + (1-t^p)^{\frac{1}{p}} b$ and $y = (1-t^p)^{\frac{1}{p}} a + tb$ with $t \in [0, 1]$ above, we deduce

$$f\left(\frac{a+b}{2^{\frac{1}{p}}}\left[t+(1-t^p)^{\frac{1}{p}}\right]\right) \leq \frac{1}{2^{\frac{1}{p}}}\left[f\left(ta+(1-t^p)^{\frac{1}{p}} b\right) + f\left((1-t^p)^{\frac{1}{p}} a + tb\right)\right].$$

If each side of inequality is integrated on [0,1] over t, then

$$\int_0^1 f\left(\frac{a+b}{2^{\frac{1}{p}}}\left[t+(1-t^p)^{\frac{1}{p}}\right]\right)$$
$$\leq \frac{1}{2^{\frac{1}{p}}}\left[\int_0^1 f\left(ta+(1-t^p)^{\frac{1}{p}} b\right) dt + \int_0^1 f\left((1-t^p)^{\frac{1}{p}} a + tb\right) dt\right]. \tag{3.5}$$

By changing variable $v = (1-t^p)^{\frac{1}{p}}$ as indicated in the proof of Theorem 3.2 we have,

$$\frac{1}{2^{\frac{1}{p}}}\left[\int_0^1 f\left(ta+(1-t^p)^{\frac{1}{p}} b\right) dt + \int_0^1 f\left((1-t^p)^{\frac{1}{p}} a + tb\right) dt\right]$$
$$= \frac{1}{2^{\frac{1}{p}}} \int_0^1 f\left((1-t^p)^{\frac{1}{p}} a + tb\right)\left[1+(1-t^p)^{\frac{1}{p}-1} t^{p-1}\right] dt \ .$$

Thus

$$\int_0^1 f\left(\frac{a+b}{2^{\frac{1}{p}}}\left[t+(1-t^p)^{\frac{1}{p}}\right]\right)dt \le \frac{1}{2^{\frac{1}{p}}}\int_0^1 f\left((1-t^p)^{\frac{1}{p}}a+tb\right)\left[1+(1-t^p)^{\frac{1}{p}-1}t^{p-1}\right]dt$$

is obtained. □

Combining Theorem 3.2 with Theorem 3.4 and Theorem 3.3, we conclude the following result:

Corollary 3.5. *Let $U\subseteq\mathbb{R}_+$ be a $p-$convex set and $f:U\to\mathbb{R}$ be a nondecreasing $p-$convex function. For $a,b\in U$ with $a<b$, the following inequality holds*

$$2^{\frac{1}{p}-1}f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\le\left(1+\frac{\Gamma^2(\frac{1}{p})}{p\Gamma(\frac{2}{p})}\right)\left[\frac{f(a)+f(b)}{2}\right]\le\psi(p)\left[\frac{f(a)+f(b)}{2}\right]$$

where $\psi(p)$ is the same as in Theorem 3.3.

Theorem 3.6. *Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be an increasing $p-$convex function. If $a,b\in\mathbb{R}_+$ with $a<b$, then*

$$\int_0^1 f(t^{\frac{1}{p}}a+(1-t)^{\frac{1}{p}}b)dt\le\frac{f(a)+f(b)}{2}.$$

Proof. From $p-$convexity of f, for all $x,y\in\mathbb{R}_+$ and $t\in[0,1]$,

$$f(t^{\frac{1}{p}}a+(1-t)^{\frac{1}{p}}b)\le t^{\frac{1}{p}}f(a)+(1-t)^{\frac{1}{p}}f(b).$$

Since $t^{\frac{1}{p}}<t$ and $(1-t)^{\frac{1}{p}}<1-t$, we have

$$f(t^{\frac{1}{p}}a+(1-t)^{\frac{1}{p}}b)\le tf(a)+(1-t)f(b).$$

Integration on t gives

$$\int_0^1 f(t^{\frac{1}{p}}a+(1-t)^{\frac{1}{p}}b)dt\le\frac{f(a)+f(b)}{2}.$$

□

Theorem 3.7. *Let $f:\mathbb{R}_+\to\mathbb{R}$ be an increasing $p-$convex function. If $a,b\in\mathbb{R}_+$ with $a<b$, then the following inequality holds:*

$$\begin{aligned}
f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right) &\le \int_0^1 f\left(\frac{a+b}{2^{\frac{1}{p}}}\left[t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right]\right)dt\\
&\le 2^{1-\frac{1}{p}}\int_0^1 f\left(at^{\frac{1}{p}}+b(1-t)^{\frac{1}{p}}\right)dt\\
&\le \frac{2p}{2^{\frac{1}{p}}(p+1)}[f(a)+f(b)].
\end{aligned}$$

Proof. The fact that $h(x)=x^{\frac{1}{p}}$ is convex function for $p\in(0,1]$ on $[0,\infty)$ implies

$$\frac{1}{2^{\frac{1}{p}}}\le\frac{t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}}{2},\ \forall t\in[0,1].$$

From the monotonicity of f, one can get

$$f\left(\frac{a+b}{2^{\frac{1}{p}}}\cdot\frac{2}{2^{\frac{1}{p}}}\right)=f\left(\frac{a+b}{2^{\frac{2}{p}-1}}\right)\le f\left(\frac{a+b}{2^{\frac{1}{p}}}\left[t^{\frac{1}{p}}+(1-t)^{\frac{1}{p}}\right]\right)$$

for all $t\in[0,1]$. Thus the first inequality (from left to right) is obtained.

By using the $p-$convexity of f, we have

$$f\left(\frac{a+b}{2^{\frac{1}{p}}}\left[t^p+(1-t)^{\frac{1}{p}}\right]\right) \leq \frac{1}{2^{\frac{1}{p}}}\left[f\left(at^{\frac{1}{p}}+b(1-t)^{\frac{1}{p}}\right)+f\left(a(1-t)^{\frac{1}{p}}+bt^{\frac{1}{p}}\right)\right]. \quad (3.6)$$

From the changing of variables, it is clear that

$$\int_0^1 f\left(a(1-t)^{\frac{1}{p}}+bt^{\frac{1}{p}}\right)dt = \int_0^1 f\left(at^{\frac{1}{p}}+b(1-t)^{\frac{1}{p}}\right)dt.$$

So (3.6) yields to the second inequality (from left to right). On the other hand, the $p-$convexity of f on $\mathbb{R}_+$ implies

$$f\left(t^{\frac{1}{p}}a+(1-t)^{\frac{1}{p}}b\right) \leq t^{\frac{1}{p}}f(a)+(1-t)^{\frac{1}{p}}f(b)\ ,\ \forall t\in[0,1].$$

By integrating this inequality over t in $[0,1]$, one can have that

$$\int_0^1 f\left(t^{\frac{1}{p}}a+(1-t)^{\frac{1}{p}}b\right)dt \leq \frac{p}{p+1}[f(a)+f(b)].$$

By multiplying each side with $2^{1-\frac{1}{p}}$, we get the last part of the inequality. □

Theorem 3.8. *Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be a $p-$convex integrable function and let $0<a<b$. If*

$$\int_a^{\infty} x^{\frac{2}{p-1}}f(x)dx$$

is finite, then

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{p}{1-p}\left[a^{\frac{1+p}{1-p}}\int_a^{\infty} x^{\frac{2}{p-1}}f(x)dx + b^{\frac{1+p}{1-p}}\int_b^{\infty} x^{\frac{2}{p-1}}f(x)dx\right].$$

Proof. Using the $p-$convexity of f on $\mathbb{R}_+$ for any $z,y\geq 0$ and $u\in[0,1]$, we have

$$f\left(u^{\frac{1}{p}}z+(1-u)^{\frac{1}{p}}y\right) \leq u^{\frac{1}{p}}f(z)+(1-u)^{\frac{1}{p}}f(y).$$

By making substitutions $z=u^{1-\frac{1}{p}}a, u\in(0,1]$ and $y=(1-u)^{1-\frac{1}{p}}b, u\in[0,1)$, the following inequality is obtained:

$$f(ua+(1-u)b) \leq u^{\frac{1}{p}}f\left(u^{1-\frac{1}{p}}a\right)+(1-u)^{\frac{1}{p}}f\left((1-u)^{1-\frac{1}{p}}b\right),\ \forall u\in(0,1). \quad (3.7)$$

To show the validity of the inequality above, first, let us testify that $\int_0^1 u^{\frac{1}{p}}f\left(u^{1-\frac{1}{p}}a\right)du$ is finite. The change of the variable $x=u^{1-\frac{1}{p}}a, u\in(0,1]$ yields

$$u=\left(\frac{x}{a}\right)^{\frac{1}{1-\frac{1}{p}}}=\left(\frac{x}{a}\right)^{\frac{p}{p-1}}=\frac{x^{\frac{p}{p-1}}}{a^{\frac{p}{p-1}}}$$

and

$$du=\frac{p}{p-1}\cdot\frac{1}{a^{\frac{p}{p-1}}}x^{\frac{p}{p-1}-1}dx=\frac{p}{p-1}\cdot\frac{1}{a^{\frac{p}{p-1}}}x^{\frac{1}{p-1}}dx.$$

Thus,

$$\begin{aligned}\int_0^1 u^{\frac{1}{p}}f\left(u^{1-\frac{1}{p}}a\right)du &= \int_{\infty}^{a}\left[\frac{x^{\frac{1}{p-1}}}{a^{\frac{1}{p-1}}}\cdot\frac{p}{p-1}\cdot\frac{x^{\frac{1}{p-1}}}{a^{\frac{p}{p-1}}}f(x)\right]dx \\ &= \frac{p}{1-p}\cdot a^{\frac{1+p}{1-p}}\int_a^{\infty}x^{\frac{2}{p-1}}f(x)dx < \infty.\end{aligned}$$

For the integral

$$\int_0^1 (1-u)^{\frac{1}{p}} f\left((1-u)^{1-\frac{1}{p}} b\right) du,$$

the change of variable $t = 1-u$, $u \in [0,1)$ yields to

$$\int_0^1 t^{\frac{1}{p}} f\left(t^{1-\frac{1}{p}} b\right) dt.$$

In a similar way above, by the substitution $x = t^{1-\frac{1}{p}} b$, $t \in (0,1]$,

$$\int_0^1 t^{\frac{1}{p}} f\left(t^{1-\frac{1}{p}} b\right) dt = \frac{p}{1-p} \cdot b^{\frac{p+1}{1-p}} \int_b^\infty x^{\frac{2}{p-1}} f(x)dx < \infty$$

is deduced. Integrating the inequality (3.7) on (0,1) over u, taking into account that

$$\int_0^1 f(ua + (1-u)b)du = \frac{1}{b-a} \int_a^b f(x)dx$$

and

$$\int_0^1 uf\left(u^{1-\frac{1}{p}} a\right) du = \frac{p}{1-p} \cdot a^{\frac{1+p}{1-p}} \int_a^\infty x^{\frac{2}{p-1}} f(x)dx$$

$$\int_0^1 (1-u) f\left((1-u)^{1-\frac{1}{p}} b\right) du = \frac{p}{1-p} \cdot b^{\frac{1+p}{1-p}} \int_b^\infty x^{\frac{2}{p-1}} f(x)dx$$

respectively, one can obtain the desired inequality. □

Theorem 3.9. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a $p-$convex function and let $0 < a < b$. Then*
(i) *Let $g : [0,1] \to \mathbb{R}$ be a function defined as follows*

$$g(t) = \frac{1}{b-a} \int_a^b f(tx + (1-t)\frac{a+b}{2})dx.$$

If f is decreasing (or nonincreasing) function then g is $p-$convex function.
(ii) *If f is integrable on $[a,b]$, then the following inequality holds for $t \in (0,1]$,*

$$g(t) \geq 2^{\frac{1}{p}-1} f\left(\frac{a+b}{2^{\frac{1}{p}}}\right).$$

Proof. (i) Let $x, y \in [0,1]$ and $\lambda, \mu \geq 0$ with $\lambda^p + \mu^p = 1$. Using $\lambda + \mu < 1$, monotonicity and $p-$convexity of f, we can write

$$\begin{aligned}
g(\lambda t_1 + \mu t_2) &= \frac{1}{b-a}\int_a^b f\left((\lambda t_1 + \mu t_2)x + [1-(\lambda t_1 + \mu t_2)]\frac{a+b}{2}\right)dx \\
&= \frac{1}{b-a}\int_a^b f\left(\lambda t_1 x - \lambda t_1 \frac{a+b}{2} + \mu t_2 x - \mu t_2 \frac{a+b}{2} + \frac{a+b}{2}\right)dx \\
&\leq \frac{1}{b-a}\int_a^b f\left(\lambda t_1 x - \lambda t_1 \frac{a+b}{2} + \mu t_2 x - \mu t_2 \frac{a+b}{2} + (\lambda+\mu)\frac{a+b}{2}\right)dx \\
&= \frac{1}{b-a}\int_a^b f\left(\lambda\left[t_1 x + (1-t_1)\frac{a+b}{2}\right] + \mu\left[t_2 x + (1-t_2)\frac{a+b}{2}\right]\right)dx \\
&\leq \frac{1}{b-a}\int_a^b \left[\lambda f\left(t_1 x + (1-t_1)\frac{a+b}{2}\right) + \mu f\left(\left[t_2 x + (1-t_2)\frac{a+b}{2}\right]\right)\right]dx \\
&= \lambda g(t_1) + \mu g(t_2).
\end{aligned}$$

(ii) Assume that $t \in (0,1]$. By changing variable $u = tx + (1-t)\frac{a+b}{2}$, we have

$$g(t) = \frac{1}{t(b-a)}\int_{ta+(1-t)\frac{a+b}{2}}^{tb+(1-t)\frac{a+b}{2}} f(u)du = \frac{1}{m-n}\int_n^m f(u)du$$

where $m = tb+(1-t)\frac{a+b}{2}$ and $n = ta+(1-t)\frac{a+b}{2}$. The right part of the Hermite-Hadamard inequality gives

$$\frac{1}{m-n}\int_n^m f(u)du \geq 2^{\frac{1}{p}-1} f\left(\frac{m+n}{2^{\frac{1}{p}}}\right) = 2^{\frac{1}{p}-1} f\left(\frac{a+b}{2^{\frac{1}{p}}}\right)$$

hence, we get the required inequality. $\square$

Theorem 3.10. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a $p-$convex function. Let $a, b \in \mathbb{R}_+$ with $a < b$. Consider the function*

$$h(t) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx + (1-t)y)dxdy\ ,\ t \in [0,1]$$

(i) *If f is decreasing function, then $h(t)$ is also $p-$ convex function on $[0,1]$.*

(ii) *If f is integrable on $[a,b]$, then the following inequality holds:*

$$2^{1-\frac{1}{p}} h(t) \geq \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2^{\frac{1}{p}}}\right)dxdy\ ,\ t \in [0,1].$$

Proof. (i) Let $x, y \in [0,1]$ and $\lambda, \mu \geq 0$ with $\lambda^p + \mu^p = 1$. Taking into account that $\lambda^{p-1} - t_1 \geq 1 - t_1$ for $t_1, \lambda \in [0,1]$ and $\mu^{p-1} - t_2 \geq 1 - t_2$ for $t_2, \mu \in [0,1]$, then using the monotonicity and $p-$convexity of f, respectively, we have

$$\begin{aligned} h\left(\lambda t_1+\mu t_2\right) &= \frac{1}{(b-a)^2}\int_a^b\int_a^b f((\lambda t_1+\mu t_2)\,x+(1-(\lambda t_1+\mu t_2))y)dxdy \\ &= \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\lambda\left[t_1x+\left(\lambda^{p-1}-t_1\right)y\right]+\mu\left[t_2x+\left(\mu^{p-1}-t_2\right)y\right]\right)dxdy \\ &\le \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\lambda\left[t_1x+\left(1-t_1\right)y\right]+\mu\left[t_2x+\left(1-t_2\right)y\right]\right)dxdy \\ &\le \frac{1}{(b-a)^2}\int_a^b\int_a^b \{\lambda f\left(\left[t_1x+\left(1-t_1\right)y\right]\right)+\mu f(\left[t_2x+\left(1-t_2\right)y\right])\}dxdy \\ &= \lambda h\left(t_1\right)+\mu h\left(t_2\right). \end{aligned}$$

(ii) From (3.4), we can write

$$f\left(\frac{x+y}{2^{\frac{1}{p}}}\right)\le\frac{f(tx+(1-t)y)+f(ty+(1-t)x)}{2^{\frac{1}{p}}}$$

for all $t\in[0,1]$ and $x,y\in[a,b]$. Integrating this inequality on $[a,b]^2$ we get

$$\frac{1}{2^{\frac{1}{p}}}\left[\int_a^b\int_a^b f(tx+(1-t)y)dxdy+\int_a^b\int_a^b f((1-t)x+ty)dxdy\right]\ge\int_a^b\int_a^b f\left(\frac{x+y}{2^{\frac{1}{p}}}\right)dxdy$$

since

$$\int_a^b\int_a^b f(tx+(1-t)y)dxdy=\int_a^b\int_a^b f((1-t)x+ty)dxdy$$

the above inequality gives us the desired result. □

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