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On Ideal Invariant Convergence of Double Sequences in Regularly Sense

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Abstract: In this paper, we defined concepts of $r(\sigma,\sigma_2)$-convergence, $r[\sigma,\sigma_2]$-convergence, $r[\sigma,\sigma_2]_p$-convergence, $r(\mathcal{I}_\sigma,\mathcal{I}_2^\sigma)$-convergence of double sequences . Also we research the relationships among them.

Keywords: Double sequence, invariant convergence, regularly ideal convergence.

1 Introduction and Background

Throughout the paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. The concept of convergence of real sequences has been extended to statistical convergence independently by Fast [16] and Schoenberg [32]. This concept was extended to the double sequences by Mursaleen and Edely [19].

The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [17] as a generalization of statistical convergence. Das et al. [3] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. Tripathy and Tripathy [34] studied on $\mathcal{I}$-convergent and regularly $\mathcal{I}$-convergent double sequences. Dündar and Altay [5] introduced $\mathcal{I}_2$-convergence and regularly $\mathcal{I}$-convergence of double sequences. Also, Dündar [**?**] introduced regularly $\mathcal{I}$-convergence and regularly $\mathcal{I}$-Cauchy double sequences of functions. Recently, Dündar and P. Akın [15] introduced the notions of $R(\mathcal{I}_{W_2},\mathcal{I}_W)$-convergence, $R(\mathcal{I}^*_{W_2},\mathcal{I}^*_W)$-convergence, $R(\mathcal{I}_{W_2},\mathcal{I}_W)$-Cauchy and $R(\mathcal{I}^*_{W_2},\mathcal{I}^*_W)$-Cauchy double sequence of sets and investigate the relationship among them. A lot of development have been made in this area after the works of [6–8, 11, 18, 23, 25**?**].

Several authors have studied invariant convergent sequences (see, [2, 20–22, 26, 28–31, 35]). Recently, the concepts of σ-uniform density of the set $A\subseteq\mathbb{N}$, $\mathcal{I}_\sigma$-convergence and $\mathcal{I}^*_\sigma$-convergence of sequences of real numbers were defined by Nuray et al. [26]. The concept of σ-convergence of double sequences was studied by Çakan et al. [2] and the concept of σ-uniform density of $A\subseteq\mathbb{N}\times\mathbb{N}$ was defined by Tortop and Dündar [35]. Dündar et al. [13] studied ideal invariant convergence of double sequences and some properties.

Now, we recall the basic definitions and concepts (See [1–3, 5–11, 14, 17, 18, 23, 25, 27, 34**? ?** , 35]).

A double sequence $x=(x_{kj})_{k,j\in\mathbb{N}}$ of real numbers is said to be convergent to $L\in\mathbb{R}$ in Pringsheim's sense if for any $\varepsilon>0$, there exists $N_\varepsilon\in\mathbb{N}$ such that $|x_{kj}-L|<\varepsilon$, whenever $k,j>N_\varepsilon$. In this case, we write $P-\lim\limits_{k,j\to\infty}x_{kj}=\lim\limits_{k,j\to\infty}x_{kj}=L$.

A family of sets $\mathcal{I}\subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(i) $\emptyset\in\mathcal{I}$, (ii) For each $A,B\in\mathcal{I}$ we have $A\cup B\in\mathcal{I}$, (iii) For each $A\in\mathcal{I}$ and each $B\subseteq A$ we have $B\in\mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N}\notin\mathcal{I}$ and non-trivial ideal is called admissible if $\{n\}\in\mathcal{I}$ for each $n\in\mathbb{N}$.

Throughout the paper we take $\mathcal{I}$ as an admissible ideal in $\mathbb{N}$.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N}\times\mathbb{N}$ is called strongly admissible ideal if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times\{i\}$ belong to $\mathcal{I}_2$ for each $i\in N$.

It is evident that a strongly admissible ideal is admissible also.

Throughout the paper we take $\mathcal{I}_2$ as a strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$.

$\mathcal{I}_2^0=\big\{A\subset\mathbb{N}\times\mathbb{N}:(\exists m(A)\in\mathbb{N})(i,j\geq m(A)\Rightarrow(i,j)\notin A)\big\}$. Then, $\mathcal{I}_2^0$ is a strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0\subset\mathcal{I}_2$.

Let σ be a mapping of the positive integers into themselves. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if it satisfies following conditions:

1. $\phi(x)\geq 0$, when the sequence $x=(x_n)$ has $x_n\geq 0$, for all n,
2. $\phi(e)=1$, where $e=(1,1,1,...)$ and
3. $\phi(x_{\sigma(n)})=\phi(x_n)$, for all $x\in\ell_\infty$.

The mappings σ are assumed to be one-to-one and such that $\sigma^m(n)\neq n$, for all positive integers n and m, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus, ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x)=\lim x$, for all $x\in c$.

In the case σ is translation mappings $\sigma(n) = n + 1$, the σ-mean is often called a Banach limit and the space V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences $\hat{c}$.

It can be shown that

$$V_\sigma = \left\{ x = (x_n) \in \ell_\infty : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} = L, \text{ uniformly in } n \right\}.$$

Let $A \subseteq \mathbb{N}$ and

$$s_m = \min_n \left| A \cap \{\sigma(n), \sigma^2(n), \cdots, \sigma^m(n)\} \right|$$

and

$$S_m = \max_n \left| A \cap \{\sigma(n), \sigma^2(n), \cdots, \sigma^m(n)\} \right|.$$

If the limits $\underline{V}(A) = \lim_{m\to\infty} \frac{s_m}{m}$ $\overline{V}(A) = \lim_{m\to\infty} \frac{S_m}{m}$ exist, then they are called a lower and upper σ-uniform density of the set A, respectively. If $\underline{V}(A) = \overline{V}(A)$, then $V(A) = \underline{V}(A) = \overline{V}(A)$ is called σ-uniform density of A.

Denote by $\mathcal{I}_\sigma$ the class of all $A \subseteq \mathbb{N}$ with $V(A) = 0$.

Let $\mathcal{I}_\sigma \subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence $x = (x_k)$ is said to be $\mathcal{I}_\sigma$-convergent to the number L, if for every $\varepsilon > 0$ $A_\varepsilon = \{k : |x_k - L| \geq \varepsilon\} \in \mathcal{I}_\sigma$, that is, $V(A_\varepsilon) = 0$. In this case, we write $\mathcal{I}_\sigma - \lim_k = L$.

Let $A \subseteq \mathbb{N} \times \mathbb{N}$ and

$$s_{mn} = \min_{k,j} \left| A \cap \Big\{ \big(\sigma(k), \sigma(j)\big), \big(\sigma^2(k), \sigma^2(j)\big), ..., \big(\sigma^m(k), \sigma^n(j)\big) \Big\} \right|$$

and

$$S_{mn} = \max_{k,j} \left| A \cap \Big\{ \big(\sigma(k), \sigma(j)\big), \big(\sigma^2(k), \sigma^2(j)\big), ..., \big(\sigma^m(k), \sigma^n(j)\big) \Big\} \right|.$$

If the limits exists $\underline{V_2}(A) = \lim_{m,n\to\infty} \frac{s_{mn}}{mn}$, $\overline{V_2}(A) = \lim_{m,n\to\infty} \frac{S_{mn}}{mn}$ exist, then they are called a lower and an upper σ-uniform density of the set A, respectively. If $\underline{V_2}(A) = \overline{V_2}(A)$, then $V_2(A) = \underline{V_2}(A) = \overline{V_2}(A)$ is called the σ-uniform density of A.

Denote by $\mathcal{I}_2^\sigma$ the class of all $A \subseteq \mathbb{N} \times \mathbb{N}$ with $V_2(A) = 0$.

Throughout the paper we let $\mathcal{I}_2^\sigma \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal.

A double sequence $x = (x_{kj})$ is said to be $\mathcal{I}_2$-invariant convergent or $\mathcal{I}_2^\sigma$-convergent to L, if for every $\varepsilon > 0$ $A(\varepsilon) = \big\{(k, j) : |x_{kj} - L| \geq \varepsilon\big\} \in \mathcal{I}_2^\sigma$ that is, $V_2\big(A(\varepsilon)\big) = 0$. In this case, we write $\mathcal{I}_2^\sigma - \lim x = L$ or $x_{kj} \to L(\mathcal{I}_2^\sigma)$.

A double sequence $x = (x_{kj})$ is said to be regularly $(\mathcal{I}_2, \mathcal{I})$-convergent ($r(\mathcal{I}_2, \mathcal{I})$-convergent) if it is $\mathcal{I}_2$-convergent in Pringsheim's sense and for every $\varepsilon > 0$, the followings hold:

$$\{k \in \mathbb{N} : |x_{kj} - L_j| \geq \varepsilon\} \in \mathcal{I}, \text{ for some } L_j \in X \text{ and each } j \in \mathbb{N},$$

$$\{j \in \mathbb{N} : |x_{kj} - M_k| \geq \varepsilon\} \in \mathcal{I}, \text{ for some } M_k \in X \text{ and each } k \in \mathbb{N}.$$

Theorem 1. *[13] Suppose that $x = (x_{kj})$ is a bounded double sequence. If $x = (x_{kj})$ is $\mathcal{I}_2^\sigma$-convergent to L, then $x = (x_{kj})$ is invariant convergent to L.*

Theorem 2. *[13] Let $0 < p < \infty$.*

(i) If $x_{kj} \to L\big([V_\sigma^2]_p\big)$, then $x_{kj} \to L\big(\mathcal{I}_2^\sigma\big)$.
(ii) If $(x_{kj}) \in \ell_\infty^2$ and $x_{kj} \to L\big(\mathcal{I}_2^\sigma\big)$, then $x_{kj} \to L\big([V_\sigma^2]_p\big)$.
(iii) If $(x_{kj}) \in \ell_\infty^2$, then $x_{kj} \to L\big(\mathcal{I}_2^\sigma\big)$ if and only if $x_{kj} \to L\big([V_\sigma^2]_p\big)$.

1.1 Main Results

Definition 1. *A double sequence $x = (x_{kj})$ is said to be regularly invariant convergent ($r(\sigma, \sigma_2)$-convergent) if it is invariant convergent in Pringsheim's sense and the following limits hold:*

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m} x_{\sigma^k(s),\sigma^j(t)} = L_j, \text{ uniformly in } s,$$

for some $L_j \in X$ and each $j \in \mathbb{N}$ and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} x_{\sigma^k(s),\sigma^j(t)} = M_k, \text{ uniformly in } t,$$

for some $M_k \in X$ and each $k \in \mathbb{N}$. Note that if $x = (x_{kj})$ is $r(\sigma, \sigma_2)$-convergent to L, the following limits hold:

$$\lim_{m\to\infty} \lim_{n\to\infty} \frac{1}{mn} \sum_{k=0}^{m} \sum_{j=0}^{n} x_{\sigma^k(s),\sigma^j(t)} = L, \text{ uniformly in } s, t$$

and

$$\lim_{n\to\infty}\lim_{m\to\infty}\frac{1}{mn}\sum_{j=0}^{n}\sum_{k=0}^{m}x_{\sigma^k(s),\sigma^j(t)}=L,\ \textit{uniformly in } s,t.$$

In this case, we write

$$r(\sigma,\sigma_2)-\lim_{m,n\to\infty}\sum_{k=0}^{m}\sum_{j=0}^{n}x_{\sigma^k(s),\sigma^j(t)}=L \textit{ or } x_{kj}\overset{r(\sigma,\sigma_2)}{\longrightarrow}L,\ \textit{uniformly in } s,t.$$

Definition 2. *A double sequence $x=(x_{kj})$ is said to be regularly strongly invariant convergent ($r[\sigma,\sigma_2]$-convergent) if it is strongly invariant convergent in Pringsheim's sense and the following limits hold:*

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m}|x_{\sigma^k(s),\sigma^j(t)}-L_j|=0,\ \textit{ uniformly in } s,$$

for some $L_j\in X$ and each $j\in\mathbb{N}$ and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n}|x_{\sigma^k(s),\sigma^j(t)}-M_k|=0,\ \textit{ uniformly in } t,$$

for some $M_k\in X$ and each $k\in\mathbb{N}$.

Note that if $x=(x_{kj})$ is $r[\sigma,\sigma_2]$-convergent to L, the following limits hold:

$$\lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{mn}\sum_{k=0}^{m}\sum_{j=0}^{n}|x_{\sigma^k(s),\sigma^j(t)}-L|=0,\ \textit{uniformly in } s,t$$

and

$$\lim_{n\to\infty}\lim_{m\to\infty}\frac{1}{mn}\sum_{j=0}^{n}\sum_{k=0}^{m}|x_{\sigma^k(s),\sigma^j(t)}-L|=0,\ \textit{uniformly in } s,t.$$

In this case, we write

$$r[\sigma,\sigma_2]-\lim_{m,n\to\infty}\sum_{k=0}^{m}\sum_{j=0}^{n}|x_{\sigma^k(s),\sigma^j(t)}-L|=0 \textit{ or } x_{kj}\overset{r[\sigma,\sigma_2]}{\longrightarrow}L,\ \textit{uniformly in } s,t.$$

Definition 3. *Let $0<p<\infty$. A double sequence $x=(x_{kj})$ is said to be regularly p-strongly invariant convergent ($r[\sigma,\sigma_2]_p$-convergent), if it is p-strongly invariant convergent in Pringsheim's sense and the following limits hold:*

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m}|x_{\sigma^k(s),\sigma^j(t)}-L_j|^p=0,\ \textit{uniformly in } s,$$

for some $L_j\in X$ each $j\in\mathbb{N}$ and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n}|x_{\sigma^k(s),\sigma^j(t)}-M_k|^p=0,\ \textit{uniformly in } t,$$

for some $M_k\in X$ each $k\in\mathbb{N}$.

Note that if $x=(x_{kj})$ is $r[\sigma,\sigma_2]_p$-convergent to L, the following limits hold:

$$\lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{mn}\sum_{k=0}^{m}\sum_{j=0}^{n}|x_{\sigma^k(s),\sigma^j(t)}-L|^p=0,\ \textit{uniformly in } s,t$$

and

$$\lim_{n\to\infty}\lim_{m\to\infty}\frac{1}{mn}\sum_{j=0}^{n}\sum_{k=0}^{m}|x_{\sigma^k(s),\sigma^j(t)}-L|^p=0,\ \textit{uniformly in } s,t.$$

In this case, we write

$$r[\sigma,\sigma_2]_p-\lim_{m,n\to\infty}\sum_{k=0}^{m}\sum_{j=0}^{n}|x_{\sigma^k(s),\sigma^j(t)}-L|=0 \textit{ or } x_{kj}\overset{r[\sigma,\sigma_2]_p}{\longrightarrow}L,\ \textit{uniformly in } s,t.$$

Definition 4. *A double sequence $x=(x_{kj})$ is said to be regularly ideal invariant convergent ($r(\mathcal{I}_\sigma,\mathcal{I}_2^\sigma)$-convergent) if it is ideal invariant convergent in Pringsheim's sense and for every $\varepsilon>0$ the followings hold:*

$$\{k\in\mathbb{N}:|x_{kj}-L_j|\geq\varepsilon\}\in\mathcal{I}_\sigma$$

for some $L_j\in X$ each $j\in\mathbb{N}$ and

$$\{j\in\mathbb{N}:|x_{kj}-M_k|\geq\varepsilon\}\in\mathcal{I}_\sigma$$

for some $M_k\in X$ each $k\in\mathbb{N}$.

Note that if $x=(x_{kj})$ is $r(\mathcal{I}_\sigma,\mathcal{I}_2^\sigma)$-convergent to L, then we write

$$r(\mathcal{I}_\sigma,\mathcal{I}_2^\sigma)-\lim x=L \text{ or } x_{kj}\overset{r(\mathcal{I}_\sigma,\mathcal{I}_2^\sigma)}{\longrightarrow}L$$

Theorem 3. *Suppose that $x=(x_{kj})$ is a bounded double sequence. If $x=(x_{kj})$ is $r(\mathcal{I}_\sigma,\mathcal{I}_2^\sigma)$-convergent, then $x=(x_{kj})$ is $r(\sigma,\sigma_2)$-convergent.*

Theorem 4. *Let $0<p<\infty$.*

(i) If (x_{kj}) is $r[\sigma,\sigma_2]_p$-convergent, then (x_{kj}) is $r(\mathcal{I}_\sigma,\mathcal{I}_2^\sigma)$-convergent.
(ii) If $(x_{kj})\in\ell_\infty^2$ and (x_{kj}) is $r(\mathcal{I}_\sigma,\mathcal{I}_2^\sigma)$-convergent, then (x_{kj}) is $r[\sigma,\sigma_2]_p$-convergent.
(iii) If $(x_{kj})\in\ell_\infty^2$, then (x_{kj}) is $r[\sigma,\sigma_2]_p$-convergent if and only if (x_{kj}) is $r(\mathcal{I}_\sigma,\mathcal{I}_2^\sigma)$ is convergent.

2 References

1 B. Altay, F. Başar, *Some new spaces of double sequences*, J. Math. Anal. Appl. **309** (1) (2005), 70–90.
2 C. Çakan, B. Altay, M. Mursaleen, *The σ-convergence and σ-core of double sequences,* Applied Mathematics Letters, **19** (2006), 1122–1128.
3 P. Das, P. Kostyrko, W. Wilczyński, P. Malik, *$\mathcal{I}$ and $\mathcal{I}^*$-convergence of double sequences*, Math. Slovaca, **58**(5) (2008), 605–620.
4 K. Dems, On $\mathcal{I}$-Cauchy sequences, *Real Anal. Exchange*, **30** (2004/2005), 123–128.
5 E. Dündar, B. Altay, *$\mathcal{I}_2$-convergence and $\mathcal{I}_2$-Cauchy of double sequences*, Acta Mathematica Scientia, **34**B(2) (2014), 343–353.
6 E. Dündar, B. Altay, On some properties of $\mathcal{I}_2$-convergence and $\mathcal{I}_2$-Cauchy of double sequences, Gen. Math. Notes, 7 (1)(2011) 1–12.
7 E. Dündar, B. Altay, $\mathcal{I}_2$-convergence of double sequences of functions, Electronic Journal of Mathematical Analysis and Applications, Vol. 3(1) Jan. 2015, pp. 111–121.
8 E. Dündar, B. Altay *$\mathcal{I}_2$-uniform convergence of double sequences of functions*, Filomat **30**(5) (2016), 1273–1281.
9 E. Dündar, B. Altay, *Multipliers for bounded $\mathcal{I}_2$-convergent of double sequences*, Math. Comput. Modelling, **55**(3-4) (2012), 1193–1198.
10 E. Dündar. Ö. Talo, *$\mathcal{I}_2$-convergence of double sequences of fuzzy numbers*, Iranian Journal of Fuzzy Systems, **10**(3) (2013), 37–50.
11 E. Dündar, M. R. Türkmen and N. Pancaroğlu Akın, *Regularly ideal convergence of double sequences in fuzzy normed spaces*, (in review).
12 E. Dündar, *Regularly ($\mathcal{I}_2,\mathcal{I}$)-Convergence and ($\mathcal{I}_2,\mathcal{I}$)-Cauchy Double Sequences of Functions*, Pioneer Journal of Algebra, Number Theory and its Applications **1**(2) (2011), 85–98.
13 E. Dündar, U. Ulusu, F. Nuray, *On ideal invariant convergence of double sequences and some properties*, Creative Mathematics and Informatics, **27**(2)(2018), 161–169.
14 E. Dündar, U. Ulusu, N. Pancaroğlu, *Strongly $\mathcal{I}_2$-lacunary Convergence and $\mathcal{I}_2$-lacunary Cauchy Double Sequences of Sets*, The Aligarh Bulletin of Mathematics, **35**(1-2)(2016), 1–15.
15 E. Dündar and N. Pancaroğlu Akın, *Wijsman Regularly Ideal Convergence of Double Sequences of Sets*, Journal of Intelligent and Fuzzy Systems, (in press), DOI:10.3233/JIFS-190626
16 H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.
17 P. Kostyrko, T. Šalát, W. Wilczyński, *$\mathcal{I}$-Convergence,* Real Anal. Exchange, **26**(2) (2000), 669–686.
18 V. Kumar, *On $\mathcal{I}$ and $\mathcal{I}^*$-convergence of double sequences*, Math. Commun. **12** (2007), 171–181.
19 M. Mursaleen, O. H. H. Edely, *Statistical convergence of double sequences*, J. Math. Anal. Appl. **288** (2003) 223–231.
20 M. Mursaleen, *Matrix transformation between some new sequence spaces*, Houston J. Math., **9** (1983), 505–509.
21 M. Mursaleen, *On finite matrices and invariant means,* Indian J. Pure Appl. Math., **10** (1979), 457–460.
22 M. Mursaleen, O. H. H. Edely, *On the invariant mean and statistical convergence*, Appl. Math. Lett., **22**(11) (2009), 1700–1704.
23 A. Nabiev, S. Pehlivan, M. Gürdal, *On $\mathcal{I}$-Cauchy sequence*, Taiwanese J. Math. **11** (2) (2007), 569–576.
24 A. Nabiev, S. Pehlivan, M. Gürdal, *On $\mathcal{I}$-Cauchy sequences,* Taiwanese J. Math., **11**(2) (2007), 569–576.
25 F. Nuray, W.H. Ruckle, *Generalized statistical convergence and convergence free spaces*, J. Math. Anal. Appl. **245** (2000), 513–527.
26 F. Nuray, H. Gök, U. Ulusu, *$\mathcal{I}_\sigma$-convergence*, Math. Commun., **16** (2011), 531–538.
27 A. Pringsheim, *Zur theorie der zweifach unendlichen Zahlenfolgen*, Math. Ann. **53** (1900), 289–321.
28 R. A. Raimi, *Invariant means and invariant matrix methods of summability*, Duke Math. J., **30**(1) (1963), 81–94.
29 E. Savaş, *Some sequence spaces involving invariant means*, Indian J. Math., **31** (1989), 1–8.
30 E. Savaş, *Strongly σ-convergent sequences*, Bull. Calcutta Math., **81** (1989), 295–300.
31 P. Schaefer, *Infinite matrices and invariant means*, Proc. Amer. Math. Soc., **36** (1972), 104–110.
32 I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly **66** (1959) 361–375.
33 Y. Sever, E. Dündar, *Regularly Ideal Convergence and Regularly Ideal Cauchy Double Sequences in 2-Normed Spaces*, Filomat **28**:5 (2015), 907–915.
34 B. Tripathy, B.C. Tripathy, *On $\mathcal{I}$-convergent double sequences*, Soochow J. Math. **31** (2005), 549–560.
35 Ş. Tortop, E. Dündar, *Wijsman $\mathcal{I}_2$-invariant convergence of double sequences of sets*, Journal of Inequalities and Special Functions, **9**(4) (2018),90-100 .
36 U. Ulusu, E. Dündar,*Asymptotically lacunary $\mathcal{I}_2$-invariant equivalence*, Journal of Intelligent and Fuzzy Systems, **36**(1) (2019), 467-472, DOI:10.3233/JIFS-181796