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Research Article

On ideals of prime rings involving n-skew commuting additive mappings with applications

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Abstract

Let $n > 1$ be a fixed positive integer and S be a subset of a ring R. A mapping ζ of a ring R into itself is called n-skew-commuting on S if $\zeta(x)x^n + x^n\zeta(x) = 0$, $\forall\ x \in S$. The main aim of this paper is to describe n-skew-commuting mappings on appropriate subsets of R. With this, many known results can be either generalized or deduced. In particular, this solves the conjecture in [M. Nadeem, M. Aslam and M.A. Javed, On 2-skew commuting additive mappings of prime rings, Gen. Math. Notes, 2015]. The second main result of this paper is concerned with a pair of linear mappings of C^*-algebras. We show that here, if C^*-Algebra admits a pair of linear mappings f and g such that $f(x)x^* + x^*g(x) \in Z(A)$ for all $x \in A$, then both f and g must be zero. As the applications of first main result (Theorem 2.1) and apart from proving some other results, we characterize the linear mappings on primitive C^*-algebras. Furthermore, we provide an example to show that the assumed restrictions cannot be relaxed.

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1. Introduction

We will employ the following notations in the study. We let R denote an associative ring, $Z(R)$ denote the center of R and A represent a C^*-algebra. Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies either $a = 0$ or $b = 0$, and is semiprime if for $a \in R$, $aRa = (0)$ implies $a = 0$. For $x, y \in R$, the symbol $[x, y]$ will denote the Lie product $xy - yx$ and the symbol $x \circ y$ will denote the Jordan product $xy + yx$.

This research has been motivated by the recent work of S. Ali et al. [2]. An additive map d from R to R is called a derivation of R if $d(xy) = d(x)y + xd(y)$ holds, $\forall\ x, y \in R$ and called a Jordan derivation if $d(x^2) = d(x)x + xd(x)$ holds, $\forall\ x \in R$. Let S be a nonempty subset of R. An additive mapping $T : R \longrightarrow R$ is called a left centralizer (resp.

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Jordan left centralizer) if $T(xy) = T(x)y$ (resp. $T(x^2) = T(x)x$) holds, $\forall\ x, y \in R$. A mapping $\zeta : R \to R$ is called centralizing (resp. commuting) on S if $[\zeta(x), x] \in Z(R)$, $\forall\ x \in S$ (resp. $[\zeta(x), x] = 0$, $\forall\ x \in S$). In [22], Deng and Bell extended the above notions as follows: For a positive integer n, the mapping ζ is called n-centralizing (resp. n-commuting) on S, if $[\zeta(x), x^n] \in Z(R)$, $\forall\ x \in S$ (resp. $[\zeta(x), x^n] = 0$, $\forall\ x \in S$). The study of centralizing and commuting mapping goes back to Posner [36]. A classical result of Posner (Posner's second theorem) states that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Mayne [31] proved the analogous result for centralizing automorphisms. In [11], Bell and Martindale proved that if a semiprime ring R admits a derivation d centralizing on a nonzero left ideal I of R, then R contains a nonzero central ideal, provided $d(I) \neq 0$. A variety of results on centralizing and commuting mappings and their applications have been obtained by a number of authors (see [3,5–7,13,15,16,21,23,25,26,30,32,37,39] where further references can be found).

Following [14], a mapping ζ of a ring R into itself is called skew-centralizing (resp. skew-commuting) on a subset S of R if $\zeta(x)x + x\zeta(x) \in Z(R)$, $\forall\ x \in S$ (resp. $\zeta(x)x + x\zeta(x) = 0$, $\forall\ x \in S$). A mapping ζ of a ring R into itself is called semi-commuting on a subset S of R if either $\zeta(x)x + x\zeta(x) = 0$, $\forall\ x \in S$ or $\zeta(x)x - x\zeta(x) = 0$, $\forall\ x \in S$. Motivated by the definition of n-commuting mapping, Bell and Lucier [10] called a mapping ζ of a ring R into itself n-skew-commuting on a subset S of R if $\zeta(x)x^n + x^n\zeta(x) = 0$, $\forall\ x \in S$. In particular, for $n = 1, 2$, we call them 1-skew-commuting and 2-skew commuting. In [14], Brešar studied 1-skew-commuting mappings and proved that if R is 2-torsion free semiprime ring and $\zeta : R \to R$ an additive mapping such that $\zeta(x)x + x\zeta(x) = 0$, $\forall\ x \in R$, then $\zeta = 0$. Recently, Fošner [24] studied the above mentioned result in the case of 2-skew commuting mappings. For results concerning skew-commuting mappings and their generalizations (such as semi-commuting, skew-centralizing, semi-centralizing mappings) we refer the reader to ([15,17–19,27–29,35,38]) where further references can be found. In [34], Nadeem et al. proved that if R is a prime ring with $char(R) \neq 2, 3$, I is an ideal of R and $\zeta : R \to R$ an additive mapping such that $\zeta(x)x^2 + x^2\zeta(x) = 0$, $\forall\ x \in I$, then $\zeta = 0$ on I. Moreover, they concluded the paper with following conjecture.

Conjecture 1.1. [34, Conjecture] *Let $n \geq 2$ be a fixed integer and R be a prime ring with suitable torsion restrictions. Suppose that an additive mapping $\zeta : R \to R$ satisfies the functional identity*

$$\zeta(x)x^n + x^n\zeta(x) = 0, \forall x \in R.$$

Then, $\zeta = 0$.

The principal aim of the present paper is to prove Conjecture 1.1 just mentioned above. With this, many known results can be either generalized or deduced (see for example, [2], [14] and [34]). As the applications of the first main result, we established the following result: let n be a fixed positive integer, and R be a prime ring such that $char(R) = 0$ or $char(R) \geq n$. Suppose there exists a Jordan left $*$-centralizer $T : R \to R$ such that $T(x)x^n \pm x^nT(x) = 0$, $\forall\ x \in R$. Then $T = 0$. Moreover, we characterizes a pair of linear mappings on C^*-algebras. In fact, we prove that under mild conditions, if C^*-algebra A admits a pair of linear mappings f and g such that $f(x)x^* + x^*g(x) \in Z(A)$, $\forall\ x \in A$, then $f = 0$ and $g = 0$. Furthermore, we provide an example to show that the assumed restrictions cannot be relaxed. Finally, we conclude our paper with some open problems.

2. Results on rings

The main goal of this paper is to prove the following theorem.

Theorem 2.1. *Let n be a fixed positive integer, R be a prime ring such that $char(R) = 0$ or $char(R) \geq n$ and I be a nonzero ideal of R. Suppose that an additive mapping $\zeta : R \to R$*

satisfies the relation

$$\zeta(x)x^n + x^n\zeta(x) = 0, \ \forall x \in I.$$

In this case $\zeta = 0$.

Proof. Here we use some ideas similar to [14]. By the hypothesis, we have

$$\zeta(x)x^n + x^n\zeta(x) = 0, \ \forall x \in I. \tag{2.1}$$

For $n = 1, 2$, result follows by Theorem 1 of [14] and Lemma 4 of [24]. Now we assume that $n > 2$. Left multiplication by x^n to (1.1) yields that

$$x^n\zeta(x)x^n + x^{2n}\zeta(x) = 0, \ \forall x \in I. \tag{2.2}$$

Also, right multiplication by x^n to (1.1) yields that

$$\zeta(x)x^{2n} + x^n\zeta(x)x^n = 0, \ \forall x \in I. \tag{2.3}$$

Calculating (2.3) – (2.2) gives

$$\zeta(x)x^{2n} - x^{2n}\zeta(x) = 0, \ \forall x \in I. \tag{2.4}$$

This can be rewritten as $[\zeta(x), x^{2n}] = 0, \ \forall x \in I$. By Theorem 1.1 of [8], we conclude that

$$[\zeta(x), x] = 0, \ \forall x \in I. \tag{2.5}$$

Application of relation (2.5) gives

$$\zeta(x)x^n = x^n\zeta(x) = 0, \ \forall x \in I. \tag{2.6}$$

Therefore, expression (2.1) forces that $2x^n\zeta(x) = 0, \forall \ x \in I$. Since $char(R) \geq n$, so

$$x^n\zeta(x) = 0, \ \forall x \in I. \tag{2.7}$$

Also, we have

$$\zeta(x)x^n = 0, \ \forall x \in I. \tag{2.8}$$

For any positive integer k, replace x by $x + ky$ in (2.8) to get

$$\zeta(x + ky)(x + ky)^n = 0, \ \forall x, y \in I. \tag{2.9}$$

The above relation can be written as

$$k(\zeta(x)\sum_{i=0}^{n-1} x^i y x^{n-i-1} + \zeta(y)x^n) + ... + k^n(\zeta(y)\sum_{i=0}^{n-1} y^i x y^{n-i-1} + \zeta(x)y^n) = 0, \ \forall x, y \in I.$$

By Lemma 1 of [20], we get

$$\zeta(y)\sum_{i=0}^{n-1} y^i x y^{n-i-1} + \zeta(x)y^n = 0, \ \forall x, y \in I. \tag{2.10}$$

Linearization of equation (2.5) gives

$$\zeta(x)y + \zeta(y)x - x\zeta(y) - y\zeta(x) = 0, \ \forall x, y \in I. \tag{2.11}$$

Replacing y by y^n in (2.11)

$$\zeta(x)y^n + \zeta(y^n)x - x\zeta(y^n) - y^n\zeta(x) = 0, \ \forall x, y \in I. \tag{2.12}$$

This can be written as

$$\zeta(x)y^{2n} + \zeta(y^n)xy^n - x\zeta(y^n)y^n - y^n\zeta(x)y^n = 0, \ \forall x, y \in I. \tag{2.13}$$

Left multiplication by y^n to (2.10) yields

$$y^n\zeta(y)\sum_{i=0}^{n} y^i x y^{n-i-1} + y^n\zeta(x)y^n = 0, \ \forall x, y \in I. \tag{2.14}$$

From (2.7) and (2.14), we obtain

$$y^n\zeta(x)y^n = 0, \ \forall x, y \in I. \tag{2.15}$$

Interchanging the role of x and y, we find that

$$x^n\zeta(y)x^n = 0, \ \forall x, y \in I. \tag{2.16}$$

Replace y by y^n in (2.16) to get

$$x^n\zeta(y^n)x^n = 0, \ \forall x, y \in I. \tag{2.17}$$

Then,

$$x^{2n}\zeta(y^n)x^n = 0, \ \forall x, y \in I. \tag{2.18}$$

Taking $y = x$ in (2.16), it follows that

$$x^n\zeta(x)x^n = 0, \ \forall x \in I. \tag{2.19}$$

By (2.15), relation (2.13) reduces to

$$\zeta(x)y^{2n} + \zeta(y^n)xy^n - x\zeta(y^n)y^n = 0, \ \forall x, y \in I. \tag{2.20}$$

By (2.6), expression (2.19) can be written as

$$x^{2n}\zeta(x) = 0, \ \forall x \in I. \tag{2.21}$$

Putting x^n for x in (2.20), we obtain

$$\zeta(x^n)y^{2n} + \zeta(y^n)x^ny^n - x^n\zeta(y^n)y^n = 0, \ \forall x, y \in I. \tag{2.22}$$

Multiplying by x^{2n} to (2.20) from left side, we get

$$x^{2n}\zeta(x)y^{2n} + x^{2n}\zeta(y^n)xy^n - x^{3n}\zeta(y^n)y^n = 0, \ \forall x, y \in I. \tag{2.23}$$

Application of (2.17) and (2.21) yields that

$$x^{3n}\zeta(y^n)y^n = 0, \ \forall x, y \in I. \tag{2.24}$$

Application of [14, Lemma 1] yields

$$\zeta(y^n)y^n = 0, \ \forall y \in I. \tag{2.25}$$

From relation (2.20) and (2.25), we get

$$\zeta(x)y^{2n} + \zeta(y^n)xy^n = 0, \ \forall x, y \in I. \tag{2.26}$$

Replacing x by xy^n in (2.26), we find

$$\zeta(xy^n)y^{2n} + \zeta(y^n)xy^{2n} = 0, \ \forall x, y \in I. \tag{2.27}$$

Right multiplication by y^n to (2.26) yields

$$\zeta(x)y^{3n} + \zeta(y^n)xy^{2n} = 0, \ \forall x, y \in I. \tag{2.28}$$

Calculating (2.28)$-$(2.27) gives

$$\zeta(x)y^{3n} - \zeta(xy^n)y^{2n} = 0, \ \forall x, y \in I. \tag{2.29}$$

Left multiplication by y^n to (2.12) gives

$$y^n\zeta(x)y^n + y^n\zeta(y^n)x - y^nx\zeta(y^n) - y^{2n}\zeta(x) = 0, \ \forall x, y \in I. \tag{2.30}$$

Application of (2.15) and (2.25) yields that

$$y^nx\zeta(y^n) + y^{2n}\zeta(x) = 0, \ \forall x, y \in I. \tag{2.31}$$

Replacing x by xy^n in (2.31), we find

$$y^nxy^n\zeta(y^n) + y^{2n}\zeta(xy^n) = 0, \ \forall x, y \in I. \tag{2.32}$$

From the equation (2.25), (2.32) becomes

$$y^{2n}\zeta(xy^n) = 0, \ \forall x, y \in I. \tag{2.33}$$

Putting $sy^{3n}\zeta(t)$ for y in equation (2.11), we obtain

$$\zeta(x)sy^{3n}\zeta(t) + \zeta(sy^{3n}\zeta(t))x - x\zeta(sy^{3n}\zeta(t)) - sy^{3n}\zeta(t)\zeta(x) = 0 \tag{2.34}$$

$\forall\ s, t, x, y \in I$. Right multiplication by y^{3n} to (2.34) yields

$$\zeta(x)sy^{3n}\zeta(t)y^{3n} + \zeta(sy^{3n}\zeta(t))xy^{3n} - x\zeta(sy^{3n}\zeta(t))y^{3n} - sy^{3n}\zeta(t)\zeta(x)y^{3n} = 0 \tag{2.35}$$

$\forall\ s, t, x, y \in I$. Replace y by sy^n in equation (2.11), we obtain

$$\zeta(x)sy^n + \zeta(sy^n)x - x\zeta(sy^n) - sy^n\zeta(x) = 0, \ \forall s, x, y \in R. \tag{2.36}$$

Multiplying by y^{2n} to (2.36) from left as well as from right, we find that

$$y^{2n}\zeta(x)sy^{3n} + y^{2n}\zeta(sy^n)xy^{2n} - y^{2n}x\zeta(sy^{2n})y^{2n} - y^{2n}sy^n\zeta(x)y^{2n} = 0 \tag{2.37}$$

$\forall\ s, x, y \in I$. Application of (2.15) and (2.33) yields that

$$y^{2n}\zeta(x)sy^{3n} - y^{2n}x\zeta(sy^{2n}) = 0, \ \forall s, x, y \in I. \tag{2.38}$$

From the equation (2.29), (2.38) reduces to

$$y^{2n}\zeta(x)sy^{7n} - y^{2n}x\zeta(s)y^{6n} = 0, \ \forall s, x, y \in I. \tag{2.39}$$

Then,

$$(y^{2n}\zeta(x)sy^{4n} - y^{2n}x\zeta(s)y^{3n})y^{3n} = 0, \ \forall s, x, y \in I.$$

Application of [14, Lemma 1] yields that

$$y^{2n}\zeta(x)sy^{4n} - y^{2n}x\zeta(s)y^{3n} = 0, \ \forall s, x, y \in I.$$

This can be written as

$$y^{3n}\zeta(x)sy^{4n} - y^{3n}x\zeta(s)y^{3n} = 0, \ \forall s, x, y \in I. \tag{2.40}$$

Replacing s by $sy^{3n}\zeta(t)$ in expression (2.40), we find that

$$y^{3n}\zeta(x)sy^{3n}\zeta(t)y^{4n} - y^{3n}x\zeta(sy^{3n}\zeta(t))y^{3n} = 0, \ \forall s, t, x, y \in I.$$

Application of (2.15) gives

$$y^{3n}x\zeta(sy^{3n}\zeta(t))y^{3n} = 0, \ \forall s, t, x, y \in I. \tag{2.41}$$

This implies $y^{3n}IR\zeta(sy^{3n}\zeta(t))y^{3n} = (0)$, $\forall\ s, t, y \in I$. The primeness of R yields either $y^{3n}I = (0)$ or $\zeta(sy^{3n}\zeta(t))y^{3n} = 0$, $\forall\ s, t, y \in I$. There is nothing to prove if $y^{3n} = 0$, $\forall$ $y \in I$, and henceforth we conclude that

$$\zeta(sy^{3n}\zeta(t))y^{3n} = 0, \ \forall s, t, y \in I. \tag{2.42}$$

By (2.15) and (2.42), the relation (2.35) reduces to

$$\zeta(sy^{3n}\zeta(t))xy^{3n} - sy^{3n}\zeta(t)\zeta(x)y^{3n} = 0, \ \forall s, t, x, y \in I. \tag{2.43}$$

Replace s by rs in equation (2.43) to get

$$\zeta(rsy^{3n}\zeta(t))xy^{3n} - rsy^{3n}\zeta(t)\zeta(x)y^{3n} = 0, \ \forall r, s, t, x, y \in I. \tag{2.44}$$

Left multiplication by r to (2.43) gives

$$r\zeta(sy^{3n}\zeta(t))xy^{3n} - rsy^{3n}\zeta(t)\zeta(x)y^{3n} = 0, \ \forall r, s, t, x, y \in I. \tag{2.45}$$

Calculating (2.45)−(2.44) gives

$$\zeta(rsy^{3n}\zeta(t))xy^{3n} - rsy^{3n}\zeta(t)\zeta(x)y^{3n} = 0, \ \forall r, s, t, x, y \in I.$$

This can be written as

$$\zeta(rsy^{3n}\zeta(t))xy^{3n} - r\zeta(sy^{3n}\zeta(t))xy^{3n} = 0, \ \forall r, s, t, x, y \in I. \tag{2.46}$$

Left multiplication by z^n to relation (2.46) yields

$$z^n\zeta(rsy^{3n}\zeta(t))xy^{3n} - z^n r\zeta(sy^{3n}\zeta(t))xy^{3n} = 0, \ \forall r, s, t, x, y, z \in I. \tag{2.47}$$

Replace x by $z^n x$ in (2.47) to get

$$z^n\zeta(rsy^{3n}\zeta(t))z^n xy^{3n} - z^n r\zeta(sy^{3n}\zeta(t))z^n xy^{3n} = 0, \ \forall r, s, t, x, y, z \in I. \tag{2.48}$$

Application of (2.15) gives

$$z^n r\zeta(sy^{3n}\zeta(t))z^n xy^{3n} = 0, \ \forall r, s, t, x, y, z \in I. \tag{2.49}$$

This implies $z^n IR\zeta(sy^{3n}\zeta(t))z^n xy^{3n} = (0)$, $\forall\ s, t, x, y, z \in I$. Since R is prime, the last expression forces that

$$\zeta(sy^{3n}\zeta(t))z^n = 0, \ \forall s, t, y, z \in I. \tag{2.50}$$

Application of [14, Lemma 1] gives that $\zeta(sy^{3n}\zeta(t)) = 0$, $\forall\ s, t, y \in I$. The last relation is similar to equation (13) of [14], and henceforth the rest of the proof runs on similar lines as in [14]. We present the proof for the reader's convenience. Now we assume that $\zeta(t) = 0$ for some $t \in I$. Therefore, we have $0 \neq a = y^{3n}\zeta(t)$ for some $t \in I$. Then, $L = Ra$ is a nonzero left ideal of R. Thus from the last relation, we conclude that $\zeta(L) = (0)$. Replace y by l in (2.11), where $l \in L$ and using the fact that $\zeta(l) = 0$, we obtain $\zeta(x)l - l\zeta(x) = 0$, $\forall\ x \in R$ and $l \in L$. Replacing l by rl, where $r \in R$ and $l \in L$, we get $\zeta(x)rl - rl\zeta(x) = 0$, $\forall\ x, r \in R$ and $l \in L$. Then,

$$(\zeta(x)r - r\zeta(x))l = 0, \ \forall x, r \in R; \ l \in L. \tag{2.51}$$

Now, replace r by $x^n r$ in (2.51) to get $(\zeta(x)x^n r - x^n r\zeta(x))l = 0$, $\forall\ x, r \in R$ and $l \in L$. By the relation (2.8), we conclude that $x^n r\zeta(x)l = 0$, $\forall\ r, x \in R$ and $l \in L$. Hence, we have $x^n Rl\zeta(x) = (0)$, $\forall\ x \in R$ and $l \in L$. The primeness of R gives $l\zeta(x) = 0$, $\forall\ x \in R$ and $l \in L$. Then, we have $lr\zeta(x) = 0$ *i.e.*, $LR\zeta(x) = (0)$, $\forall\ x \in R$. From the last relation we get $\zeta(x) = 0$, $\forall\ x \in R$, since $L \neq 0$ and R is prime. Thereby the proof is completed. □

The following corollaries recaptures some knows results (viz.; [2], [14] and [34]).

Corollary 2.2. [14, Theorem 1] *Let R be a prime ring with char$(R) \neq 2$. Suppose that an additive mapping $\zeta : R \to R$ satisfies the relation $\zeta(x)x + x\zeta(x) = 0$, $\forall x \in I$. In this case $\zeta = 0$.*

Corollary 2.3. [34, Theorem 3.1] *Let R be a prime ring with char$(R) \neq 2$ and I be a non-zero ideal of R. Suppose that an additive mapping $\zeta : R \to R$ satisfies the relation $\zeta(x)x^2 + x^2\zeta(x) = 0$, $\forall x \in I$. In this case $\zeta = 0$.*

Corollary 2.4. *Let n be a fixed positive integer, R be a prime ring such that char$(R) = 0$ or char$(R) \geq n$. Suppose that an additive mapping $\zeta : R \to R$ satisfies the relation*

$$\zeta(x)x^n + x^n\zeta(x) = 0, \ \forall x \in R.$$

In this case $\zeta = 0$.

Notice that in case of prime rings with characteristic two, skew-commuting mappings behave like commuting mappings. The following result justifies this fact.

Theorem 2.5. *Let R be a prime ring with char$(R) = 2$. Suppose that an additive mapping $\zeta : R \to R$ satisfies the relation $\zeta(x)x + x\zeta(x) = 0$, $\forall\ x \in R$. Then, ζ has the form $\zeta(x) = \lambda x + \mu(x)$ where λ is an element in C, the extended centroid of R (see [9] for details) and $\mu : R \to C$ is an additive mapping.*

Proof. By the assumption, we have $\zeta(x)x + x\zeta(x) = 0$, $\forall\, x \in R$. Since $char(R) = 2$, so we have $x = -x$, $\forall\, x \in R$. Thus, the last relation can be rewritten as $\zeta(x)x - (-x)\zeta(x) = 0$, $\forall\, x \in R$. Then, $\zeta(x)x - x\zeta(x) = 0$, $\forall\, x \in R$. This implies that $[\zeta(x), x] = 0$, $\forall\, x \in R$. Thus, ζ is commuting on R. By Theorem 3.6 of [16], we can write that $\zeta(x) = \lambda x + \mu(x)$ where λ is an element in C, the extended centroid of R and $\mu : R \to C$ is an additive mapping. □

Corollary 2.6. *Let R be a prime ring with $char(R) = 2$. Then every semi-commuting mappings ζ of a ring R into itself must have the form $\zeta(x) = \lambda x + \mu(x)$ where λ is an element in C, the extended centroid of R and $\mu : R \to C$ is an additive mapping.*

Proof. As consequences of Theorem 2.5 above and Theorem 3.6 of [16]. □

Let R be a ring with involution $'*'$. An additive mapping $d : R \longrightarrow R$ is called a $*$-derivation if $d(xy) = d(x)y^* + xd(y)$ holds, $\forall\, x, y \in R$, and is called a Jordan $*$-derivation if $d(x^2) = d(x)x^* + xd(x)$ holds, $\forall\, x \in R$. An additive mapping $T : R \longrightarrow R$ is called a left $*$-centralizer (resp. Jordan left $*$-centralizer) if $T(xy) = T(x)y^*$ (resp. $T(x^2) = T(x)x^*$) holds, $\forall\, x, y \in R$ (see [1] and [4] for details). In [13], Brešar considered a pair of additive mappings (derivations) and proved the following result: If a noncommutative prime ring R admits a pair of derivations d and g such that $d(x)x - xg(x) \in Z(R)$, $\forall\, x \in U$ or $d(x)x + xg(x) \in Z(R)$, $\forall\, x \in U$, where U is a nonzero left ideal of R, then $d = g = 0$. Further, Chaudhary and Thaheem [18] extended the above mentioned results for semiprime rings and showed that if R is a semiprime ring and f, g a pair of derivations of R such that $f(x)x + xg(x) \in Z(R)$, $\forall\, x \in R$, then f and g are central. Inspired by these work's, Ali et al. [1] established the following result.

Theorem 2.7. [1, Theorem 4.4] *Let m, n be fixed positive integers, and R be a $(m+n)!$-torsion free noncommutative prime ring with involution $'*'$ of the second kind having the identity element e. Suppose there exist Jordan $*$-derivations $d, g : R \to R$ such that $d(x^m)x^n \pm x^n g(x^m) = 0$, $\forall\, x \in R$. Then $d = g = 0$.*

Our next theorem is motivated by the above mentioned result.

Theorem 2.8. *Let n be a fixed positive integer, and let R be a prime ring with involution $'*'$ such that $char(R) = 0$ or $char(R) \geq n$. Suppose there exists a Jordan left $*$-centralizer $T : R \to R$ such that $T(x)x^n \pm x^n T(x) = 0$, $\forall\, x \in R$. Then $T = 0$.*

Proof. First we consider the situation, $T(x)x^n + x^n T(x) = 0$, $\forall\, x \in R$. Since every Jordan left $*$-centralizer $T : R \to R$ is an additive map, so application of Corollary 2.4 yields the required result. □

The similar arguments can be adapted in the case $T(x)x^n - x^n T(x) = 0$, $\forall x \in R$. This proves the result.

3. Results on C^*-algebras

In this section, we present the applications of Theorem 2.1 to certain special classes of algebras, some of which are related to C^*-algebras. Further, we characterizes a linear mapping $f : A \to A$ which satisfies the following relation

$$f(xy) = f(y)x^* + y^* f(x), \forall x, y \in A \text{ (where } '*' : A \to A \text{ is an involution).} \tag{3.1}$$

In fact, these mappings appeared first time in the recent paper due to Ali et al. [1]. A Banach algebra is a linear associative algebra which, as a vector space, is a Banach space with norm $\|\cdot\|$ satisfying the multiplicative inequality; $\|xy\| \leq \|x\|\|y\|$, $\forall\, x$ and y in A. The Jacobson radical of A is the intersection of all primitive ideals of A and is denoted by $rad(A)$. An additive mapping $* : A \to A$ mapping x to x^* is called an involution if the following conditions are satisfied: (i) $(xy)^* = y^* x^*$, (ii) $(x^*)^* = x$, and *(iii)* $(\lambda x)^* = \bar{\lambda} x^*$, $\forall\ x, y \in A$ and $\lambda \in \mathbb{C}$ the field of complex numbers, where $\bar{\lambda}$ is the conjugate of λ. An algebra equipped with an involution is called a $*$-algebra or algebra with involution. A C^*-algebra A is a Banach $*$-algebra with the additional norm condition $\|x^*x\| = \|x\|^2$, $\forall\ x \in A$. A C^*-algebra A is primitive if its zero ideal is primitive, that is, if A has a faithful nonzero irreducible representation (see [33] for details). Throughout the present section, C^*-algebras are assumed to be nonunital unless indicated otherwise.

Theorem 3.1. *Let n be a fixed positive integer. Next, let A be a primitive C^*-algebra. Suppose that a linear mapping $\zeta : R \to R$ satisfies the relation*

$$\zeta(x)x^n + x^n\zeta(x) = 0, \forall x \in A.$$

In this case $\zeta = 0$.

Proof. It is well known that every primitive C^*-algebra is prime (viz., [33, Theorem 5.4.5]). Thus, A is a prime C^*-algebra and so a prime ring. Therefore by Theorem 2.1, we get the required result. □

Corollary 3.2. *Let A be a primitive C^*-algebra. Then, zero is only linear mapping which is skew-commuting on A.*

In [18], Chaudhary and Thaheem studied the situation regarding a pair of derivations of semiprime rings. Especially, they proved that if R is a semiprime ring and f, g a pair of derivations of R such that $f(x)x + xg(x) \in Z(R)$, $\forall\ x \in R$, then f and g must be central. So, our next theorem is related to a pair of linear mappings of C^*-algebras. Precisely, we prove the following result.

Theorem 3.3. *Let A be a C^*-algebra. Next, let f and g be a pair of linear mappings of A which satisfies (3.1) and the relation*

$$f(x)x^* + x^*g(x) \in Z(A), \forall x \in A.$$

In this case $f = 0$ and $g = 0$.

Proof. We are given that $f,\ g : A \to A$ a pair of additive mappings of A which satisfies (3.1) and $f(x)x^* + x^*g(x) \in Z(A)$, $\forall\ x \in A$. Replacing x by x^* in the last relation, we get $f(x^*)x + xg(x^*) \in Z(A)$, $\forall\ x \in A$. Since an involution $'*'$, f and g are additive mappings, so we can define the maps $f_1 : A \to A$ by $f_1(x) = f(x^*)$ and $g_1(x) = g(x^*)$, $\forall\ x \in A$. Thus, the last expression yields that $f_1(x)x + xg_1(x) \in Z(A)$, $\forall\ x \in A$. It is easy to verify that f_1, g_1 are derivations of A and notice that A is a C^*-algebra and every C^*-algebra is semiprime ring, application of [18, Theorem 2.2] yields that f_1 and g_1 are central. Consequently, f_1 and g_1 are commuting as well as centralizing on A. From [12, Corollary 3.7], we obtain f_1 and g_1 maps A into $Z(A) \cap rad(A)$. Hereafter, we have $f_1 = 0$ and $g_1 = 0$, since A is C^*-algebra and it is well know that every C^*-algebra is semisimple $(i.e., rad(A) = 0)$ (see [33] for details). Thus $f = 0$ and $g = 0$. □

Corollary 3.4. *Let A be a C^*-algebra. Next, let f be a linear mapping of A which satisfies (3.1) and the relation*

$$f(x)x^* + x^*f(x) \in Z(A), \forall x \in A.$$

In this case $f = 0$.

Using similar approach with necessary variations as we have used in Theorem 3.3, we can prove the following result.

Theorem 3.5. *Let n be a fixed positive integer. Next, let A be a primitive C^*-algebra. Suppose that a linear mapping $\zeta : R \to R$ satisfies the relation*

$$\zeta(x)x^{*n} + x^{*n}\zeta(x) = 0, \forall x \in A.$$

In this case $\zeta = 0$.

The following example shows that the above results are not true in the case of arbitrary Banach $*$-algebras.

Example 3.6. Let $A = \left\{ \begin{pmatrix} 0 & x_{12} & x_{13} \\ 0 & 0 & x_{23} \\ 0 & 0 & 0 \end{pmatrix} \middle| \ x_{12}, x_{13}, x_{23} \in \mathbb{C} \right\}$, where $\mathbb{C}$ is the field of complex numbers. Clearly, A is a Banach algebra under the norm $\|X\| = \max\limits_{1\leq j\leq 3} \sum\limits_{i=1}^{3} |x_{ij}|$, $\forall\ X = (x_{ij}) \in A$. Define the mappings f, g and involution $'*'$ on A such that

$$f\begin{pmatrix} 0 & x_{12} & x_{13} \\ 0 & 0 & x_{23} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x_{12} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ g\begin{pmatrix} 0 & x_{12} & x_{13} \\ 0 & 0 & x_{23} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & x_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $\begin{pmatrix} 0 & x_{12} & x_{13} \\ 0 & 0 & x_{23} \\ 0 & 0 & 0 \end{pmatrix}^{*} = \begin{pmatrix} 0 & -x_{12} & x_{13} \\ 0 & 0 & -x_{23} \\ 0 & 0 & 0 \end{pmatrix}$.

Then it can be easily check that A is a Banach $*$-algebra and (for $f = \zeta$) ζ satisfies the conditions $\zeta(X)X^n + X^n\zeta(X) = 0$ and $\zeta(X)(X^*)^n + (X^*)^n\zeta(X) = 0$, $\forall\ X \in A$, but $\zeta \neq 0$. Further, it is straightforward to check that the mappings f, g satisfies the relation (3.1) and $f(x)x^* + x^*g(x) \in Z(A)$, $\forall\ x \in A$. However, $f \neq 0$ and $g \neq 0$. Hence, in Theorems 3.1, 3.3 and 3.5, the hypothesis of C^*-algebra is crucial.

We conclude the paper with the following open problems for further studies.

Problem 3.7. Let n be a fixed positive integer and A be a C^*-algebra. Next, let f and g be a pair of additive mappings of A such that

$$f(x)x^n + x^n g(x) = 0 \text{ or } \in Z(A), \forall x \in A.$$

Then what we can say about the behavior of f and g?

Problem 3.8. Let m, n be fixed positive integers and A be a C^*-algebra. Next, let f and g be a pair of additive mappings of A such that

$$f(x^m)x^n + x^n g(x^m) = 0 \text{ or } \in Z(A), \forall x \in A.$$

Then what we can say about the behavior f and g?

Problem 3.9. Let m, n be fixed positive integers and A be a C^*-algebra. Next, let f and g be a pair of additive mappings of A such that

$$f(x^m)x^{*n} + x^{*n} g(x^m) = 0 \text{ or } \in Z(A), \forall x \in A.$$

Then what we can say about the behavior of f and g?

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