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Some Results on Statistical Hypersurfaces of Sasakian Statistical Manifolds and Holomorphic Statistical Manifolds

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(Dedicated to the memory of **Prof. Dr. Aurel BEJANCU (1946 - 2020)**)

ABSTRACT

In this paper, we study the statistical immersion of codimension one from a Sasakian statistical manifold of constant $\phi-$curvature to a holomorphic statistical manifold of constant holomorphic curvature and its converse. We prove that in both cases the constant $\phi-$curvature equals to one and the constant holomorphic curvature must be zero. Moreover, we construct several examples of statistical manifolds, Sasakian statistical manifolds and holomorphic statistical manifolds of constant holomorphic curvature zero.

Keywords: *Sasakian statistical manifolds, holomorphic statistical manifolds, constant $\phi-$curvature, constant holomorphic curvature, statistical hypersurface.*

AMS Subject Classification (2020): *53C15; 53B25; 53D10.*

1. Introduction

The notion of statistical structure was initially introduced from the treatment of statistical inference problems in information geometry by S. Amari in 1985[1]. From then on, the geometry of statistical manifolds has developed in close relations with affine differential geometry[13] and Hessian geometry[16]. By definition, a statistical structure can be viewed as a generalization of a Riemannian structure containing a Riemannian metric and its Levi-Civita connection. Inspired from this idea, in 2004, T. Kurose introduced the notion of holomorphic statistical structure as a generalization of Kähler structure in [12]. Several years later, H. Furuhata introduced the notion of Sasakian statistical structure as a generalization of Sasakian structure in [10] and Kenmotsu statistical structure as a generalization of Kenmotsu structure in [9].

Since a statistical structure can be considered as a generalization of a Riemannian structure, it is natural to consider whether the results in Riemannian geometry still hold in the geometry of statistical manifolds or not. For example, in 1999, B. Y. Chen[6] obtained a sharp relationship between the squared mean curvature and the Ricci curvature for a Riemannian submanifold of a real space form, which is known as the Chen-Ricci inequality. Later in 2015, M. E. Aydin, A. Mihai and I. Mihai[3] established the Chen-Ricci inequality for a statistical submanifold of a statistical manifold of constant curvature, which generalized Chen's classical result. Moreover, many other geometric inequalities in classical Riemannian geometry have been generalized to various statistical manifolds. For instance, M. E. Aydin, A. Mihai and I. Mihai[4] obtained the generalized Wintgen inequality for statistical submanifolds in statistical manifolds of constant curvature; B. Y. Chen, A. Mihai, and I. Mihai[7] proved the Chen first inequality for statistical submanifolds in Hessian manifolds of constant Hessian curvature. Also, some other results in Riemannian geometry can be generalized to the geometry of statistical manifolds. For example, in 2015, M. Milijević[14] generalized a classical result[11] on totally real submanifolds of complex space forms to totally real statistical submanifolds of holomorphic statistical manifolds. Recently, M. Milijević[15] proved the non-existence of CR submanifolds of maximal

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CR dimension with umbilical shape operators in holomorphic statistical manifolds of nonzero constant holomorphic sectional curvature. This result corresponds to the theorem about the non-existence of umbilical hypersurfaces in non-flat complex space forms due to Y. Tashiro and S. Tachibana[18].

In 2009, H. Furuhata[8] considers a holomorphic statistical manifold of constant holomorphic curvature as a statistical hypersurface of a statistical manifold of constant curvature, and proved that the constant holomorphic curvature must be zero. Inspired by H. Furuhata's result, in this paper we investigate the statistical immersion of codimension one from a Sasakian statistical manifold of constant $\phi-$curvature to a holomorphic statistical manifold of constant holomorphic curvature and its inverse. We prove that in both cases the constant $\phi-$curvature of the Sasakian statistical manifold must be equal to one and the constant holomorphic curvature of the holomorphic statistical manifold must be equal to zero(see Theorem 3.1 and Theorem 4.1).

This paper is organized as follows. In Section 2 we briefly recall some basic knowledge about Sasakian statistical manifolds, holomorphic statistical manifolds and statistical immersions. Our two main theorems are presented in Section 3 and Section 4. Besides, we also construct several examples of statistical structures, Sasakian statistical structures and holomorphic statistical structures in Section 5. Some of these structures depend on several functions. By choosing these functions adequately we obtain the holomorphic statistical structures of constant holomorphic curvature zero.

2. Preliminaries

Let (M, g) be a Riemannian manifold and ∇^0 be the Levi-Civita connection of g on M. Throughout this paper, we denote the set of sections of vector bundle $E \to M$ by $C^\infty(E)$. For instance, we denote the set of all smooth tangent vector fields on M by $C^\infty(TM)$ and the set of all smooth normal vector fields on M by $C^\infty(T^\perp M)$. Besides, $C^\infty(M, \mathbb{R})$ denotes the set of all smooth functions on M.

Let ∇ be an affine connection on a Riemannian manifold (M, g). The affine connection ∇^* is called the dual connection of ∇ with respect to g, if

$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla^*_Z Y) \tag{2.1}$$

holds for any $X, Y, Z \in C^\infty(TM)$.

Definition 2.1. [13] Let (M, g) be a Riemannian manifold and ∇ an affine connection on M. The pair (g, ∇) is called a statistical structure or a Codazzi structure, if ∇ is torsion free and $(\nabla_X g)(Y,Z) = (\nabla_Y g)(X,Z)$ holds for any $X, Y, Z \in C^\infty(TM)$. In this case, (M, g, ∇) is said to be a statistical manifold or a Codazzi manifold.

By definition, a Riemannian structure (g, ∇^0) is a special statistical structure, which is called a Riemannian statistical structure or a trival statistical structure. In fact, the Levi-Civita connection ∇^0 is self-dual with respect to the Riemannian metric g. Besides, if (g, ∇) is a statistical structure, so is (g, ∇^*).

It is known that the statistical manifold originated from statistics and here we give a classical statistical structure on a parametric statistical model.

Example 2.1. [13] Let (Ω, β) be a measurable space and $\mathcal{M}$ a parametric statistical model on Ω. Namely, $\mathcal{M}$ is a set of probability distributions on (Ω, β) parametrized by $\zeta = \left(\zeta^1, \ldots, \zeta^n\right) \in U \subset \mathbb{R}^n$:

$$\mathcal{M} = \left\{ p(x,\zeta) | p(x,\zeta) > 0, \int_\Omega p(x,\zeta)dx = 1 \right\}.$$

Under suitable conditions(see [2]), $\mathcal{M}$ is regarded as a manifold with a local coordinate system $(\zeta^1, \ldots, \zeta^n)$.

We set

$$g(\zeta) := \sum \{ \int_\Omega \left(\frac{\partial \log p}{\partial \zeta^i}(x,\zeta) \right) \left(\frac{\partial \log p}{\partial \zeta^j}(x,\zeta) \right) p(x,\zeta)dx \} d\zeta^i d\zeta^j,$$

and

$$\Gamma^{(\alpha)}_{ijk}(\zeta) := \int_\Omega \left\{ \frac{\partial^2 \log p}{\partial \zeta^i \partial \zeta^j}(x,\zeta) + \frac{1-\alpha}{2} \frac{\partial \log p}{\partial \zeta^i}(x,\zeta) \frac{\partial \log p}{\partial \zeta^j}(x,\zeta) \right\} \frac{\partial \log p}{\partial \zeta^k}(x,\zeta) p(x,\zeta) dx (\alpha \in \mathbb{R}).$$

Define an affine connection $\nabla^{(\alpha)}$ by $g(\nabla^{(\alpha)}_{\frac{\partial}{\partial \zeta^i}} \frac{\partial}{\partial \zeta^j}, \frac{\partial}{\partial \zeta^k}) = \Gamma^{(\alpha)}_{ijk}(\zeta)$. It can be proved that $(\mathcal{M}, g, \nabla^{(\alpha)})$ is a statistical manifold. In fact, g is known as the Fisher metric and $\nabla^{(\alpha)}$ the $\alpha-$connection with respect to g.

Proposition 2.1. [8] *Let (M, g, ∇) be a statistical manifold and ∇^0 the Levi-Civita connection of g on M. For any $X, Y, Z \in C^\infty(TM)$, the difference tensor field K of type $(1,2)$ defined by $K_X Y = \nabla_X Y - \nabla^0_X Y$ satisfies:*

$$K_X Y = K_Y X, \; g(K_X Y, Z) = g(K_X Z, Y). \tag{2.2}$$

Conversely, if a $(1,2)-$tensor field K on M satisfies (2.2), then $(M, g, \nabla^0 + K)$ is a statistical manifold.

Next we introduce the notion of statistical immersion and give some basic properties of statistical submanifolds.

Definition 2.2. [8] Let $(\tilde{M}, \tilde{g}, \tilde{\nabla})$ be a statistical manifold and $f : M \to \tilde{M}$ be an immersion. Denote the tangent mapping and the cotagent mapping of f by f_* and f^* respectively. Define g and ∇ on M by

$$g = f^*\tilde{g}, \;\; g(\nabla_X Y, Z) = \tilde{g}(\tilde{\nabla}_X f_* Y, f_* Z), \;\; \forall X, Y, Z \in C^\infty(TM).$$

Then the pair (g, ∇) is a statistical structure on M, which is called the induced statistical structure by f from $(\tilde{g}, \tilde{\nabla})$.

Let (M, g, ∇) and $(\tilde{M}, \tilde{g}, \tilde{\nabla})$ be two statistical manifolds. An immersion $f : M \to \tilde{M}$ is called a statistical immersion if (g, ∇) coincides with the induced statistical structure[8]. Also, (M, g, ∇) is called a statistical submanifold of $(\tilde{M}, \tilde{g}, \tilde{\nabla})$. Similar to the Riemannian submanifolds, the Gauss and the Weingarten formulas in statistical submanifolds are as follows[20]:

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \; \tilde{\nabla}^*_X Y = \nabla^*_X Y + h^*(X, Y), \tag{2.3}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^\perp_X N, \; \tilde{\nabla}^*_X N = -A^*_N X + \nabla^{*\perp}_X N, \tag{2.4}$$

where $X, Y \in C^\infty(TM), \; N \in C^\infty(T^\perp M)$. In the above formulas, h and h^* are the second fundamental forms with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively; A and A^* are the shape operators with respect to $\tilde{\nabla}$ and $\tilde{\nabla}^*$, respectively. Besides, there are relationships among them similar to the Riemannian case[20]:

$$h(X, Y) = h(Y, X), \;\; h^*(X, Y) = h^*(Y, X), \tag{2.5}$$

$$g(A_N X, Y) = \tilde{g}(h^*(X, Y), N), \;\; g(A^*_N X, Y) = \tilde{g}(h(X, Y), N). \tag{2.6}$$

Proposition 2.2. [20] *Let $f : (M, g, \nabla) \to (\tilde{M}, \tilde{g}, \tilde{\nabla})$ be a statistical immersion. Denote the curvature tensor field of $\tilde{\nabla}(resp. \; \nabla)$ by $\tilde{R}(resp. \; R)$. Then the Gauss equation is*

$$[\tilde{R}(X, Y)Z]^\top = R(X, Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X, \tag{2.7}$$

where any $X, Y, Z \in C^\infty(TM), \; N \in C^\infty(T^\perp M)$, and $[\,]^\top$ denotes the tangential component of $[\,]$.

Remark 2.1. By (2.7), if M is a statistical hypersurface of $\tilde{M}$ and N is the unit normal vector field on M, then the Gauss equation can be written as

$$[\tilde{R}(X, Y)Z]^\top = R(X, Y)Z + g(A^*X, Z)AY - g(A^*Y, Z)AX, \tag{2.8}$$

where we respectively denote A_N and A^*_N by A and A^* for simplicity.

Next, we review the definition of Kähler manifold, which is known as an important object in Riemannian geometry.

Definition 2.3. [21] Let M be an even dimensional differential manifold and J a $(1,1)-$tensor field on M. J is called an almost complex structure if $J^2 = -Id$, where Id denotes the identity transformation. A manifold endowed with an almost complex structure is called an almost complex manifold.

Definition 2.4. [21] Let (M, J) be an almost complex manifold and g a Riemannian metric on M. If $g(JX, JY) = g(X, Y)$ holds for any $X, Y \in C^\infty(TM)$, then g is called an almost Hermitian metric and (M, J, g) is called an almost Hermitian manifold.

Proposition 2.3. [21] *Let (M, J, g) be an almost Hermitian manifold. For any $X, Y \in C^\infty(TM)$, we define*

$$\omega(X, Y) = g(X, JY). \tag{2.9}$$

Then ω is a $2-$form on M. Especially, $\omega(X, X) = 0$.

Definition 2.5. [21] An almost Hermitian manifold (M, J, g) is called a Kähler manifold if ω is closed, namely, $d\omega = 0$.

As we know, a statistical structure can be considered as a generalization of a Riemannian structure. Motivated by this idea, T. Kurose[12] introduced the notion of holomorphic statistical manifold by endowing a Kähler manifold with a suitable statistical structure.

Definition 2.6. [8] (M, J, g, ∇) is called a holomorphic statistical manifold if (M, J, g) is a Kähler manifold, (∇, g) is a statistical structure on M and $\nabla\omega = 0$, where ω is defined by (2.9).

Proposition 2.4. [8] *Let (M, J, g) be a Kähler manifold and (g, ∇) a statistical structure on M. Then (M, J, g, ∇) is a holomorphic statistical manifold if and only if the difference tensor field K satisfies*

$$K_X JY + JK_X Y = 0 \tag{2.10}$$

for any $X, Y \in C^\infty(TM)$.

Conversely, if a $(1, 2)-$tensor field K on M satisfies (2.2) and (2.10), then $(M, J, g, \nabla^0 + K)$ is a holomorphic statistical manifold, where ∇^0 denotes the Levi-Civita connection of g.

Definition 2.7. [8] A holomorphic statistical manifold is said to be of constant holomorphic curvature $k \in \mathbb{R}$ if

$$R(X, Y)Z = \frac{k}{4}\{g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - g(JX, Z)JY + 2g(X, JY)JZ\} \tag{2.11}$$

holds for any $X, Y \in C^\infty(TM)$, where R is the curvature tensor field of ∇.

Now we recall the notion of Sasakian manifold, which is another classical topic in differential geometry.

Definition 2.8. [5] Let ϕ, ξ, η respectively represent a $(1, 1)-$tensor field, a vector field and a $1-$form on an odd dimensional manifold M. If the following equations hold for any $X \in C^\infty(TM)$:

$$\phi^2 X = -X + \eta(X)\xi,\ \eta(\xi) = 1,\ \eta(\phi X) = 0, \tag{2.12}$$

then the triple (ϕ, ξ, η) is called an almost contact structure. In an almost contact structure (ϕ, ξ, η), ξ is called the structure vector field.

Definition 2.9. [5] Let (ϕ, ξ, η) be an almost contact structure and g be a Riemannian metric on an odd dimensional manifold M. For any $X, Y \in C^\infty(TM)$, if

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\ \eta(X) = g(X, \xi), \tag{2.13}$$

then the quadruple (ϕ, ξ, η, g) is called an almost contact metric structure and (M, ϕ, ξ, η, g) is called an almost contact metric manifold.

Proposition 2.5. [5] *Let (M, ϕ, ξ, η, g) be an almost contact metric manifold. For any $X, Y \in C^\infty(TM)$, we define*

$$\Omega(X, Y) := g(X, \phi Y). \tag{2.14}$$

Then Ω is a 2-form on M. In particular, $\Omega(X, X) = 0$ and $\phi\xi = 0$.

Definition 2.10. [5] An almost contact metric manifold (M, ϕ, ξ, η, g) is called a Sasakian manifold if

$$(\nabla^0_X \phi)Y = g(X, Y)\xi - \eta(Y)X \tag{2.15}$$

holds for any $X, Y \in C^\infty(TM)$, where ∇^0 is the Levi-Civita connection of g on M.

Similarly, H. Furuhata introduced the notion of Sasakian statistical manifold by endowing a Sasakian manifold with a suitable statistical structure[10].

Definition 2.11. [10] $(M, \phi, \xi, \eta, g, \nabla)$ is called a Sasakian statistical manifold if (M, ϕ, ξ, η, g) is a Sasakian manifold, (∇, g) is a statistical structure on M and $\nabla\Omega = \nabla^*\Omega$, where Ω is defined by (2.14).

Proposition 2.6. [10] *Let (M, ϕ, ξ, η, g) be a Sasakian manifold and (g, ∇) a statistical structure on M. Then $(M, \phi, \xi, \eta, g, \nabla)$ is a Sasakian statistical manifold if and only if the difference tensor field K satisfies*

$$K_X \phi Y + \phi K_X Y = 0 \tag{2.16}$$

for any $X, Y \in C^\infty(TM)$.

Conversely, if a $(1,2)-$tensor field K on M satisfies (2.2) and (2.16), then $(M, \phi, \xi, \eta, g, \nabla^0 + K)$ is a Sasakian statistical manifold, where ∇^0 denotes the Levi-Civita connection of g.

Definition 2.12. A Sasakian statistical manifold $(M, \phi, \xi, \eta, g, \nabla)$ is said to be of constant $\phi-$curvature $c \in \mathbb{R}$ if

$$\begin{aligned} R(X,Y)Z =& \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \end{aligned} \tag{2.17}$$

holds for any $X, Y, Z \in C^\infty(TM)$, where R is the curvature tensor field of ∇.

3. Sasakian statistical manifolds as hypersurfaces of holomorphic statistical manifolds

Theorem 3.1. *Let $(\tilde{M}, J, \tilde{g}, \tilde{\nabla})$ be a $(2m+2)-$dimensional holomorphic statistical manifold of constant holomorphic curvature k, $m \geq 4$, and $(M, \phi, \xi, \eta, g, \nabla)$ be a $(2m+1)-$dimensional Sasakian statistical manifold of constant $\phi-$curvature c. If M is a statistical hypersurface of $\tilde{M}$, then $k = 0, c = 1$.*

Proof. Since M is a statistical hypersurface of $\tilde{M}$, substituting (2.11) and (2.17) into the Gauss equation (2.8), we have

$$\begin{aligned} &\frac{k}{4}\{\tilde{g}(Y,Z)X - \tilde{g}(X,Z)Y + \tilde{g}(JY,Z)JX - \tilde{g}(JX,Z)JY + 2\tilde{g}(X,JY)JZ\}^\top \\ &= \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi \\ &\quad + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} + g(A^*X, Z)AY - g(A^*Y, Z)AX. \end{aligned} \tag{3.1}$$

Denote the left-hand side of (3.1) and the right-hand side of (3.1) by A_1 and B_1 respectively. Let $\{e_i\}$ be a local orthonormal frame field on M and N be a unit normal vector field on M. Putting $Y = Z = e_i$ in (3.1) and summing with respect to i, one obtains

$$\begin{aligned} A_1 &= \frac{k}{4}\sum_{i=1}^{2m+1}\{\tilde{g}(e_i,e_i)X - \tilde{g}(X,e_i)e_i + \tilde{g}(Je_i,e_i)JX - \tilde{g}(JX,e_i)Je_i + 2\tilde{g}(X,Je_i)Je_i\}^\top \\ &= \frac{k}{4}\{(2m+1)X - X - 3J\sum_{i=1}^{2m+1}\tilde{g}(JX,e_i)e_i\}^\top \\ &= \frac{k}{4}\{2mX - 3J[JX - \tilde{g}(JX,N)N]\}^\top \\ &= \frac{k}{4}\{(2m+3)X - 3g(X,JN)JN\}, \end{aligned}$$

where we have used the fact that $\tilde{g}(X, JY) = -\tilde{g}(JX, Y)$ and JN is a tangent vector field since $\tilde{g}(N, JN) = 0$. Besides,

$$\begin{aligned}
B_1 = &\frac{c+3}{4}\sum_{i=1}^{2m+1}\{g(e_i,e_i)X - g(X,e_i)e_i\} + \frac{c-1}{4}\sum_{i=1}^{2m+1}\{\eta(X)\eta(e_i)e_i - \eta(e_i)\eta(e_i)X \\
&+ g(X,e_i)\eta(e_i)\xi - g(e_i,e_i)\eta(X)\xi + g(\phi e_i,e_i)\phi X - g(\phi X,e_i)\phi e_i - 2g(\phi X,e_i)\phi e_i\} \\
&+ g(A^*X,e_i)Ae_i - g(A^*e_i,e_i)AX \\
= &\frac{c+3}{4}\{(2m+1)X - X\} + \frac{c-1}{4}\{\eta(X)\sum_{i=1}^{2m+1} g(e_i,\xi)e_i - g(\sum_{i=1}^{2m+1} g(e_i,\xi)e_i,\xi)X \\
&+ g(X,\sum_{i=1}^{2m+1} g(e_i,\xi)e_i)\xi - (2m+1)\eta(X)\xi - 3\phi\sum_{i=1}^{2m+1} g(\phi X,e_i)e_i\} + AA^*X - (trA^*)AX \\
= &\frac{c+3}{4}(2mX) + \frac{c-1}{4}\{\eta(X)\xi - X + \eta(X)\xi - (2m+1)\eta(X)\xi + 3X - 3\eta(X)\xi\} \\
&+ AA^*X - (trA^*)AX \\
= &\frac{c+3}{4}(2mX) + \frac{c-1}{4}\{(-2-2m)\eta(X)\xi + 2X\} + AA^*X - (trA^*)AX,
\end{aligned}$$

where we have used the equation $g(X, \phi Y) = -g(\phi X, Y)$. Then we have

$$\frac{k}{4}\{(2m+3)X - 3g(X,JN)JN\} = \frac{c+3}{4}(2mX) + \frac{c-1}{4}\{(-2-2m)\eta(X)\xi + 2X\} + AA^*X - (trA^*)AX.$$

For simplicity, we write

$$p = \frac{(2m+3)k - 2m(c+3) - 2(c-1)}{4},\ q = \frac{(c-1)(2m+2)}{4},\ r = -\frac{3k}{4},$$

then the above formula becomes

$$AA^*X - (trA^*)AX = pX + q\eta(X)\xi + rg(X,JN)JN. \tag{3.2}$$

Choose $\{e_i\}$ such that $Ae_i = \lambda_i e_i$ and assume that $A^*e_i = \sum_l a_i^l e_l$. Putting $X = e_i$ in (3.2), we obtain

$$\sum_l a_i^l\lambda_l e_l - (trA^*)\lambda_i e_i = pe_i + q\eta(e_i)\sum_l \eta(e_l)e_l + rg(e_i,JN)\sum_l g(e_l,JN)e_l,$$

which implies:

$$\begin{aligned}
a_i^i\lambda_i - (trA^*)\lambda_i &= p + q\eta^2(e_i) + rg^2(e_i,JN), \\
a_i^l\lambda_l &= q\eta(e_i)\eta(e_l) + rg(e_i,JN)g(e_l,JN) \ \ (l \neq i).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
AA^*e_i &= \sum_l a_i^l\lambda_l e_l = a_i^i\lambda_i e_i + \sum_{l(\neq i)} a_i^l\lambda_l e_l \\
&= a_i^i\lambda_i e_i + \sum_{l(\neq i)}[q\eta(e_i)\eta(e_l) + rg(e_i,JN)g(e_l,JN)]e_l, \\
A^*Ae_i &= \lambda_i\sum_l a_i^l e_l = \lambda_i a_i^i e_i + \sum_{l(\neq i)}\lambda_i a_i^l e_l = \lambda_i a_i^i e_i + \sum_{l(\neq i)}\lambda_i a_l^i e_l \\
&= a_i^i\lambda_i e_i + \sum_{l(\neq i)}[q\eta(e_i)\eta(e_l) + rg(e_i,JN)g(e_l,JN)]e_l,
\end{aligned}$$

so that $AA^* = A^*A$. Therefore, we can choose a local orthonormal frame field $\{e_i\}$ such that $Ae_i = \lambda_i e_i$ and $A^*e_i = \lambda_i^* e_i$. Putting $X = e_i$ and $Y = e_j$ in (3.1), we have

$$\begin{aligned}
&\frac{k}{4}\{\tilde{g}(e_j,Z)e_i - \tilde{g}(e_i,Z)e_j + \tilde{g}(Je_j,Z)Je_i - \tilde{g}(Je_i,Z)Je_j + 2\tilde{g}(e_i,Je_j)JZ\}^\top \\
= &\frac{c+3}{4}\{g(e_j,Z)e_i - g(e_i,Z)e_j\} + \frac{c-1}{4}\{\eta(e_i)\eta(Z)e_j - \eta(e_j)\eta(Z)e_i + g(e_i,Z)\eta(e_j)\xi - g(e_j,Z)\eta(e_i)\xi \\
&+ g(\phi e_j,Z)\phi e_i - g(\phi e_i,Z)\phi e_j - 2g(\phi e_i,e_j)\phi Z\} + g(\lambda_i^* e_i,Z)\lambda_j e_j - g(\lambda_j^* e_j,Z)\lambda_i e_i.
\end{aligned}$$

If Z is normal to span $\{e_i, e_j\}$, then it follows that

$$\begin{aligned}&\frac{k}{4}\{\tilde{g}(Je_j, Z)Je_i - \tilde{g}(Je_i, Z)Je_j + 2\tilde{g}(e_i, Je_j)JZ\}^{\top}\\&= \frac{c-1}{4}\{\eta(e_i)\eta(Z)e_j - \eta(e_j)\eta(Z)e_i + g(\phi e_j, Z)\phi e_i - g(\phi e_i, Z)\phi e_j - 2g(\phi e_i, e_j)\phi Z\}.\end{aligned} \tag{3.3}$$

If $g(\phi e_i, e_j) = 0$ for any i, j, then $\phi e_i = \sum\limits_k g(\phi e_i, e_k)e_k = 0$, which means $\phi = 0$. Obviously, this is impossible. Thus, there exist i, j such that $g(\phi e_i, e_j) \neq 0$. Assume that $c \neq 1$, then it follows from (3.3) that $\phi Z \in span\{e_i, e_j, \phi e_i, \phi e_j, (Je_i)^{\top}, (Je_j)^{\top}, (JZ)^{\top}\}$. Noting that Z is normal to span $\{e_i, e_j\}$, we see that $rank\phi \leq 7$. It contradicts the fact that $rank\phi = 2m$(see Theorem 4.1 in [5]) and $m \geq 4$. This proves that $c = 1$. Then, we have

$$\frac{k}{4}\{\tilde{g}(Je_j, Z)Je_i - \tilde{g}(Je_i, Z)Je_j + 2\tilde{g}(e_i, Je_j)JZ\}^{\top} = 0.$$

If $\tilde{g}(e_i, Je_j) = 0$ holds for any i, j, then $(Je_j)^{\top} = \sum\limits_i g(e_i, Je_j)e_i = 0$, which implies that for any tangent vector field X, JX is parallel with the unit normal vector field N. This is impossible. Hence, there exist i, j such that $\tilde{g}(e_i, Je_j) \neq 0$. If $k \neq 0$, then $JZ \in span\{Je_i, Je_j, N\}$, which gives that $rankJ \leq 4$. It contradicts the fact that $rankJ = 2m + 2$(see page 7 of [21]) and $m \geq 4$ similarly. Thus, $k = 0$. This completes the proof.

□

4. Holomorphic statistical manifolds as hypersurfaces of Sasakian statistical manifolds

Theorem 4.1. *Let $(\tilde{M}, \phi, \xi, \eta, \tilde{g}, \tilde{\nabla})$ be a $(2m+1)-$dimensional Sasakian statistical manifold of constant $\phi-$curvature k, $m \geq 5$, and (M, J, g, ∇) be a $2m-$dimensional holomorphic statistical manifold of constant holomorphic curvature c. If M is a statistical hypersurface of $\tilde{M}$, then $k = 1, c = 0$.*

Proof. Since M is a statistical hypersurface of Sasakian statistical manifold $\tilde{M}$, substituting (2.11) and (2.17) into the Gauss equation (2.8), we have

$$\begin{aligned}&\frac{k+3}{4}\{\tilde{g}(Y,Z)X - \tilde{g}(X,Z)Y\}^{\top} + \frac{k-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \tilde{g}(X,Z)\eta(Y)\xi - \tilde{g}(Y,Z)\eta(X)\xi\\&+ \tilde{g}(\phi Y, Z)\phi X - \tilde{g}(\phi X, Z)\phi Y - 2\tilde{g}(\phi X, Y)\phi Z\}^{\top}\\&= \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY + 2g(X,JY)JZ\}\\&\quad + g(A^*X, Z)AY - g(A^*Y, Z)AX.\end{aligned} \tag{4.1}$$

Denote the left-hand side of (4.1) and the right-hand side of (4.1) by A_2 and B_2 respectively. Let $\{e_i\}$ be a local orthonormal frame field on M and N be the normal vector field on M. Putting $Y = Z = e_i$ in (4.1) and summing with respect to i, one obtains

$$\begin{aligned}A_2 =& \frac{k+3}{4}\{2mX - X\} + \frac{k-1}{4}\{\eta(X)\sum_{i=1}^{2m}\eta(e_i)e_i - \sum_{i=1}^{2m}\eta(e_i)\eta(e_i)X + \sum_{i=1}^{2m}\tilde{g}(X,e_i)\eta(e_i)\xi\\&- \sum_{i=1}^{2m}\tilde{g}(e_i,e_i)\eta(X)\xi + \sum_{i=1}^{2m}\tilde{g}(\phi e_i, e_i)\phi X - \sum_{i=1}^{2m}\tilde{g}(\phi X, e_i)\phi e_i - 2\sum_{i=1}^{2m}\tilde{g}(\phi X, e_i)\phi e_i\}^{\top}\\=& \frac{k+3}{4}(2m-1)X + \frac{k-1}{4}\{\eta(X)\xi^{\top} - |\xi^{\top}|^2 X + \eta(X)\xi - 2m\eta(X)\xi - 3\phi[\sum_{i=1}^{2m}\tilde{g}(\phi X, e_i)e_i]\}^{\top}\\=& \frac{k+3}{4}(2m-1)X + \frac{k-1}{4}\{\eta(X)\xi^{\top} - |\xi^{\top}|^2 X + \eta(X)\xi - 2m\eta(X)\xi - 3\phi[\phi X - \tilde{g}(\phi X, N)N]\}^{\top}\\=& \frac{k+3}{4}(2m-1)X + \frac{k-1}{4}\{(3 - |\xi^{\top}|^2)X - (2m+1)\eta(X)\xi^{\top} - 3g(X, \phi N)\phi N\},\end{aligned}$$

where we have used the fact that $\tilde{g}(X,\phi Y)=-\tilde{g}(\phi X,Y)$ and ϕN is a tangent vector field since $\tilde{g}(\phi N,N)=0$. Besides,

$$\begin{aligned} B_2 &= \frac{c}{4}\{2mX-X-3J\sum_{i=1}^{2m} g(JX,e_i)e_i\}+g(A^*X,e_i)Ae_i-g(A^*e_i,e_i)AX \\ &= \frac{c}{4}\{2mX-X-3J(JX)\}+AA^*X-(trA^*)AX \\ &= \frac{c}{4}(2m+2)X+AA^*X-(trA^*)AX. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\frac{k+3}{4}(2m-1)X+\frac{k-1}{4}\{(3-|\xi^\top|^2)X-(2m+1)\eta(X)\xi^\top-3g(X,\phi N)\phi N\} \\ &=\frac{c}{4}(2m+2)X+AA^*X-(trA^*)AX. \end{aligned}$$

For simplicity, we write

$$p=\frac{(k+3)(2m-1)+(k-1)(3-|\xi^\top|^2)-c(2m+2)}{4}, q=\frac{(2m+1)(1-k)}{4}, r=\frac{3(1-k)}{4},$$

then the above formula becomes

$$AA^*X-(trA^*)AX=pX+q\eta(X)\xi^\top+rg(X,\phi N)\phi N. \tag{4.2}$$

Choose $\{e_i\}$ such that $Ae_i=\lambda_i e_i$ and assume that $A^*e_i=\sum_l a_i^l e_l$. Putting $X=e_i$ in (4.2), we obtain

$$\sum_l a_i^l\lambda_l e_l-(trA^*)\lambda_i e_i=pe_i+q\eta(e_i)\sum_l \eta(e_l)e_l+rg(e_i,\phi N)\sum_l g(e_l,\phi N)e_l,$$

which implies:

$$\begin{aligned} a_i^i\lambda_i-(trA^*)\lambda_i &= p+q\eta^2(e_i)+rg^2(e_i,\phi N), \\ a_i^l\lambda_l &= q\eta(e_i)\eta(e_l)+rg(e_i,\phi N)g(e_l,\phi N) \ \ (l\neq i). \end{aligned}$$

Hence, we have

$$\begin{aligned} AA^*e_i &= A\sum_l a_i^l e_l=\sum_l a_i^l\lambda_l e_l=a_i^i\lambda_i e_i+\sum_{l(\neq i)} a_i^l\lambda_l e_l \\ &= a_i^i\lambda_i e_i+\sum_{l(\neq i)}[q\eta(e_i)\eta(e_l)+rg(e_i,\phi N)g(e_l,\phi N)]e_l, \\ A^*Ae_i &= A^*\lambda_i e_i=\lambda_i\sum_l a_i^l e_l=\lambda_i a_i^i e_i+\sum_{l(\neq i)}\lambda_i a_i^l e_l=\lambda_i a_i^i e_i+\sum_{l(\neq i)}\lambda_i a_l^i e_l \\ &= a_i^i\lambda_i e_i+\sum_{l(\neq i)}[q\eta(e_i)\eta(e_l)+rg(e_i,\phi N)g(e_l,\phi N)]e_l, \end{aligned}$$

so that $AA^*=A^*A$. Therefore, we can choose a local orthonormal frame field $\{e_i\}$ such that $Ae_i=\lambda_i e_i$ and $A^*e_i=\lambda_i^* e_i$. Putting $X=e_i$ and $Y=e_j$ in (4.1), we obtain

$$\begin{aligned} &\frac{k+3}{4}\{\tilde{g}(e_j,Z)e_i-\tilde{g}(e_i,Z)e_j\}+\frac{k-1}{4}\{\eta(e_i)\eta(Z)e_j-\eta(e_j)\eta(Z)e_i+\tilde{g}(e_i,Z)\eta(e_j)\xi-\tilde{g}(e_j,Z)\eta(e_i)\xi \\ &+\tilde{g}(\phi e_j,Z)\phi e_i-\tilde{g}(\phi e_i,Z)\phi e_j-2\tilde{g}(\phi e_i,e_j)\phi Z\}^\top \\ &=\frac{c}{4}\{g(e_j,Z)e_i-g(e_i,Z)e_j+g(Je_j,Z)Je_i-g(Je_i,Z)Je_j+2g(e_i,Je_j)JZ\} \\ &\ \ +g(\lambda_i^*e_i,Z)\lambda_j e_j-g(\lambda_j^*e_j,Z)\lambda_i e_i. \end{aligned}$$

If Z is normal to span $\{e_i,e_j\}$, then

$$\begin{aligned} &\frac{k-1}{4}\{\eta(e_i)\eta(Z)e_j-\eta(e_j)\eta(Z)e_i+\tilde{g}(\phi e_j,Z)\phi e_i-\tilde{g}(\phi e_i,Z)\phi e_j-2\tilde{g}(\phi e_i,e_j)\phi Z\}^\top \\ &=\frac{c}{4}\{g(Je_j,Z)Je_i-g(Je_i,Z)Je_j+2g(e_i,Je_j)JZ\}. \end{aligned} \tag{4.3}$$

If $\tilde{g}(\phi e_i, e_j) = 0$ for any i, j, then $(\phi e_i)^\top = \sum_k \tilde{g}(\phi e_i, e_k)e_k = 0$, which implies that for any $X \in C^\infty(TM)$, ϕX is parallel with N. This is impossible. Thus, there exist i, j such that $\tilde{g}(\phi e_i, e_j) \neq 0$. Assume that $k \neq 1$, then it follows from (4.3) that $\phi Z \in span\{e_i, e_j, \phi e_i, \phi e_j, N, Je_i, Je_j, JZ\}$. Noting that Z is normal to span$\{e_i, e_j, N\}$, we see that $rank\phi \leq 9$. It contradicts the fact that $rank\phi = 2m$(see Theorem 4.1 in [5]) and $m \geq 5$. This proves that $k = 1$. Then,

$$\frac{c}{4}\{g(Je_j, Z)Je_i - g(Je_i, Z)Je_j + 2g(e_i, Je_j)JZ\} = 0.$$

If $g(e_i, Je_j) = 0$ holds for any i, j, then $Je_j = \sum_i g(e_i, Je_j)e_i = 0$, which means that $JX = 0$ for any tangent vector field X. This is impossible. Hence, there exist i, j such that $g(e_i, Je_j) \neq 0$. If $c \neq 0$, then $JZ \in span\{Je_i, Je_j\}$, which gives that $rankJ \leq 2$. It contradicts the fact that $rankJ = 2m$(see page 7 of [21]) and $m \geq 5$ similarly. Thus, $c = 0$. This completes the proof. □

5. Examples of statistical manifolds

In this section we show some examples of statistical manifolds and provide several methods to construct Sasakian statistical manifolds and holomorphic statistical manifolds(see Example 5.2 and Example 5.3). Especially, we give all the Sasakian statistical structures on the usual Sasakian manifold $\mathbb{R}^3$ in terms of three independent functions(see Proposition 5.1). Moreover, we find out all the holomorphic statistical structures of constant holomorphic curvature 0 on a Kähler manifold due to A. N. Siddiqui and M. H. Shahid[17](see Proposition 5.2).

Example 5.1. Consider a 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 | y \neq 0\}$, where (x, y, z) are the standard coordinate system in $\mathbb{R}^3$. Let g be the Riemannian metric defined by $g = y^2(dx^2 + dy^2 + dz^2)$. Then (M, g) is a Riemannian manifold. Take $e_1 = \frac{\partial}{\partial x}, e_2 = \frac{\partial}{\partial y}, e_3 = \frac{\partial}{\partial z}$, then $\{e_i\}$ is an orthogonal frame field on M. Define an affine connection ∇ on M by

$$\begin{aligned}
&\nabla_{e_1}e_1 = 0, \quad &&\nabla_{e_1}e_2 = \frac{2}{y}e_1, \quad &&\nabla_{e_1}e_3 = \frac{1}{y}e_3,\\
&\nabla_{e_2}e_1 = \frac{2}{y}e_1, \quad &&\nabla_{e_2}e_2 = 0, \quad &&\nabla_{e_2}e_3 = 0,\\
&\nabla_{e_3}e_1 = \frac{1}{y}e_3, \quad &&\nabla_{e_3}e_2 = 0, \quad &&\nabla_{e_3}e_3 = \frac{1}{y}e_1 - \frac{2}{y}e_2.
\end{aligned}$$

It can be easily proved that $(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z)$ and $\nabla_X Y - \nabla_Y X - [X, Y] = 0$ hold for any $X, Y, Z \in C^\infty(TM)$. By Definition 2.1, (M, g, ∇) is a statistical manifold.

Example 5.2. Let (M, ϕ, ξ, η, g) be a Sasakian manifold. Take a vector field $\Psi \in C^\infty(TM)$ orthogonal to ξ and define a tensor field K of type $(1, 2)$ on M by

$$K(X, Y) = \{g(\phi\Psi, X)g(\Psi, Y) + g(\phi\Psi, Y)g(\Psi, X)\}\Psi + \{g(\Psi, X)g(\Psi, Y) - g(\phi\Psi, X)g(\phi\Psi, Y)\}\phi\Psi, \tag{5.1}$$

where $X, Y \in C^\infty(TM)$, then K satisfies (2.2) and (2.16). According to Proposition 2.6, $(M, \phi, \xi, \eta, g, \nabla^0 + K)$ is a Sasakian statistical manifold, where ∇^0 is the Levi-Civita connection of g on M.

Proposition 5.1. *Consider a 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3\}$, where (x, y, z) is the standard coordinate system in $\mathbb{R}^3$. The usual Sasakian structure on M is given by[5]*

$$\xi = 2\frac{\partial}{\partial z}, \eta = \frac{1}{2}(dz - ydx),$$

$$g = \eta \otimes \eta + \frac{1}{4}(dx \otimes dx + dy \otimes dy),$$

$$\phi(X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}) = Y\frac{\partial}{\partial x} - X\frac{\partial}{\partial y} + Yy\frac{\partial}{\partial z}.$$

Take $e_1 = \frac{\partial}{\partial x}, e_2 = \frac{\partial}{\partial y}, e_3 = \frac{\partial}{\partial z}$, then $\{e_i\}$ is an orthogonal frame field on M. Denote by $\{\omega_i\}$ the dual frame field of $\{e_i\}$. Let $K = \sum_{i,j,l=1}^{3} K_{ij}^l e_l \otimes \omega_i \otimes \omega_j$ be a $(1, 2)-$tensor field on M. By Proposition 2.6, $(M, \phi, \xi, \eta, g, \nabla^0 + K)$ is a Sasakian *statistical manifold if and only if K satisfies:*

$$K(e_i,e_j)=K(e_j,e_i),$$
$$g(K(e_i,e_j),e_k)=g(K(e_i,e_k),e_j),$$
$$K(e_i,\phi e_j)+\phi K(e_i,e_j)=0,$$

or equivalently, the coefficients $\{K^l_{ij}\}$ satisfy:

$$\begin{aligned}&K^1_{11}=-K^1_{22}=-K^2_{12}=-K^2_{21}=a,\ \ K^1_{12}=K^1_{21}=K^2_{11}=-K^2_{22}=b,\ \ K^3_{33}=c,\\&K^3_{12}=K^3_{21}=yb,\ \ K^3_{13}=K^3_{31}=-yc,\ \ K^3_{22}=-ay,\ \ K^3_{11}=ya+y^2c,\\&K^1_{13}=K^1_{23}=K^1_{31}=K^1_{32}=K^1_{33}=K^2_{13}=K^2_{23}=K^2_{31}=K^2_{32}=K^2_{33}=K^3_{23}=K^3_{32}=0,\end{aligned}\tag{5.2}$$

where $a,b,c\in C^\infty(M,\mathbb{R})$.

Example 5.3. Let (M,J,g) be a Kähler manifold. Take a vector field $\Psi\in C^\infty(TM)$ and define a tensor field K of type $(1,2)$ on M by

$$\begin{aligned}K(X,Y)=&\{\lambda[g(\Psi,X)g(\Psi,Y)-g(\Psi,JX)g(\Psi,JY)]-\mu[g(\Psi,JX)g(\Psi,Y)+g(\Psi,X)g(\Psi,JY)]\}\Psi\\&+\{\lambda[g(\Psi,X)g(\Psi,JY)+g(\Psi,JX)g(\Psi,Y)]-\mu[g(\Psi,JX)g(\Psi,JY)-g(\Psi,X)g(\Psi,Y)]\}J\Psi,\end{aligned}\tag{5.3}$$

where $\lambda,\mu\in C^\infty(M,\mathbb{R}),X,Y\in C^\infty(TM)$, then K satisfies (2.2) and (2.10). According to Proposition 2.4, $(M,\phi,\xi,\eta,g,\nabla^0+K)$ is a holomorphic statistical manifold, where ∇^0 is the Levi-Civita connection of g on M.

Lemma 5.1. [17] *Consider a $2-$dimensional manifold $M=\{(x,y)\in\mathbb{R}^2|x>0,y>0\}$, where (x,y) is the standard coordinate system in $\mathbb{R}^2$. The Riemannian metric g on M is given by $g=x\{(\mathrm{d}x)^2+(\mathrm{d}y)^2\}$ and the complex structure J is defined by*

$$J\frac{\partial}{\partial x}=\frac{\partial}{\partial y},J\frac{\partial}{\partial y}=-\frac{\partial}{\partial x},$$

then (M,g,J) is a Kähler manifold.

Proposition 5.2. *Let (M,g,J) be the Kähler manifold defined in Lemma 5.1 and ∇ be an affine connection on M. Then (M,g,J,∇) is a holomorphic statistical manifold of constant holomorphic curvature 0 if and only if*

$$\begin{aligned}&\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x}=\frac{1-2c_2}{2x}\frac{\partial}{\partial x}+\frac{c_1}{x}\frac{\partial}{\partial y},\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y}=\frac{c_1}{x}\frac{\partial}{\partial x}+\frac{1+2c_2}{2x}\frac{\partial}{\partial y},\\&\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x}=\frac{c_1}{x}\frac{\partial}{\partial x}+\frac{1+2c_2}{2x}\frac{\partial}{\partial y},\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y}=\frac{-1+2c_2}{2x}\frac{\partial}{\partial x}-\frac{c_1}{x}\frac{\partial}{\partial y},\end{aligned}\tag{5.4}$$

where c_1 and c_2 are constants satisfying $c_1^2+c_2^2=\frac{1}{4}$.

Proof. Take $e_1=\frac{\partial}{\partial x},e_2=\frac{\partial}{\partial y}$ for simplicity, then $\{e_i\}$ is an orthogonal frame field on M. Denote the Levi-Civita connection of g on M by ∇^0. By using Koszul's formula[19]:

$$2g(\nabla^0_XY,Z)=Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)-g(X,[Y,Z])+g(Y,[Z,X])+g(Z,[X,Y]),$$

we get:

$$\nabla^0_{e_1}e_1=\frac{1}{2x}e_1,\nabla^0_{e_1}e_2=\frac{1}{2x}e_2,$$
$$\nabla^0_{e_2}e_1=\frac{1}{2x}e_2,\nabla^0_{e_2}e_2=-\frac{1}{2x}e_1.$$

Let $\{\omega_i\}$ be the dual frame field of $\{e_i\}$ and $K=\sum_{i,j,l=1}^{2}K^l_{ij}e_l\otimes\omega_i\otimes\omega_j$ be a $(1,2)$-tensor field on M. By Proposition 2.4, (M,g,J,∇^0+K) is a holomorphic statistical manifold if and only if K satisfies:

$$K(e_i,e_j)=K(e_j,e_i),$$
$$g(K(e_i,e_j),e_k)=g(K(e_i,e_k),e_j),$$
$$K(e_i,Je_j)+JK(e_i,e_j)=0,$$

or equivalently the coefficients $\{K_{ij}^l\}$ satisfy:

$$\begin{aligned} &K_{12}^1 = K_{21}^1 = K_{11}^2 = -K_{22}^2 = a, \\ &-K_{11}^1 = K_{22}^1 = K_{12}^2 = K_{21}^2 = b, \end{aligned} \tag{5.5}$$

where any $a, b \in C^\infty(M, \mathbb{R})$. Therefore, $(M, g, J, \nabla = \nabla^0 + K)$ is a holomorphic statistical manifold if and only if the affine connection ∇ is given by

$$\begin{aligned} &\nabla_{e_1} e_1 = (\frac{1}{2x} - b)e_1 + ae_2, \nabla_{e_1} e_2 = ae_1 + (\frac{1}{2x} + b)e_2, \\ &\nabla_{e_2} e_1 = ae_1 + (\frac{1}{2x} + b)e_2, \nabla_{e_2} e_2 = (-\frac{1}{2x} + b)e_1 - ae_2. \end{aligned} \tag{5.6}$$

According to equation (2.11), (M, g, J, ∇) is of constant holomorphic curvature 0 if and only if $R(X, Y)Z = 0$ holds for any $X, Y, Z \in C^\infty(TM)$, which is equivalent to

$$R(e_1, e_2)e_1 = 0, \quad R(e_1, e_2)e_2 = 0.$$

By (5.6), we calculate

$$\begin{aligned} R(e_1, e_2)e_1 &= \nabla_{e_1}\nabla_{e_2} e_1 - \nabla_{e_2}\nabla_{e_1} e_1 \\ &= \nabla_{e_1}(ae_1 + (\frac{1}{2x} + b)e_2) - \nabla_{e_2}((\frac{1}{2x} - b)e_1 + ae_2) \\ &= e_1(a)e_1 + a[(\frac{1}{2x} - b)e_1 + ae_2] - \frac{1}{2x^2}e_2 + \frac{1}{2x}[ae_1 + (\frac{1}{2x} + b)e_2] + e_1(b)e_2 \\ &\quad + b[ae_1 + (\frac{1}{2x} + b)e_2] - \frac{1}{2x}[ae_1 + (\frac{1}{2x} + b)e_2] + e_2(b)e_1 + b[ae_1 + (\frac{1}{2x} + b)e_2] \\ &\quad - e_2(a)e_2 - a[(-\frac{1}{2x} + b)e_1 - ae_2] \\ &= [e_1(a) + e_2(b) + \frac{a}{x}]e_1 + [e_1(b) - e_2(a) + 2(a^2 + b^2) + \frac{b}{x} - \frac{1}{2x^2}]e_2, \end{aligned}$$

and similarly,

$$R(e_1, e_2)e_2 = [e_1(b) - e_2(a) - 2(a^2 + b^2) + \frac{b}{x} + \frac{1}{2x^2}]e_1 + [-e_1(a) - e_2(b) - \frac{a}{x}]e_2.$$

Hence, (M, g, J, ∇) is of constant holomorphic curvature 0 if and only if

$$e_1(a) + e_2(b) + \frac{a}{x} = 0, \tag{5.7}$$

$$e_1(b) - e_2(a) + 2(a^2 + b^2) + \frac{b}{x} - \frac{1}{2x^2} = 0, \tag{5.8}$$

$$e_1(b) - e_2(a) - 2(a^2 + b^2) + \frac{b}{x} + \frac{1}{2x^2} = 0. \tag{5.9}$$

In the following we solve the system of equations above.
(I)If $b = 0$, then we get $a = \frac{1}{2x}$ from (5.7), which satisfying (5.8) and (5.9).
(II)If $a = 0$, then we get $b = \frac{1}{2x}$from (5.7), which satisfying (5.8) and (5.9).
(III)If $a \neq 0$ and $b \neq 0$, then it can be proved that $e_2(a) = e_2(b) = 0$. In fact, subtracting (5.9) from (5.8), we get

$$a^2 + b^2 = \frac{1}{4x^2}, \tag{5.10}$$

which implies that (5.8) and (5.9) are equivalent to

$$e_1(b) - e_2(a) = -\frac{b}{x}. \tag{5.11}$$

Taking the derivative of (5.10) with respect to e_1 and e_2 respectively, we obtain

$$ae_1(a) + be_1(b) = -\frac{1}{4x^3}, \tag{5.12}$$

$$ae_2(a) + be_2(b) = 0. \tag{5.13}$$

Multiplying (5.11) by b and (5.7) by a, we obtain

$$be_1(b) - be_2(a) = -\frac{b^2}{x},$$

$$ae_1(a) + ae_2(b) + \frac{a^2}{x} = 0.$$

Adding the above two equations up, one gets

$$ae_1(a) + be_1(b) + ae_2(b) - be_2(a) = -\frac{a^2 + b^2}{x},$$

which together with (5.10), (5.12) and (5.13) gives

$$-\frac{a^2 + b^2}{b} e_2(a) = 0.$$

From the above equation and (5.13), we see that $e_2(a) = 0$ and $e_2(b) = 0$. Hence, (5.7) and (5.11) can be rewritten as

$$e_1(a) = -\frac{a}{x},\ e_1(b) = -\frac{b}{x}.$$

One can easily solve the above ordinary differential equations:

$$a = \frac{c_1}{x}, b = \frac{c_2}{x},$$

where c_1 and c_2 are two constant real numbers. Substituting them into (5.10), one has $c_1^2 + c_2^2 = \frac{1}{4}$.

Combining (I), (II) and (III), we prove that (M, g, J, ∇) is of constant holomorphic curvature 0 if and only if

$$a = \frac{c_1}{x}, b = \frac{c_2}{x},$$

where c_1, c_2 are two constants satisfying $c_1^2 + c_2^2 = \frac{1}{4}$. Substituting them into (5.6), we will get (5.4). This completes the proof.

□

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