Conference Proceeding Science and Technology, 3(1), 2020, 156–159

Conference Proceeding of 3rd International E-Conference on Mathematical Advances and Applications (ICOMAA-2020)

On $\rho-$ Statistical Convergence of Sequences of Sets

ISSN: 2651-544X
http://dergipark.gov.tr/cpost

Nazlım Deniz Aral[1,] Hacer Şengül Kandemir[2] Mikail Et[3]*

[1] *Department of Mathematics, Bitlis Eren University, Bitlis, Turkey, ORCID:0000-0002-8984-2620*
[2] *Faculty of Education, Harran University, Osmanbey Campus 63190, Şanlıurfa, Turkey, ORCID:0000-0003-4453-0786*
[3] *Department of Mathematics, Fırat University, 23119 Elazığ ; Turkey,ORCID:0000-0001-8292-7819*
** Corresponding Author E-mail: ndaral@beu.edu.tr*

Abstract: In this paper we introduce the concepts of Wijsman $\rho-$statistical convergence, Wijsman strongly $\rho-$statistical convergence and Wijsman $\rho-$strongly $p-$ summability. Also, the relationship between these concepts are given.

Keywords: Cesàro summability, Statistical convergence, Strongly $p-$Cesàro summability, Wijsman convergence.

1 Introduction

The concept of statistical convergence was introduced by Steinhaus [24] and Fast [16]. Schoenberg [23] established some basic properties of statistical convergence and studied the concept as a summability method. Later on it was further investigated from the sequence space point of view and linked with summability theory by Altınok et al. [1], Bhardwaj and Dhawan [2], Caserta et al. [3], Çınar et al. [9], Connor [4], Çakallı et al. ([5]-[6]-[7]), Çolak ([10]-[11]), Et et al. ([12]-[13]-[14]-[15]), Fridy [17], Gadjiev and Orhan [18], Işık and Akbaş [19], Salat [21], Savaş and Et [22], Şengül [25] and many others. A real or complex number sequence $x = (x_k)$ is said to be statistically convergent to ℓ if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |x_k - \ell| \geq \varepsilon\}| = 0.$$

Let (X, σ) be a metric space. The distance $d(x, A)$ from a point x to a non-empty subset A of (X, σ) is defined to be

$$d(x, A) = \inf_{y\in A} \sigma(x, y).$$

If $\sup_k d(x, A_k) < \infty$(for each $x \in X$), then we say that the sequence $\{A_k\}$ is bounded. The set of all bounded sequences of sets denoted L_∞. The concepts of Wijsman statistical convergence and boundedness for the sequence $\{A_k\}$ were given by Nuray and Rhoades [20] as follows.

Let (X, σ) be a metric space. For any non-empty closed subsets $A, A_k \subset X (k \in \mathbb{N})$ we say that the sequence $\{A_k\}$ is Wijsman statistical convergent to A if the sequence $(d(x, A_k))$ is statistically convergent to $d(x, A)$, i.e., for $\varepsilon > 0$ and for each $x \in X$

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leqslant n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.$$

Ulusu and Nuray ([26],[27]) defined Wijsman lacunary statistical convergence of sequence of sets, and considered its relation with Wijsman statistical convergence.

The concept of $\rho-$ statistical convergence was defined by Çakallı [8]. A sequence (x_k) of points in $\mathbb{R}$, the set of real numbers, is called $\rho-$statistically convergent to ℓ if

$$\lim_{n\to\infty} \frac{1}{\rho_n}|\{k \leq n : |x_k - \ell| \geq \varepsilon\}| = 0$$

for each $\varepsilon > 0$ where $\rho = (\rho_n)$ is a non-decreasing sequence of positive real numbers tending to ∞ such that $\limsup_n \frac{\rho_n}{n} < \infty, \Delta\rho_n = O(1)$ and $\Delta\rho_n = \rho_{n+1} - \rho_n$ for each positive integer n. In this case we write $S_\rho - \lim x_k = \ell$ or $x_k \to \ell\,(S_\rho)$.

If α is a sequence such that α_k satisfies property P for all k except a set of natural density zero, then we say that α_k satisfies P for "almost all k ", and we abbreviate this by "a.a.k."

2 Main Results

Definition 1. *Let (X, σ) be a metric space. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is Wijsman $\rho-$summable to A if for each $x \in X$*

$$\lim_{n\to\infty}\frac{1}{\rho_n}\sum_{k=1}^{n}d\left(x,A_k\right)=d\left(x,A\right)$$

where $\rho=(\rho_n)$ is a non-decreasing sequence of positive real numbers tending to ∞ such that $\limsup_n \frac{\rho_n}{n}<\infty$, $\Delta\rho_n=O(1)$ and $\Delta\rho_n=\rho_{n+1}-\rho_n$ for each positive integer n.

In this case, we write $A_k \longrightarrow A(WN_\rho)$.

Definition 2. *Let (X,σ) be a metric space. For any non-empty closed subsets $A, A_k\subset X$, we say that the sequence $\{A_k\}$ is Wijsman $\rho-$ statistical convergent to A (or $WS_\rho-$ convergent to A) if for each $\varepsilon>0$ and $x\in X$,*

$$\lim_{n\to\infty}\frac{1}{\rho_n}\left|\left\{k\leqslant n:\left|d\left(x,A_k\right)-d\left(x,A\right)\right|\geq\varepsilon\right\}\right|=0$$

where $\rho=(\rho_n)$ is a non-decreasing sequence of positive real numbers tending to ∞ such that $\limsup_n \frac{\rho_n}{n}<\infty$, $\Delta\rho_n=O(1)$ and $\Delta\rho_n=\rho_{n+1}-\rho_n$ for each positive integer n.

In this case, we write $A_k \longrightarrow A(WS_\rho)$.

If $\rho=(\rho_n)=n$, for all $n\in\mathbb{N}$, Wijsman ρ- statistical convergent is coincide Wijsman statistical convergence defined by Nuray and Rhoades [20].

Definition 3. *Let (X,σ) be a metric space. For any non-empty closed subset $A_k\subset X$, we say that the sequence $\{A_k\}$ is Wijsman $\rho-$ statistically Cauchy if for each $\varepsilon>0$, there exists a number $N(=N_\varepsilon)$ such that for each $x\in X$,*

$$\lim_{n\to\infty}\frac{1}{\rho_n}\left|\left\{k\leqslant n:\left|d\left(x,A_k\right)-d\left(x,A_N\right)\right|\geq\varepsilon\right\}\right|=0.$$

Definition 4. *Let (X,σ) be a metric space. For any non-empty closed subsets $A, A_k\subset X$, we say that the sequence $\{A_k\}$ is said to be Wijsman $\rho-$ strongly $p-$ summable to A for each positive real number p and for each $x\in X$,*

$$\lim_{n\to\infty}\frac{1}{\rho_n}\sum_{k=1}^{n}\left|d\left(x,A_k\right)-d\left(x,A\right)\right|^p=0.$$

If $p=1$, Wijsman $\rho-$ strongly $p-$ summable reduces to Wijsman $\rho-$ strongly summable and we write $A_k \longrightarrow A\left((WS,[\rho])\right)$.

Theorem 1. *(X,σ) be a metric space and A, A_k (for all $k\in\mathbb{N}$) be non-empty closed subsets of X, then*
i) $\{A_k\}\to A(WS,[\rho])\Rightarrow A_k\to A(WS_\rho)$ and $(WS,[\rho])$ is a proper subset of WS_ρ,
ii)$\{A_k\}\in L_\infty$ and $A_k\to A(WS_\rho)\Rightarrow A_k\to A\left((WS,[\rho])\right)$,
iii) $WS_\rho\cap L_\infty=(WS,[\rho])\cap L_\infty$.

Proof: i) The inclusion part of proof is easy. In order to show that the inclusion $(WS,[\rho])\subseteq WS_\rho$ is proper, we define a sequence $\{A_k\}$ as follows

$$A_k=\begin{cases}\{\sqrt{k}\}, & \text{if } \ k=n^2\\ \{0\}, & \text{if } \ otherwise.\end{cases}$$

Let $(\mathbb{R},d)$ be a metric space such that for $x,y\in X$, $d\left(x,y\right)=|x-y|$ and $\rho=(\rho_n)=n$. We have for every $\varepsilon>0, x>0$

$$\frac{1}{\rho_n}\left|\left\{k\leqslant n:\left|d\left(x,A_k\right)-d\left(x,\{0\}\right)\right|\geq\varepsilon\right\}\right|\leq\frac{\sqrt{n}}{n}\to0,$$

as $n\to\infty$, we get

$$\lim_{n\to\infty}\frac{1}{\rho_n}\left|\left\{k\leqslant n:\left|d\left(x,A_k\right)-d\left(x,\{0\}\right)\right|\geq\varepsilon\right\}\right|=0$$

i.e. $A_k\to\{0\}\left(WS_\rho\right)$.

On the other hand, for $x>0$,

$$\frac{1}{\rho_n}\sum_{k\leqslant n}\left|d\left(x,A_k\right)-d\left(x,\{0\}\right)\right|=\frac{\sqrt{n}\left(\sqrt{n}+1\right)}{n}\to1.$$

So $A_k\nrightarrow\{0\}\left((WS,[\rho])\right)$.

ii))$\{A_k\}\in L_\infty$ and $A_k\to A(WS_\rho)$. Then, we have $\left|d\left(x,A_k\right)-d\left(x,A\right)\right|\leqslant M$
for each $x\in X$ and all $k\leqslant n$. Given $\varepsilon>0$, we get

$$\frac{1}{\rho_n}\sum_{k\leqslant n}|d(x,A_k)-d(x,A)| = \frac{1}{\rho_n}\sum_{\substack{k=1\\|d(x,A_k)-d(x,A)|\geqslant\varepsilon}}^{n}|d(x,A_k)-d(x,A)|$$

$$+\frac{1}{\rho_n}\sum_{\substack{k=1\\|d(x,A_k)-d(x,A)|<\varepsilon}}^{n}|d(x,A_k)-d(x,A)|$$

$$\leqslant \frac{M}{\rho_n}|\{k\leqslant n : |d(x,A_k)-d(x,A)|\geq\varepsilon\}|+\varepsilon.$$

Therefore we have the result.
iii) Follows from i) and ii).

□

Corollary 1. *If $\liminf\frac{\rho_n}{n}>1$, then $W-\lim A_k=A\Rightarrow A_k\longrightarrow A(WS_\rho)$.*

Remark 1. *The converse of Corollary 1 is not true, in general. For this, let $X=\mathbb{R}$ consider a sequence $\{A_k\}$ as*

$$A_k:=\begin{cases}\{x\in\mathbb{R}:2\leqslant x\leqslant k\}, & \text{if}\ \ k\geqslant 2\ \ \text{and}\ \ k\ \ \text{is a square integer,}\\ \{1\}, & \text{if}\ \ \text{otherwise}.\end{cases}$$

This sequence is not Wijsman convergent. But if we consider $(\rho_n)=(n)$,

$$\frac{1}{\rho_n}|\{k\leq n:|d(x,A_k)-d(x,\{1\})|\geq\varepsilon\}|\leqslant\frac{\sqrt{n}}{n}\to 0\ (n\to\infty).$$

This sequence is Wijsman $\rho-$ statistically convergent to set $A=\{1\}$.

Theorem 2. *Let $\rho=(\rho_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ such that $\limsup_n\frac{\rho_n}{n}<\infty,\Delta\rho_n=O(1)$ and $\Delta\rho_n=\rho_{n+1}-\rho_n$ for each positive integer n and $\frac{\rho_n}{n}\geq 1$ for all $n\in\mathbb{N}$. If the sequence $\{A_k\}$ is Wijsman strongly $\rho-$ summable to A, then $\{A_k\}$ is Wijsman $\rho-$ statistically convergent to A.*

Proof:
Let $st-\lim_W A_k=A$. Given $\varepsilon>0$, we get

$$\frac{1}{n}\sum_{k\leqslant n}|d(x,A_k)-d(x,A)| = \frac{\rho_n}{n}\frac{1}{\rho_n}\sum_{k\leqslant n}|d(x,A_k)-d(x,A)|$$

$$\geqslant\frac{1}{\rho_n}|\{k\leqslant n:|d(x,A_k)-d(x,A)|\geq\varepsilon\}|.$$

This proves the proof.

□

Theorem 3. *Let (X,σ) be a metric space. The following statements are equivalent;*
i) $\{A_k\}$ is a Wijsman $\rho-$ statistically convergent,
ii) $\{A_k\}$ is a Wijsman $\rho-$ statistically Cauchy sequence,
iii) $\{A_k\}$ is a sequence for which there is a Wijsman convergent sequence $\{B_k\}$ such that $\{A_k\}=\{B_k\}$ a.a.k.

Proof: Omitted. □

Definition 5. *Let (X,σ) be a metric space. For any non-empty closed subsets $A,A_k\subset X$, we say that the sequence $\{A_k\}$ is Wijsman ρ-almost convergent to A if for each $x\in X$*

$$\lim_{n\to\infty}\frac{1}{\rho_n}\sum_{k=1}^{n}d(x,A_{k+i})=d(x,A)$$

uniformly in i and we write $A_k\longrightarrow A\,([WN_\rho])$.

Definition 6. *Let (X,σ) be a metric space. For any non-empty closed subsets $A,A_k\subset X$, we say that the sequence $\{A_k\}$ is said to be Wijsman ρ-strongly $p-$ almost convergent to A if p positive real number and for each $x\in X$,*

$$\lim_{n\to\infty}\frac{1}{\rho_n}\sum_{k=1}^{n}|d(x,A_{k+i})-d(x,A)|^p=0$$

uniformly in i.

If $p=1$, Wijsman $\rho-$ strongly $p-$ almost convergent is said to be Wijsman strongly $\rho-$ almost convergent and we write $A_k \longrightarrow A\left([WS,[\rho]]\right)$.

Definition 7. *Let (X,σ) be a metric space. For any non-empty closed subset $A_k \subset X$, we say that the sequence $\{A_k\}$ is Wijsman almost $\rho-$ statistically convergent to A if for each $\varepsilon > 0$ and for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{\rho_n} \left|\{k \leqslant n : \left|d\left(x, A_{k+i}\right) - d\left(x, A\right)\right| \geq \varepsilon\}\right| = 0$$

uniformly in i.

Theorem 4. *Let (X,σ) be a metric space and p be a positive real number. Then, for any non-empty closed subsets $A, A_k \subset X$,*
i) $\{A_k\}$ is Wijsman almost ρ-statistical convergent to A if it is Wijsman $\rho-$ strongly $p-$ almost convergent to A,
ii) If $\{A_k\}$ is bounded and Wijsman almost $\rho-$ statistical convergent to A, then it is Wijsman $\rho-$ strongly $p-$ almost convergent to A.

Proof: The proof is similar to the Theorem 1. □

It is easy to see that $C \subset [WN_\rho] \subset [WS,[\rho]] \subset L_\infty$ where $C, [WN_\rho], [WS,[\rho]]$ and L_∞ denote the sets of the all Wijsman convergent, Wijsman $\rho-$ almost convergent, Wijsman strongly $\rho-$ almost convergent and bounded sequences of sets.

3 References

1 H. Altınok, M. Et, R. Çolak, *Some remarks on generalized sequence space of bounded variation of sequences of fuzzy numbers*, Iran. J. Fuzzy Syst. **11**(5) (2014), 39–46.
2 V. K. Bhardwaj, S. Dhawan, *$f-$statistical convergence of order α and strong Cesàro summability of order α with respect to a modulus*, J. Inequal. Appl. **332** (2015), 14 pp.
3 A. Caserta, G. Di Maio, L. D. R. Kočinac, *Statistical convergence in function spaces*, Abstr. Appl. Anal., (2011), Article ID 420419, 11 pp.
4 J. S. Connor, *The Statistical and strong $p-$Cesàro convergence of sequences*, Analysis **8** (1988), 47–63.
5 H. Çakallı, *Lacunary statistical convergence in topological groups*, Indian J. Pure Appl. Math. **26**(2) (1995), 113–119.
6 H. Çakallı , B. Hazarika, *Ideal quasi-Cauchy sequences*, J. Inequal. Appl. **234** (2012), 11 pp.
7 H. Çakallı, *A variation on ward continuity*, Filomat **27**(8) (2013), 1545–1549.
8 H. Çakallı, *A variation on statistical ward continuity*, Bull. Malays. Math. Sci. Soc. **40** (2017), 1701-1710.
9 M. Çınar, M. Karakaş, M. Et, *On pointwise and uniform statistical convergence of order α for sequences of functions*, Fixed Point Theory Appl. **33** (2013), 11 pp.
10 R. Çolak, *Statistical convergence of order α*, Modern Methods in Analysis and Its Applications, New Delhi, India: Anamaya Pub. **2010** (2010) 121–129.
11 R. Çolak, *On λ-Statisitical Convergence*, Conference on Summability and Applications, (2011) Istanbul Commerce Univ. May 12-13, İstanbul.
12 M. Et, A. Alotaibi, S. A. Mohiuddine, *On $(\Delta^m, I)-$statistical convergence of order α*, The Scientific World Journal, (2014), Article ID 535419, 5 pages.
13 M. Et, H. Şengül, *Some Cesaro-type summability spaces of order α and lacunary statistical convergence of order α*, Filomat, **28**(8), (2014), 1593–1602.
14 M. Et, B. C. Tripathy, A. J. Dutta, *On pointwise statistical convergence of order α of sequences of fuzzy mappings*, Kuwait J. Sci. **41**(3) (2014), 17–30.
15 M. Et, R. Çolak, Y. Altın, *Strongly almost summable sequences of order α*, Kuwait J. Sci. **41**(2) (2014), 35–47.
16 H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.
17 J. A. Fridy, *On statistical convergence*, Analysis **5** (1985), 301–313.
18 A. D. Gadjiev, C. Orhan, *Some approximation theorems via statistical convergence*, Rocky Mountain J. Math. **32**(1) (2002), 129–138.
19 M. Işık, K. E. Akbaş, *On $\lambda-$statistical convergence of order α in probability*,J. Inequal. Spec. Funct. **8**(4) (2017), 57–64.
20 F. Nuray, B. E. Rhoades, *Statistical convergence of sequences of sets*, Fasc. Math. **49** (2012), 87–99.
21 T. Šalát, *On statistically convergent sequences of real numbers*, Math. Slovaca **30** (1980), 139–150.
22 E. Savaş, M. Et, *On $(\Delta_\lambda^m, I)-$statistical convergence of order α*, Period. Math. Hungar. **71**(2) (2015), 135–145.
23 I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly **66** (1959), 361–375.
24 H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math. **2** (1951), 73–74.
25 H. Şengül, *Some Cesàro-type summability spaces defined by a modulus function of order (α,β)*, Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat. **66**(2) (2017), 80–90.
26 U. Ulusu, F. Nuray, *Lacunary statistical convergence of sequence of sets*, Prog. Appl. Math. **4**(2) (2012), 99–109.
27 U. Ulusu, E. Dündar, *I-lacunary statistical convergence of sequences of sets*, Filomat **28**(8) (2014), 1567–1574.