INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 13 NO. 2 PAGE 152–159 (2020)
DOI: HTTPS://DOI.ORG/10.36890/IEJG.777046

Ricci Solitons on Multiply Warped Product Manifolds

Dilek Açıkgöz Kaya* and Leyla Onat

(Communicated by Henrique F. De Lima)

ABSTRACT

In this paper, we study Ricci solitons with the structure of multiply warped product manifolds. We also study Ricci solitons equipped with the concurrent vector fields on multiply warped product manifolds.

Keywords: *Ricci soliton; gradient Ricci soliton; warped product; multiply warped product.*
AMS Subject Classification (2020): *53C25*

1. Introduction

Einstein metrics have an important place in the frontier between mathematics and physics. In the last few decades, many researchers focused to this special class of metrics as well as some similar structures that are called Einstein type metrics [1]. One of the most famous among these which is introduced by Hamilton[9] is the concept of Ricci solitons [3, 7, 12, 13, 14, 15] . A Ricci soliton is a Riemannian metric together with a vector field X and a scalar λ which satisfies

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g. \tag{1.1}$$

(M, g, X, λ) is called shrinking, steady and expanding when $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$, respectively. When X is gradient of a function f, $(M, g, \nabla f, \lambda)$ is called a gradient Ricci soliton and equation (1.1) becomes

$$\mathrm{Ric} + \nabla^2 f = \lambda g$$

where $\nabla^2 f$ stands for the Hessian of f.

Lately, there is a special interest on the relation between warped product manifolds (introduced in [2]) and Ricci solitons. In [6, 8, 16] , the authors studied gradient Ricci solitons with the structure of warped products and considered the potential function of the gradient Ricci soliton so that the lift of a function defined on base or fiber. In [10], this idea is extended to gradient Ricci solitons with the structure of multiply warped product manifolds.

In [11], the authors investigated the inheritance properties of Ricci soliton warped product manifolds by their factor manifolds. Motivating from this, we consider Ricci solitons with the structure of multiply warped product manifolds and reach some hereditary properties to its factors.

2. Preliminaries

In this section, we give a brief summary of multiply warped products defined in [17].

Definition 2.1 **([17]).** Let (B, g_B) and (F_i, g_{F_i}) be r and s_i dimensional Riemannian manifolds where $i \in \{1, 2, \cdots, m\}$ and also $M = B \times F_1 \times F_2 \times \cdots \times F_m$ be an $n-$dimensional Riemannian manifold,

Received : *05-August-2020,* ***Accepted :*** *23-September-2020*
*** Corresponding author***

where $n = r + \sum_{i=1}^{m} s_i$. Let $b_i : B \longrightarrow \mathbb{R}^+$ be smooth functions for $i \in \{1, 2, \cdots, m\}$. The multiply warped product is the product manifold $B \times_{b_1} F_1 \times_{b_2} F_2 \times_{b_3} \cdots \times_{b_m} F_m$ furnished with the metric tensor $g = g_B \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}$ defined by

$$g = \pi^*(g_B) \oplus (b_1 \circ \pi)^2 \sigma_1^*(g_{F_1}) \oplus \cdots \oplus (b_m \circ \pi)^2 \sigma_m^*(g_{F_m})$$

where π and σ_i are the natural projections on B and F_i, respectively. The functions b_i are called the warping functions for $i \in \{1, 2, \cdots, m\}$. If $m = 1$, then we obtain a singly warped product. If all $b_i \equiv 1$, then we have a product manifold.

We denote ∇, ${}^B\nabla$ and ${}^{F_i}\nabla$; Ric, ${}^B\mathrm{Ric}$ and ${}^{F_i}\mathrm{Ric}$ the Levi-Civita connections and Ricci curvatures of the M, B and F_i, respectively.

The lift of X to M is the unique element of $\mathfrak{X}(M)$ that is π-related to X and σ_i-related to zero vector field on B. Similarly, $V_i \in \mathfrak{X}(F_i)$ can be lifted to M and the set of all such lifts is denoted by $\mathfrak{L}(F_i)$. We will use the same notation for a vector field and its lift.

Now, we recall the following formulas for multiply warped products.

Lemma 2.1 ([5]). *$M = B^r \times_{b_1} F_1^{s_1} \times_{b_2} F_2^{s_2} \times_{b_3} \cdots \times_{b_m} F_m^{s_m}$ be a multiply warped product manifold. For $X, Y \in \mathfrak{L}(B)$, $V \in \mathfrak{L}(F_i)$ and $W \in \mathfrak{L}(F_j)$,*

(i) $\nabla_X Y$ *is the lift of* ${}^B\nabla_X Y$ *on* B,

(ii) $\nabla_X V = \nabla_V X = \frac{X(b_i)}{b_i} V$,

(iii) $\nabla_V W = \begin{cases} 0 & \text{if } i \neq j, \\ {}^{F_i}\nabla_V W - \left(\frac{g(V,W)}{b_i}\right) \mathrm{grad}_B(b_i) & \text{if } i = j. \end{cases}$

Lemma 2.2 ([5]). *$M = B^r \times_{b_1} F_1^{s_1} \times_{b_2} F_2^{s_2} \times_{b_3} \cdots \times_{b_m} F_m^{s_m}$ be a multiply warped product manifold. For $X, Y \in \mathfrak{L}(B)$, $V \in \mathfrak{L}(F_i)$ and $W \in \mathfrak{L}(F_j)$,*

(i) $\mathrm{Ric}(X, Y) = {}^B\mathrm{Ric}(X, Y) - \sum_{i=1}^{m} \frac{s_i}{b_i} \mathrm{Hess}_B^{b_i}(X, Y)$,

(ii) $\mathrm{Ric}(X, V) = 0$,

(iii) For $i \neq j$, $\mathrm{Ric}(X, V) = 0$,

(iv) For $i = j$,

$$\mathrm{Ric}(V, W) = {}^{F_i}\mathrm{Ric}(V, W) - \left[\frac{\Delta_B b_i}{b_i} + (s_i - 1) \frac{|\mathrm{grad}_B b_i|^2}{b_i^2} + \sum_{\substack{k=1 \\ k \neq i}}^{m} s_k \frac{g_B(\mathrm{grad}_B b_i, \mathrm{grad}_B b_k)}{b_i b_k} \right] g(V, W).$$

Throughout this paper, we will consider multiply warped product manifold with same warping functions, i.e., $b_i = b : B \longrightarrow \mathbb{R}$ for $i \in \{1, 2, \cdots, m\}$. Emanating from this point, we conclude the following proposition.

Proposition 2.1. *Let $M = B^r \times_b F_1^{s_1} \times_b F_2^{s_2} \times_b \cdots \times_b F_m^{s_m}$ be a multiply warped product manifold endowed with the metric $g = g_B + b^2 \sum_{i=1}^{m} g_{F_i}$ and $\bar{X} = X + V_i$ be a vector field on M such that $X \in \mathfrak{L}(B)$ and $V_i \in \mathfrak{L}(F_i)$. Then,*

$$\mathcal{L}_{\bar{X}} g = \mathcal{L}_X^B g_B + b^2 \sum_{i=1}^{m} \mathcal{L}_{V_i}^{F_i} g_{F_i} + 2bX(b) \sum_{i=1}^{m} g_{F_i}. \tag{2.1}$$

Proof. Let $\bar{X} = X + V_i, \bar{Y} = Y + W_i, \bar{Z} = Z + U_i$ be vector fields on M such that $X, Y, Z \in \mathfrak{L}(B)$ and $V_i, W_i, U_i \in \mathfrak{L}(F_i)$ for $i \in \{1, 2, \cdots, m\}$.

$$\begin{aligned}
\mathcal{L}_{\bar{X}} g(\bar{Y}, \bar{Z}) &= g(\nabla_{\bar{Y}} \bar{X}, \bar{Z}) + g(\bar{Y}, \nabla_{\bar{Z}} \bar{X}) \\
&= g(\nabla_{(Y+W_i)}(X + V_i), Z + U_i) + g(Y + W_i, \nabla_{(Z+U_i)}(X + V_i)) \\
&= g(\nabla_Y X, Z) + g(\nabla_{W_i} V_i, Z) + g(\nabla_Y V_i, U_i) + g(\nabla_{W_i} X, U_i) + g(\nabla_{W_i} V_i, U_i) \\
&\quad + g(Y, \nabla_Z X) + g(Y, \nabla_{U_i} V_i) + g(W_i, \nabla_Z V_i) + g(W_i, \nabla_{U_i} X) + g(W_i, \nabla_{U_i} V_i) \\
&= g_B(\nabla_Y X, Z) + g\left(-\frac{g(W_i, V_i)}{b} \nabla b, Z\right) + g\left(\frac{Y(b)}{b} V_i, U_i\right) + g\left(\frac{X(b)}{b} W_i, U_i\right) \\
&\quad + b^2 g_{F_i}(\nabla_{W_i} V_i, U_i) + g_B(Y, \nabla_Z X) + g\left(Y, -\frac{g(U_i, V_i)}{b} \nabla b\right) + g\left(W_i, \frac{Z(b)}{b} V_i\right) \\
&\quad + g\left(W_i, \frac{X(b)}{b} U_i\right) + b^2 g_{F_i}(W_i, \nabla_{U_i} V_i) \\
&= g_B(\nabla_Y X, Z) + g_B(Y, \nabla_Z X) + b^2 g_{F_i}(\nabla_{W_i} V_i, U_i) + b^2 g_{F_i}(W_i, \nabla_{U_i} V_i) \\
&\quad + 2bX(b) g_{F_i}(W_i, U_i)
\end{aligned}$$

which implies the equation (2.1). □

3. Main Results

In this section, we examine the inheritance properties of Ricci solitons and gradient Ricci solitons on multiply warped product manifolds in order to characterize their factor manifolds.

Theorem 3.1. *Let $M = B^r \times_b F_1^{s_1} \times_b F_2^{s_2} \times_b \cdots \times_b F_m^{s_m}$ be a multiply warped product manifold endowed with the metric $g = g_B + b^2 \sum_{i=1}^{m} g_{F_i}$. If $(M, g, \bar{X}, \lambda)$ is a Ricci soliton with $\bar{X} = X + V_i$ where $X \in \mathfrak{L}(B)$ and $V_i \in \mathfrak{L}(F_i)$, then the base manifold $\left(B, g_B, X - \left(\sum_{i=1}^{m} s_i\right) \nabla(\ln b), \lambda\right)$ is a Ricci soliton and all of the fiber manifolds $\left(F_i, g_{F_i}, b^2 V_i, \lambda b^2\right)$ are Ricci solitons when b is constant.*

Proof. Let $M = B^r \times_b F_1^{s_1} \times_b F_2^{s_2} \times_b \cdots \times_b F_m^{s_m}$ be a Ricci soliton. Then for all $\bar{Y} = Y + W_i, \bar{Z} = Z + U_i \in \mathfrak{X}(M)$, the equation

$$Ric(\bar{Y}, \bar{Z}) + \frac{1}{2} \mathcal{L}_{\bar{X}} g(\bar{Y}, \bar{Z}) = \lambda g(\bar{Y}, \bar{Z}) \tag{3.1}$$

is satisfied. From Lemma 2.2, we have

$$\begin{aligned}
\mathrm{Ric}(\bar{Y}, \bar{Z}) &= {}^B\mathrm{Ric}(Y, Z) - \frac{\sum_{i=1}^{m} s_i}{b} \mathrm{Hess}_B^b(Y, Z) + \sum_{i=1}^{m} {}^{F_i}\mathrm{Ric}(W_i, U_i) \\
&\quad - \left[b\Delta b + \left(\sum_{i=1}^{m} s_i - 1\right) |\nabla b|^2\right] \sum_{i=1}^{m} g_{F_i}(W_i, U_i).
\end{aligned}$$

Using this equation and Proposition 2.1 in equation (3.1) we get

$$\begin{aligned}
&{}^B\mathrm{Ric}(Y, Z) - \frac{\sum_{i=1}^{m} s_i}{b} \mathrm{Hess}_B^b(Y, Z) + \sum_{i=1}^{m} {}^{F_i}\mathrm{Ric}(W_i, U_i) - \left[b\Delta b + \left(\sum_{i=1}^{m} s_i - 1\right) |\nabla b|^2\right] \sum_{i=1}^{m} g_{F_i}(W_i, U_i) \\
&+ \frac{1}{2} \mathcal{L}_X^B g_B(Y, Z) + \frac{1}{2} b^2 \sum_{i=1}^{m} \mathcal{L}_{V_i}^{F_i} g_{F_i}(W_i, U_i) + bX(b) \sum_{i=1}^{m} g_{F_i}(W_i, U_i) \\
&= \lambda g_B(Y, Z) + \lambda b^2 \sum_{i=1}^{m} g_{F_i}(W_i, U_i)
\end{aligned} \tag{3.2}$$

So, we may conclude

$$ {}^{B}\mathrm{Ric}(Y,Z) - \frac{\sum\limits_{i=1}^{m} s_i}{b}\mathrm{Hess}_B^b(Y,Z) + \frac{1}{2}\mathcal{L}_X^B g_B(Y,Z) = \lambda g_B(Y,Z) \tag{3.3} $$

and

$$ \begin{aligned} \sum_{i=1}^{m} {}^{F_i}\mathrm{Ric}(W_i,U_i) - \left[b\Delta b + \left(\sum_{i=1}^{m} s_i - 1\right)|\nabla b|^2 \right] \sum_{i=1}^{m} g_{F_i}(W_i,U_i) \\ + \frac{1}{2}b^2 \sum_{i=1}^{m} \mathcal{L}_{V_i}^{F_i} g_{F_i}(W_i,U_i) + bX(b)\sum_{i=1}^{m} g_{F_i}(W_i,U_i) = \lambda b^2 \sum_{i=1}^{m} g_{F_i}(W_i,U_i) \end{aligned} \tag{3.4} $$

The equation (3.4) implies that for each $i \in \{1,2,\cdots,m\}$, the fiber F_i is a Ricci soliton with potential field $b^2 V_i$ and constant λb^2 when b is constant.

$$ \frac{1}{2}\mathcal{L}_X^B g_B(Y,Z) - \frac{\sum\limits_{i=1}^{m} s_i}{b}\mathrm{Hess}_B^b(Y,Z) = \frac{1}{2}\mathcal{L}^B_{X-\sum\limits_{i=1}^{m} s_i \nabla(\ln b)} g_B(Y,Z), $$

in equation (3.3) gives us

$$ {}^{B}\mathrm{Ric} + \frac{1}{2}\mathcal{L}^B_{X-\sum\limits_{i=1}^{m} s_i \nabla(\ln b)} g_B = \lambda g_B. $$

which means the base B is also a Ricci soliton. □

Using Theorem 3.1 we can state the following corollary.

Corollary 3.1. *Let $M = B^r \times_b F_1^{s_1} \times_b F_2^{s_2} \times_b \cdots \times_b F_m^{s_m}$ be a multiply warped product manifold endowed with the metric $g = g_B + b^2 \sum\limits_{i=1}^{m} g_{F_i}$. If $(M, g, \bar{X}, \lambda)$ is a Ricci soliton and X is a Killing vector field on B, then the base $\left(B, g_B, -\left(\sum\limits_{i=1}^{m} s_i\right)\nabla(\ln b), \lambda\right)$ is a gradient Ricci soliton.*

Remark 3.1. Let $M = B^r \times_b F_1^{s_1} \times_b F_2^{s_2} \times_b \cdots \times_b F_m^{s_m}$ be a multiply warped product manifold endowed with the metric $g = g_B + b^2 \sum\limits_{i=1}^{m} g_{F_i}$. Assume that $(M, g, \bar{X}, \lambda)$ is a Ricci soliton. Then (M, g) is Einstein if one of the following conditions holds:

(i) $\bar{X} = X$ and X is a Killing vector field on B.

(ii) $\bar{X} = \sum\limits_{i=1}^{m} V_i$ and each V_i is a Killing vector field on F_i for $i \in \{1,\cdots,m\}$.

(iii) X and each V_i are Killing vector fields on B and F_i for $i \in \{1,\cdots,m\}$, respectively and $X(b) = 0$.

In the following theorem, we consider Riemannian manifolds with certain properties so that their multiply warped product manifold has the structure of Ricci soliton.

Theorem 3.2. *Let (B, g_B, X, λ) be a Ricci soliton and $(F_i^{s_i}, g_{F_i})$ be an Einstein manifold with factor μ_i for each $i \in \{1,\cdots,m\}$. Then $\left(M = B^r \times_b F_1^{s_1} \times_b \cdots \times_b F_m^{s_m}, g = g_B + b^2 \sum\limits_{i=1}^{m} g_{F_i}, \bar{X}, \lambda\right)$ is a Ricci soliton if the following conditions hold*

(i) For each $i \in \{1,\cdots,m\}$, V_i is conformal with factor $2\rho_i$,

(ii) $\mathrm{Hess}_B^b = 0$,

(iii) $(\lambda - \rho_i)b^2 = bX(b) + \mu_i - \left(\left(\sum\limits_{i=1}^{m} s_i\right) - 1\right)|\nabla b|^2$ for each $i \in \{1,\cdots,m\}$.

Proof. Let (B, g_B, X, λ) be a Ricci soliton and $(F_i^{s_i}, g_{F_i})$ be an Einstein manifold with factor μ_i for each $i \in \{1, \cdots, m\}$. Then we have

$$^{B}\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X^B g_B = \lambda g_B$$

and $^{F_i}\mathrm{Ric} = \mu_i g_{F_i}$ for each $i \in \{1, \cdots, m\}$. Using the equation (3.2) and the conditions in the hypothesis, we arrive

$$\begin{aligned}
\mathrm{Ric} + \frac{1}{2}\mathcal{L}_{\bar{X}} g &= \lambda g_B + \sum_{i=1}^{m} \mu_i g_{F_i} - \left[\left(\left(\sum_{i=1}^{m} s_i\right) - 1\right)|\nabla b|^2\right] \sum_{i=1}^{m} g_{F_i} \\
&\quad + b^2 \sum_{i=1}^{m} \rho_i g_{F_i} + bX(b) \sum_{i=1}^{m} g_{F_i} \\
&= \lambda g_B + b^2 \sum_{i=1}^{m} \rho_i g_{F_i} + \sum_{i=1}^{m} \left[\mu_i - \left(\left(\sum_{j=1}^{m} s_j\right) - 1\right)|\nabla b|^2 + bX(b)\right] g_{F_i} \\
&= \lambda g_B + b^2 \sum_{i=1}^{m} \rho_i g_{F_i} + \sum_{i=1}^{m} (\lambda - \rho_i)\, b^2 g_{F_i} \\
&= \lambda g
\end{aligned}$$

which implies $(M, g, \bar{X}, \lambda)$ is a Ricci soliton. □

Now, we give the inheritance properties of gradient Ricci soliton multiply warped product.

Theorem 3.3. *Let $M = B \times_b F_1 \times_b F_2 \times_b \cdots \times_b F_m$ be a multiply warped product such that (M, g, ϕ, λ) is a gradient Ricci soliton. Then,*

(i) (B, g_B, u, λ) is a gradient Ricci soliton with $u = \phi_1 - \sum s_i \ln b$ and $\phi_1 = \phi$ at some fixed point of $F_1 \times F_2 \times \cdots \times F_m$,

(ii) $(F_i, g_{F_i}, u_i, \lambda b^2)$ is a gradient Ricci soliton with $u_i = \phi$ at some fixed point of B if b is constant.

Proof. Assume that (M, g, ϕ, λ) is a gradient Ricci soliton. For $\bar{Y}, \bar{Z} \in \mathfrak{X}(M)$,

$$\mathrm{Ric}(\bar{Y}, \bar{Z}) + \nabla^2\phi(\bar{Y}, \bar{Z}) = \lambda g(\bar{Y}, \bar{Z}) \tag{3.5}$$

is satisfied. Let $\bar{Y} = Y,\ \bar{Z} = Z$, then equation (3.5) becomes

$$^{B}\mathrm{Ric}(Y, Z) - \sum_{i=1}^{m} \frac{s_i}{b}\mathrm{Hess}_B^b(Y, Z) + \mathrm{Hess}_B^{\phi_1}(Y, Z) = \lambda g_B(Y, Z)$$

$$^{B}\mathrm{Ric}(Y, Z) + \mathrm{Hess}_B^{u}(Y, Z) = \lambda g_B(Y, Z)$$

where $u = \phi_1 - \sum s_i \ln b$ and $\phi_1 = \phi$ at some fixed point of $F_1 \times F_2 \times \cdots \times F_m$. Hence B is a gradient Ricci soliton. Using the same pattern, (ii) can be verified. □

4. Multiply Warped Product Ricci Solitons with Concurrent Vector Fields

In this section, we firstly recall the definition of a concurrent vector field on a Riemannian manifold, i.e., $X \in \mathfrak{X}(M)$ is said to be concurrent if it satisfies

$$\nabla_Z X = X$$

for any vector field $Z \in \mathfrak{X}(M)$[4].

The next result gives the necessary and sufficient conditions for components of concurrent vector fields on a multiply warped product manifold.

Proposition 4.1. *Let $\bar{X} = X + V_i$ be a vector field on $M^n = B^r \times_b F_1^{s_1} \times_b F_2^{s_2} \times_b \cdots \times_b F_m^{s_m}$ where $X \in \mathfrak{L}(B)$ and $V_i \in \mathfrak{L}(F_i)$. $\bar{X}$ is a concurrent vector field on M if and only if X is a concurrent vector field on B and one of the following conditions holds*

(i) V_i is a concurrent vector field on F_i and b is constant,

(ii) $V_i = 0$ and $X(b) = b$.

Proof. Suppose that $\bar{X}$ is a concurrent vector field on M and $\{\partial_i\}_{i=1}^n$ be a basis for $\mathfrak{X}(M)$. Then, for $1 \le j \le r$,

$$\partial_j = \nabla_{\partial_j}\bar{X} = \nabla_{\partial_j}(X + V_i) = \nabla^B_{\partial_j}X + \sum_{i=1}^m \frac{\partial_j(b)}{b}V_i. \tag{4.1}$$

By comparing the tangential and normal parts of the (4.1), we can conclude

$$\nabla^B_{\partial_j}X = \partial_j$$

and

$$0 = \frac{\partial_j(b)}{b}\sum_{i=1}^m V_i$$

so that X is a concurrent vector field on B and $\frac{\partial_j(b)}{b} = 0$ or $\sum\limits_{i=1}^m V_i = 0$.

Case 1: It is clear that $\frac{\partial_j(b)}{b} = 0$ for $1 \le j \le r$ implies b is constant. When $j \in \{r+1, r+2, \cdots, r+s_1, r+s_1+1, \cdots, n\}$, we get

$$\partial_j = \nabla_{\partial_j}\bar{X} = \frac{X(b)}{b}\partial_j + \sum_{i=1}^m \nabla^{F_i}_{\partial_j}V_i - b\sum_{i=1}^m g_{F_i}(\partial_j, V_i)\nabla b.$$

Since b is constant, we have $\nabla^{F_i}_{\partial_j}V_i = \partial_j$, i.e., V_i is concurrent vector field on F_i for $1 \le i \le m$.

Case 2: Assuming $\sum\limits_{i=1}^m V_i = 0$ yields

$$\partial_j = \nabla_{\partial_j}\bar{X} = \frac{X(b)}{b}\partial_j.$$

Therefore, we have $\dfrac{X(b)}{b} = 1$, i.e., (ii) holds.

The converse can be shown by direct calculation. □

Using the above proposition, we reach the following theorem.

Theorem 4.1. *Let $\bar{X} = X + V_i$ be a concurrent vector field on $M = B \times_b F_1 \times_b F_2 \times_b \cdots \times_b F_m$ where $X \in \mathfrak{L}(B)$ and $V_i \in \mathfrak{L}(F_i)$ such that $(M, g, \bar{X}, \lambda)$ is a Ricci soliton. If $V_i \neq 0$, then M, B and F_i are Ricci flat, gradient Ricci soliton with $\lambda = 1$.*

Proof. Assume that $(M, g, \bar{X}, \lambda)$ is a Ricci soliton and $\bar{X} = X + V_i$ be a concurrent vector field on M. For $\bar{Y}, \bar{Z} \in \mathfrak{X}(M)$, we have

$$\mathcal{L}_{\bar{X}}g(\bar{Y}, \bar{Z}) = g(\nabla_{\bar{Y}}\bar{X}, \bar{Z}) + g(\bar{Y}, \nabla_{\bar{Z}}\bar{X}) = g(\bar{Y}, \bar{Z}) + g(\bar{Y}, \bar{Z}) = 2g(\bar{Y}, \bar{Z}).$$

Thus, we get

$$\mathrm{Ric}(\bar{Y}, \bar{Z}) = (\lambda - 1)g(\bar{Y}, \bar{Z}). \tag{4.2}$$

In particular, for $\bar{Y} = \sum\limits_{i=1}^m W_i,\ \bar{Z} = \sum\limits_{i=1}^m U_i$ we have

$$\begin{aligned} \mathrm{Ric}(\bar{Y}, \bar{Z}) &= \sum_{i=1}^m \mathrm{Ric}(W_i, U_i) \\ &= \sum_{i=1}^m {}^{F_i}\mathrm{Ric}(W_i, U_i) - \left[b\Delta b + \left(\sum_{i=1}^m s_i - 1\right)|\nabla b|^2\right]\sum_{i=1}^m g_{F_i}(W_i, U_i). \end{aligned} \tag{4.3}$$

Combining equations (4.2) and (4.3), we have

$$(\lambda-1)b^2\sum_{i=1}^{m} g_{F_i}(W_i,U_i)=\sum_{i=1}^{m} {}^{F_i}\mathrm{Ric}(W_i,U_i)-\left[b\Delta b+\left(\sum_{i=1}^{m} s_i-1\right)|\nabla b|^2\right]\sum_{i=1}^{m} g_{F_i}(W_i,U_i).$$

From (i) of Proposition 4.1, we know that V_i is concurrent for each $i\in\{1,\cdots,m\}$ and $b=c$ for some constant c. Hence, the above equation becomes

$$\sum_{i=1}^{m} {}^{F_i}\mathrm{Ric}(W_i,U_i)=\sum_{i=1}^{m}(\lambda-1)c^2 g_{F_i}(W_i,U_i), \tag{4.4}$$

i.e., F_i is Einstein with factor $\mu=(\lambda-1)c^2$. Since this equation is valid for any vector field in $\mathfrak{X}(F_i)$, we may write

$${}^{F_i}\mathrm{Ric}(V_i,V_i)=(\lambda-1)c^2|V_i|^2_{F_i}. \tag{4.5}$$

Let $\{V_i,e_1,\cdots,e_{s_i-1}\}$ be an orthogonal basis of $\mathfrak{X}(F_i)$. Then, Riemann curvature tensor is

$$\begin{aligned}
{}^{F_i}\mathrm{R}(V_i,e_j,V_i,e_j) &= g_{F_i}({}^{F_i}\mathrm{R}(V_i,e_j)V_i,e_j)\\
&= g_{F_i}(\nabla_{V_i}\nabla_{e_j}V_i-\nabla_{e_j}\nabla_{V_i}V_i-\nabla_{[V_i,e_j]}V_i,e_j)\\
&= g_{F_i}(\nabla_{V_i}e_j-\nabla_{e_j}V_i-[V_i,e_j],e_j)\\
&= 0,
\end{aligned}$$

which implies ${}^{F_i}\mathrm{Ric}(V_i,V_i)=0$. From equation (4.5), we have $\lambda=1$. Moreover, M and F_i are Ricci flat from equations (4.2) and (4.4).

Now, suppose that $\bar{Y}=Y,\ \bar{Z}=Z$. Then

$$\begin{aligned}
0=\mathrm{Ric}(\bar{Y},\bar{Z}) &= \mathrm{Ric}(Y,Z)\\
&= {}^{B}\mathrm{Ric}(Y,Z)-\sum_{i=1}^{m}\frac{s_i}{b}\mathrm{Hess}^b_B(Y,Z)\\
&= {}^{B}\mathrm{Ric}(Y,Z)
\end{aligned}$$

since b is constant. Hence, B is Ricci flat. It is clear that M, B and F_i are Ricci soliton with $\lambda=1$.

Since $g(\bar{Y},\nabla\phi)=\bar{Y}(\phi)=g(\nabla_{\bar{Y}}\bar{X},\bar{X})=g(\bar{Y},\bar{X})$ for $\phi=\frac{1}{2}g(\bar{X},\bar{X})$, M is a gradient Ricci soliton with potential function ϕ. Similarly, one can show that B and F_i are gradient Ricci solitons with potential functions $u=\frac{1}{2}g(X,X)$ and $u_i=\frac{1}{2}g(V_i,V_i)$. □

References

[1] Besse, A.L.: Einstein Manifolds, Classics in Mathematics, Springer-Verlag, Berlin (2008).
[2] Bishop, R.L., O'Neill, B.: *Manifolds of negative curvature*. Trans. Amer. Math. Soc. **145**, 1-49 (1969).
[3] Cao, H-D., Zhou,D.: *On complete gradient shrinking Ricci solitons*. Journal of Differential Geometry. **85**, 175-186 (2010).
[4] Chen, B-Y.: *Some results on concircular vector fields and their applications to Ricci solitons*. Bull. Korean Math. Soc. **52** No. 5, 1535-1547 (2015).
[5] Dobarro, F., Ünal, B.: *Curvature of multiply warped products*. J. Geom. Phys. **55**, 75-106 (2005).
[6] Feitosa, F.E.S.,Freitas Filho A.A., Gomes, J.N.V.: *On the construction of gradient Ricci soliton warped product*. Nonlinear Analysis **161**, 30-43 (2017).
[7] Fernandez-Lopez, M., Garcia-Rio, E.: *Rigidity of shrinking Ricci solitons*. Mathematische Zeitschrift. **269**, 461-466 (2011).
[8] Günsen, S., Onat, L. and Açıkgöz Kaya, D.: *The warped product manifold as gradient Ricci soliton and relation to its components*. C. R. Acad. Bulg. Sci. **72** (8), 1015-1023 (2019).
[9] Hamilton R.S.: *Three-manifolds with positive Ricci curvature*. J. Differential Geom. **17** (2), 255-306 (1982).
[10] Karaca, F. and Özgür, C.: *Gradient Ricci Solitons on Multiply Warped Product Manifolds*. Filomat. **32** (12), 4221-4228 (2018).
[11] Mantica, C.A., Shenawy, S., Ünal, B.: *Ricci Solitons on Singly Warped Product Manifolds and Applications*. Preprint arxiv:1508.02794v2 2019).
[12] Munteanu, O., Sesum, N.: *On Gradient Ricci Solitons*. Journal of Geometric Analysis. **23** , 539-561 (2013).
[13] Petersen, P., Wylie, W.:*Rigidity of gradient Ricci solitons*. Pacific Journal of Mathematics. **241** , 329-345 (2009).
[14] Petersen, P., Wylie, W.: *On gradient Ricci solitons with symmetry*. Proc. Amer. Math. Soc. **137** , 2085-2092 (2009).
[15] Petersen, P., Wylie, W.: *On the classification of gradient Ricci solitons*. Geom. Topol. **14**, 2277-2300 (2010).
[16] Sousa, M.L., Pina, R.: *Gradient Ricci Solitons with Structure of Warped Product*. Results Math. **71**, 825-840 (2017).
[17] Ünal, B.: *Multiply warped products*. J. Geom. Phys. **34** , 287-301 (2000). MR1762779 and DOI 10.1016/S0393-0440(99)00072-8

Affiliations

Dilek Açıkgöz Kaya
Address: Aydın Adnan Menderes University, Dept. of Mathematics, 09010, Aydın-Turkey.
e-mail: dilek.acikgoz@adu.edu.tr
ORCID ID:0000-0003-1603-9658

Leyla Onat
Address: Aydın Adnan Menderes University, Dept. of Mathematics, 09010, Aydın-Turkey.
e-mail: lonat@adu.edu.tr
ORCID ID:0000-0002-9926-1467