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New Fixed Point Theorem for generalized T_F-contractive Mappings and its Application for Solving Some Polynomials

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Abstract

Let (X, d) be a complete metric space. In this paper, we study some new fixed point theorems for generalized T_F-contractive mapping defined on complete metric spaces by using graph closed concept and we proved the existence and uniqueness of a fixed point. These conditions are analogous to Ćirić conditions. In this paper, we compare the two concepts of graph closed and sequentially convergent, and we show that the concept of sequentially convergent is a special case of the concept of graph closed. Also, we provide an counterexample for Dubey et. al. and provide an example in support of our main results. Finally, by using our main results, we present an application to solving some polynomials. Our results, extend several results on the topic in the corresponding literature.

Keywords: Fixed point ; contractive mapping ; generalized T_F-contractive mapping ; graph closed.
2010 MSC: 46J10, 46J15, 47H10, 54H25.

1. Introduction

In 2002 [5], Branciari established the Banach Contractive Principle [4] in the following theorem.

Theorem 1.1. *Let (X, d) be a complete metric space, $\alpha \in [0, 1)$ and $S : X \longrightarrow X$ be a mapping such that, for each $x, y \in X$,*

$$\int_0^{d(Sx,Sy)} \phi(t)dt \leq \alpha \int_0^{d(x,y)} \phi(t)dt,$$

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where $\phi : [0,+\infty) \longrightarrow [0,+\infty)$ *is a Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of* $[0,+\infty)$*, nonnegative, and such that for each* $\varepsilon > 0$, $\int_0^\varepsilon \phi(t)dt > 0$*; then* S *has a unique fixed point* $b \in X$ *such that for each* $x \in X$, $\lim_{n\to\infty} S^n x = b$.

After this result in 2003 [17], Rhoades extended the Branciari Theorem in the following. His results also extended the Ćirić's theorem [7].

Theorem 1.2. *Let* (X, d) *be a complete metric space,* $\alpha \in [0, 1)$ *and* $f : X \longrightarrow X$ *a mapping such that, for each* $x, y \in X$,

$$\int_0^{d(fx,fy)} \phi(t)dt \leq \alpha \int_0^{m(x,y)} \phi(t)dt, \tag{1}$$

where

$$m(x, y) = \max\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}\}$$

and $\phi : [0,+\infty) \longrightarrow [0,+\infty)$ *is a Lebesgue-integrable mapping which is summable (i.e., with finite integral) on each compact subset of* $[0,+\infty)$*, nonnegative, and such that for each* $\varepsilon > 0$, $\int_0^\varepsilon \phi(t)dt > 0$. *Then* f *has a unique fixed point* $b \in X$ *such that for each* $x \in X$, $\lim_{n\to\infty} f^n x = b$.

In 2010, Moradi and Beiranvand [13] introduced a new class of contractive mappings and extend the Branciari's theorem as follows:

Theorem 1.3. *Let* (X, d) *be a complete metric space,* $\alpha \in [0, 1)$, $T, f : X \longrightarrow X$ *be two mappings such that* T *is one-to-one and graph closed and* f *is a* T_F*-contraction; that is:*

$$F(d(Tfx, Tfy)) \leq \alpha F(d(Tx, Ty)),$$

for all $x, y \in X$, *where* $F : [0,+\infty) \longrightarrow [0,+\infty)$ *is nondecreasing and continuous from the right with* $F^{-1}(0) = \{0\}$*; then* f *has a unique fixed point* $a \in X$. *Also for every* $x \in X$, *the sequence of iterates* $\{Tf^n x\}$ *converges to* Ta.

In 2015, Mehmet Kir and Hukmi Kiziltunc [11], extended Kannan fixed point theorem by using T_F-contraction mappings. After that in 2017, Dubey et. al. [9], proved some fixed point theorems for T_F type contractive conditions in the framework of complete metric spaces. Some of their results, as follows:

Theorem 1.4. *[9] Let* (X, d) *be a complete metric space and* $T, f : X \longrightarrow X$ *be mappings such that* T *is continuous, one-to-one and subsequentially convergent. If* $a, b \in [0, 1)$ *and* $x, y \in X$

$$F(d(Tfx, Tfy)) \leq a[F(d(Tx, Ty))] + b[F(d(Tx, Tfx)) + F(d(Tx, Tfy))],$$

where $F : [0,+\infty) \longrightarrow [0,+\infty)$ *is nondecreasing and continuous from the right with* $F^{-1}(0) = \{0\}$. *Then* f *has a unique fixed point* $a \in X$. *Also, if* T *is sequentially convergent then for every* $x_0 \in X$ *the sequence of iterates* $\{f^n x_0\}$ *converges to the fixed point.*

Theorem 1.5. *[9] Let* (X, d) *be a complete metric space and* $T, f : X \longrightarrow X$ *be mappings such that* T *is continuous, one-to-one and subsequentially convergent. If* $a, b, c \in [0, 1)$ *and* $x, y \in X$

$$\begin{aligned} F(d(Tfx, Tfy)) \quad \leq \quad & a[F(d(Tx, Ty))] + b[F(d(Tx, Tfy)) + F(d(Ty, Tfxy))] \\ & +c[F(d(Tx, Tfx)) + F(d(Tx, Tfy))], \end{aligned}$$

where $F : [0,+\infty) \longrightarrow [0,+\infty)$ *is nondecreasing and continuous from the right with* $F^{-1}(0) = \{0\}$. *Then* f *has a unique fixed point* $a \in X$. *Also, if* T *is sequentially convergent then for every* $x_0 \in X$ *the sequence of iterates* $\{f^n x_0\}$ *converges to the fixed point.*

Theorem 1.6. *[9] Let (X, d) be a complete metric space and $T, f : X \longrightarrow X$ be mappings such that T is continuous, one-to-one and subsequentially convergent. For all $x, y \in X$*

$$\begin{aligned} F(d(Tfx, Tfy)) \leq\ & a(x,y)[F(d(Tx,Ty))] \\ & +b(x,y)[F(d(Tx,Tfy)) + F(d(Ty,Tfxy))] \\ & +c(x,y)[F(d(Tx,Tfx)) + F(d(Tx,Tfy))], \end{aligned} \tag{2}$$

where $a(x, y), b(x, y), c(x, y) \geq 0$ and

$$\sup[a(x,y) + 2b(x,y) + 2c(x,y)] \leq \lambda < 1$$

and where $F : [0, +\infty) \longrightarrow [0, +\infty)$ is nondecreasing and continuous from the right with $F^{-1}(0) = \{0\}$. Then f has a unique fixed point $a \in X$. Also, if T is sequentially convergent then for every $x_0 \in X$ the sequence of iterates $\{f^n x_0\}$ converges to the fixed point.

Remark 1.7. *In the proof of above theorems, the boundedness of the sequence $\{Tf^n x_0\}$ is used by the authores, but not proved. Also the authores just considered $a, b, c \in [0, 1)$ for Theorem 1.4 and Theorem 1.5. In the following, we give counterexamples for these tow theorems. In the main resualts of this paper, we extend and correct the above theorems.*

Example 1.8. *Let $X = \{1, 2\}$ endowed with the Euclidean metric and let $f : X \longrightarrow X$ defined by $f(1) = 2, f(2) = 1$. Suppose that $a = b = c = \dfrac{2}{3}$ and $T(x) = F(x) = x$ for all $x \in X$. It can be easily verified that, the condition of Theorems 1.4 and 1.5 are hold. But f has not fixed point.*

In 2011, Samet and Vetro [19], generalized the Ćirić's fixed point theorem as follows:

Theorem 1.9. *Let (X, d) be a complete metric space, $f : X \longrightarrow X$ be a given mapping and $N \in \mathbb{N}$ be fixed. Let $\varphi_i : [0, +\infty) \longrightarrow [0, +\infty)(i = 1, \ldots, N)$ be a Lebesgue integrable mapping on each compact subset of $[0, +\infty)$ such that for all $\varepsilon > 0$,*

$$\int_0^{\varepsilon} \phi_i(t)dt > 0 \, for \, all \, i = 1, \ldots, N.$$

For all $t \geq 0$, we define

$$\begin{aligned} I_1(t) &= \int_0^t \phi_1(s)ds, \\ I_2(t) &= \int_0^{I_1(t)} \phi_2(s)ds, \\ &\vdots \\ I_N(t) &= \int_0^{I_{N-1}(t)} \phi_N(s)ds. \end{aligned}$$

We assume that for every $x, y \in X$ there exist non-negative numbers $q(x, y)$, $r(x, y)$, $s(x, y)$ and $t(x, y)$ such that

$$\sup_{x \in X}\{q(x,y) + r(x,y) + s(x,y) + 2t(x,y)\} = \lambda < 1,$$

and

$$\begin{aligned} I_N(d(fx, fy)) \leq\ & q(x,y)I_N(d(x,y)) + r(x,y)I_N(d(x,fx)) \\ +\ & s(x,y)I_N(d(y,fy)) \\ +\ & 2t(x,y)I_N(\frac{d(x,fy) + d(y,fx)}{2}) \end{aligned}$$

holds for every $x, y \in X$. Then f admits a unique fixed point $u \in X$ and for each $x \in X$, the sequence $\{f^n x\}$ converges to u.

Many authors have studies fixed point and established the Banach Contraction Principle and Ćirić's theorem. Among many authors, see for example, [1], [2], [6], [8], [10], [14], [15], [16], [19] and [20].

In Section 3 we extend Ćirić, Branciari, Rhoades, Moradi and Beiranvand, Dubey et. al., Mehmet Kir and Hukmi Kiziltunc and Samet-Vetro's Theorems. In Section 4, as an application, we provide a solution to find an answer to some polynomials.

2. Preliminaries

In this paper (X, d) denotes a complete metric space.

Definition 2.1. *[13] Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is said to be graph closed if for every sequence $\{x_n\}$ such that $\lim_{n\to\infty} Tx_n = a$ then for some $b \in X$, $Tb = a$.*

Definition 2.2. *[12] Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is said to be sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergent then $\{y_n\}$ also is convergent. T is said to be subsequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergent then $\{y_n\}$ has a convergent subsequence.*

Let Ψ denotes the class of all nondecreasing and continuous maps $F : [0, +\infty) \longrightarrow [0, +\infty)$ with $F^{-1}\{0\} = \{0\}$.

Definition 2.3. *Let (X, d) be a metric space. A mapping $f : X \longrightarrow X$ is said to be generalized T_F-contractive, if there exists $F \in \Psi$ and one-to-one and graph closed mapping $T : X \longrightarrow X$ such that*

$$F(d(Tfx, Tfy)) \leq \alpha F(N(x, y)),$$

for all $x, y \in X$ and some $\alpha \in [0, 1)$, where

$$N(x, y) = \max\{d(Tx, Ty), d(Tx, Tfx), d(Ty, Tfy), \frac{d(Tx, Tfy) + d(Ty, Tfx)}{2}\}.$$

For the main results of this paper, we prove the following usful lemma.

Lemma 2.4. *Let (X, d) be a complete metric space and let $T : X \longrightarrow X$ be a mapping such that T is continuous and subsequentially convergent. Then T is a graph closed map.*

Proof. Suppose that $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Tx_n = a$. Since T is subsequentially convergent, then there exists a subsequence $\{x_{n(k)}\}$ such that $\lim_{k\to\infty} x_{n(k)} = b$. Since T is continuous and $\lim_{n\to\infty} Tx_n = a$, then we conclude that $Tb = a$. This completes the proof. □

Remark 2.5. *In 2012 Aydi et. al. [3] proved that the main results of some papers; that consider the sequentially convergent; are particular results of previous existing theorems in the literature. We can not conclude that, every graph closed map is subsequentially convergent. For example, suppose that $X = \mathbb{R}$ endowed with the Euclidean metric and $T : X \longrightarrow X$ defined by, $Tx := \sin x$. Obviousley, T is continuous and graph closed, but T is not subsequentially convergent. Because the sequence $\{\sin(2n\pi)\}$ is convergent, but the sequence $\{2n\pi\}$ has not any convergent subsequence. In this paper we consider the graph closed mappins for the main results.*

3. Main Results

The following theorem is the main result of this paper.

Theorem 3.1. *Let (X, d) be a complete metric space and let $f : X \longrightarrow X$ be a mapping such that,*

$$F(d(Tfx, Tfy)) \leq \alpha F(N(x, y)), \tag{3}$$

for all $x, y \in X$ and some $\alpha \in [0, 1)$ (i.e., generalized T_F-contractive) where $F \in \Psi$ and $T : X \longrightarrow X$ is a one-to-one and graph closed map. Then f has a unique fixed point $b \in X$ and for every $x \in X$ the sequence of iterates $\{Tf^n x\}$ converges to Tb. Also if T is sequentially convergent then for every $x \in X$ the sequence of iterates $\{f^n x\}$ converges to b (the fixed point of f).

Proof. Unicity of the fixed point follows from (3)
Since $F \in \Psi$, for every $\varepsilon > 0$

$$F(\varepsilon) > 0. \tag{4}$$

From (3) if $x \neq y$ then,

$$d(Tfx, Tfy) < N(x, y). \tag{5}$$

Let $x \in X$. Define $x_n = Tf^n x$.
We break the argument into four steps.

Step 1. $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$.

proof. For every $n \in \mathbb{N}$, from (3),

$$F(d(x_n, x_{n+1})) \leq \alpha F(N(x_{n-1}, x_n)), \tag{6}$$

where,

$$\begin{aligned} N(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \\ &\quad \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2}\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2}\} \\ &\leq \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}\} \\ &= \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \ (from\ (4)\ and\ (5)) \\ &= d(x_{n-1}, x_n). \end{aligned} \tag{7}$$

Hence, using (6) and (7) and using induction,

$$F(d(x_n, x_{n+1})) \leq \alpha^n F(d(x, x_1)). \tag{8}$$

Letting $n \to \infty$ in (8), we get $\lim\limits_{n\to\infty} F(d(x_n, x_{n+1})) = 0$. Since $F \in \Psi$,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Step 2. $\{x_n\}$ is a bounded sequence.

proof. If $\{x_n\}$ were unbounded, then, we choose the sequence $\{n(k)\}_{k=1}^{\infty}$ such that $n(1) = 1$ and for each $k \in \mathbb{N}$; $n(k + 1) > n(k)$ is minimal in the sense that $d(x_{n(k+1)}, x_{n(k)}) > 1$. Obviously $n(k) \geq k$, for every $k \in \mathbb{N}$.
By using Step 1, there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$; $d(x_{k+1}, x_k) < \frac{1}{2}$. So for every $k \geq k_0$;

$$\begin{aligned} 1 &< d(x_{n(k+1)}, x_{n(k)}) \\ &\leq d(x_{n(k+1)}, x_{n(k+1)-1}) + d(x_{n(k+1)-1}, x_{n(k)}) \\ &\leq d(x_{n(k+1)}, x_{n(k+1)-1}) + 1. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} d(x_{n(k+1)}, x_{n(k)}) = 1.$$

Also,

$$\begin{aligned}
&d(x_{n(k+1)}, x_{n(k)}) - d(x_{n(k+1)+1}, x_{n(k+1)}) - d(x_{n(k)+1}, x_{n(k)})\\
&\leq d(x_{n(k+1)+1}, x_{n(k)+1})\\
&\leq d(x_{n(k+1)+1}, x_{n(k+1)}) + d(x_{n(k+1)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1}),
\end{aligned}$$

and this shows that,

$$\lim_{n\to\infty} d(x_{n(k+1)+1}, x_{n(k)+1}) = 1. \tag{9}$$

Also,

$$\begin{aligned}
&d(x_{n(k+1)}, x_{n(k)}) \leq N(x_{n(k+1)}, x_{n(k)})\\
&= \max\big\{d(x_{n(k+1)}, x_{n(k)}), d(x_{n(k+1)}, x_{n(k+1)+1}),\\
&\quad d(x_{n(k)}, x_{n(k)+1}), \frac{d(x_{n(k)}, x_{n(k+1)+1}) + d(x_{n(k+1)}, x_{n(k)+1})}{2}\big\}\\
&\leq \max\big\{d(x_{n(k+1)}, x_{n(k)}), \frac{[d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k+1)+1})]}{2}\\
&\quad + \frac{[d(x_{n(k+1)}, x_{n(k+1)+1}) + d(x_{n(k+1)+1}, x_{n(k)+1})]}{2}\big\}.
\end{aligned}$$

This shows that

$$\lim_{n\to\infty} N(x_{n(k+1)}, x_{n(k)}) = 1. \tag{10}$$

Since (9) and (10) are hold and

$$F(d(x_{n(k+1)+1}, x_{n(k)+1})) \leq \alpha F(N(x_{n(k+1)}, x_{n(k)})),$$

we conclude that,

$$F(1) \leq \alpha F(1).$$

Since $\alpha \in [0, 1)$, $F(1) = 0$ and this is a contradiction.

Step 3. $\{x_n\}$ is a Cauchy sequence.

proof. Since $F \in \Psi$, for every $m, n \in \mathbb{N}$ with $m > n$

$$\begin{aligned}
&F(d(x_m, x_n)) \leq \alpha F(N(x_{m-1}, x_{n-1}))\\
&= \alpha F(\max\{d(x_{m-1}, x_{n-1}), d(x_{m-1}, x_m), d(x_{n-1}, x_n),\\
&\quad \frac{d(x_{m-1}, x_n) + d(x_{n-1}, x_m)}{2}\})\\
&\leq \alpha F(\max\{d(x_{m-1}, x_{n-1}), d(x_{m-1}, x_m), d(x_{n-1}, x_n), d(x_{m-1}, x_n),\\
&\quad d(x_{n-1}, x_m)\})\\
&= \alpha F(d(x_{r(1)}, x_{s(1)})),
\end{aligned} \tag{11}$$

where $s(1) \geq n - 1$ and $r(1) > s(1)$.

By the same method, there exist $r(2), s(2) \in \mathbb{N}$ such that $s(2) \geq s(1) - 1 \geq n - 2$, $r(2) > s(2)$ and

$$F(d(x_{r(1)}, x_{s(1)})) \leq \alpha F(d(x_{r(2)}, x_{s(2)})). \tag{12}$$

By (11) and (12),

$$F(d(x_m, x_n)) \leq \alpha^2 F(d(x_{r(2)}, x_{s(2)})).$$

Using induction, there exist $r(n), s(n) \in \mathbb{N}$ such that $s(n) \geq n - n = 0$, $r(n) > s(n)$ and

$$F(d(x_m, x_n)) \leq \alpha^n F(d(x_{r(n)}, x_{s(n)})). \tag{13}$$

Since $\{x_n\}$ is bounded and (13) holds, we have

$$\lim_{m,n\to\infty} F(d(x_m, x_n)) = 0.$$

Hence, from $F \in \Psi$,

$$\lim_{m,n\to\infty} d(x_m, x_n) = 0.$$

Therefore $\{x_n\}$ is Cauchy.

Step 4. f has a fixed point.

proof. Since (X, d) is complete and $\{x_n\}$ is Cauchy, there exists $a \in X$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tf^n(x) = a.$$

Since T is graph closed, there exists $b \in X$ such that $Tb = a$. Now we show that b is a fixed point of f. From $F \in \Psi$ we conclude that

$$F(d(Tb, Tfb)) = \lim_{n\to\infty} F(d(Tf^{n+1}x, Tfb)) \leq \alpha \lim_{n\to\infty} F(N(Tf^n x, Tb)), \tag{14}$$

where

$$\begin{aligned} N(Tf^n x, Tb) \quad = \quad & \max\{d(x_n, Tb), d(x_n, x_{n+1}), d(Tb, Tfb), \\ & \frac{d(x_n, Tfb) + d(Tb, x_{n+1})}{2}\}. \end{aligned}$$

Hence

$$\lim_{n\to\infty} N(Tf^n x, Tb) = d(Tb, Tfb). \tag{15}$$

Hence, from (14)

$$F(d(Tb, Tfb)) \leq \alpha F(d(Tb, Tfb)),$$

and this shows that $F(d(Tb, Tfb)) = 0$. From (4), $d(Tb, Tfb) = 0$. Hence $Tb = Tfb$. Since T is one-to-one, $fb = b$.

Obviousely, if T is sequentially convergent then for every $x \in X$ the sequence of iterates $\{f^n x\}$ converges to b and this completes the proof. □

Remark 3.2. *Theorem 3.1 is a generalization of the Rhoades theorem (Theorem 1.2), by letting* $Tx = x$ *and* $F(t) = \int_0^t \phi(s)ds$.

The following example shows that (3) is indeed a proper extension of (1).

Example 3.3. *Let* $X = [1, +\infty)$ *endowed with the Euclidean metric. We consider a mapping* $S : X \longrightarrow X$ *defined by* $Sx = 4\sqrt{x}$. *Obviously* S *has a unique fixed point* $b = 16$.
If (1) holds, then for every $x, y \in X$ *such that* $x \neq y$, *we must have*

$$|Sx - Sy| < m(x, y).$$

But by taking $x = 1$ *and* $y = 4$ *we have,* $|Sx - Sy| = m(x, y) = 4$. *Therefore we can not use the Rhoades theorem (Theorem 1.2).*
Now we define $T : X \longrightarrow X$ *by* $Tx = \ln(e.x)$. *Obviously* T *is one-to-one and graph closed. By taking* $F(t) = t$, *all conditions of Theorem 3.1 are hold and therefore* S *has a unique fixed point.*

Remark 3.4. *Let $F : [0, +\infty) \longrightarrow [0, +\infty)$ define by $F(t) = I_N(t)$ in Theorem 1.3. Obviously, $F \in \Psi$. Since (2) holds and F is nondecreasing,*

$$\begin{aligned} F(d(fx, fy)) &\leq q(x, y)F(d(x, y)) + r(x, y)F(d(x, fx)) + s(x, y)F(d(y, fy)) \\ &+ 2t(x, y)F(\frac{d(x, fy) + d(y, fx)}{2}) \\ &\leq q(x, y)F(N(x, y)) + r(x, y)F(N(x, y)) + s(x, y)F(N(x, y)) \\ &+ 2t(x, y)F(N(x, y)) \\ &\leq \lambda F(N(x, y)). \end{aligned}$$

holds for every $x, y \in X$. So by letting $Tx = x$, all conditions in Theorem 3.1 are hold. Hence by using Theorem 3.1 f has a unique fixed point. Therefore Theorem 3.1 is a generalization of Theorem 1.9.

In the following, we extend the Theorem 1.6.

Corollary 3.5. *Let (X, d) be a complete metric space and let $f : X \longrightarrow X$ be a mapping such that,*

$$\begin{aligned} F(d(Tfx, Tfy)) \leq\ & a(x, y)[F(d(Tx, Ty))] \\ & +b(x, y)[F(d(Tx, Tfy)) + F(d(Ty, Tfxy))] \\ & +c(x, y)[F(d(Tx, Tfx)) + F(d(Tx, Tfy))], \end{aligned}$$

where $a(x, y), b(x, y), c(x, y) \geq 0$ for all $x, y \in X$ and

$$\sup_{x,y \in X} [a(x, y) + 2b(x, y) + 2c(x, y)] \leq \lambda < 1$$

for some $\lambda \in [0, 1)$, and where $F \in \Psi$ and $T : X \longrightarrow X$ is a one-to-one and graph closed map. Then f has a unique fixed point $b \in X$ and for every $x \in X$ the sequence of iterates $\{Tf^n x\}$ converges to Tb. Also if T is sequentially convergent then for every $x \in X$ the sequence of iterates $\{f^n x\}$ converges to b (the fixed point of f).

Proof. One can esealy shows that

$$\begin{aligned} & a(x, y)[F(d(Tx, Ty))] + b(x, y)[F(d(Tx, Tfy)) + F(d(Ty, Tfxy))] \\ & +c(x, y)[F(d(Tx, Tfx)) + F(d(Tx, Tfy))] \leq \lambda F(N(x, y)), \end{aligned}$$

for all $x, y \in X$. Now by using Theorem 3.1, the result is obtained. □

4. Application to solving polynomials

As an application of the main theorem of this paper we conclude the existence of solution of some polynomials.

Theorem 4.1. *Let $b, c > 0$ and $n > 1$. Then the equation*

$$y^n = by + c \tag{16}$$

has a unique solution on $[\sqrt[n]{c}, +\infty)$.

Proof. Let $0 < \varepsilon < b\sqrt[n]{c}$ be arbitrary. Put $\alpha = c + \varepsilon$. It is enough to show that the problem (14) has a unique solution on $[\sqrt[n]{\alpha}, +\infty)$.
There exists $\beta > 0$ such that $\ln(\alpha - c) + \beta \geq \alpha$. Suppose $f : [\alpha, +\infty) \longrightarrow [\alpha, +\infty)$ defined by $fx = b\sqrt[n]{x} + c$ and $T : [\alpha, +\infty) \longrightarrow [\alpha, +\infty)$ defined by $Tx = \ln(x - c) + \beta$. For all $x, y \in [\alpha, +\infty)$ with $x > y$ we have

$$|Tfx - Tfy| = \ln(\frac{\sqrt[n]{x}}{\sqrt[n]{y}}) = \frac{1}{n}\ln(\frac{x}{y}) < \frac{1}{n}\ln(\frac{x - c}{y - c}) = \frac{1}{n}|Tx - Ty| \leq \frac{1}{n}N(x, y).$$

Hence f is generalized T_F-contractive. So f has a unique fixed point z on $[\alpha, +\infty)$ and the sequence of iterates $\{Tf^n(c+1)\}$ converges to Tz and therefore, the sequence of iterates $\{f^n(c+1)\}$ converges to z. Therefore the equation $x = b\sqrt[n]{x} + c$ has a unique solution on $[\alpha, +\infty)$. Also there exists a unique $y > 0$ such that $y^n = z$. Obviously $y \in [\sqrt[n]{\alpha}, +\infty)$. Hence from $z = b\sqrt[n]{z} + c$ we have $y^n = by + c$ and this completes the proof. □

References

[1] A. Amini-Harandi, Endpoints of set-valued contractions in metric spaces, Nonlinear Anal., 72 (2010) 132–134.

[2] E. Analouei Adegani and M. Bota, Coupled coincidence point results for mappings without mixed monotone property in partially ordered G-metric spaces, Mathematical Analysis and Convex Optimization, 1 (2020) 93–106.

[3] H. Aydi, E. Karapınar and B. Samet, Remarks on some recent fixed point theorems, Fixed Point Theory and Applications, (2012), 2012:76.

[4] S. Banach, Sur Les Operations Dans Les Ensembles Abstraits et Leur Application Aux Equations Integrales, Fund. Math. 3(1922) 133–181 (French).

[5] A. Branciari,A fixed point theorem for mappings satisfying a general contractive condition of integral type, Int. J. Math. Math. Sci. 29 (9) (2002) 531–536.

[6] C. Chifu, Common fixed point results in extended b-metric spaces endowed with a directed graph, Results in Nonlinear Analysis, 2 (2019), 1, 18-24.

[7] Lj. B. Ćirić, Generalized contraction and fixed point theorem, Publ. Inst. Math. (Beogard) (N.S.) 12 (26) (1971) 19–26.

[8] B. Djafari Rouhani and Sirous Moradi, Common fixed point of generalized φ-weak contractive multi-valued and single valued mappings, Fixed point theory and Applications, (2010), doi:10.1155/2010/708984.

[9] A.K. Dubey, Urmila Mishra and Rita Pal, New fixed point theorems for t_f type contractive conditions, Global Journal of Pure and Applied Mathematics, 13, No. 11 (2017) 529–539.

[10] S. Gulyaz-Ozyurt, A note on Kannan type mappings with alpha F-contractive iterate, Results in Nonlinear Analysis, 2 (3) (2019), 143-146.

[11] M. Kir and H. Kiziltunc, Some generalized fixed point theorems in the context of ordered metric spaces, J. Nonlinear Sci. Appl., 8 (2015) 7955–7962.

[12] S. Moradi and D. Alimohammadi, New Extensions of Kannan Fixed-Point Theorem on Complete Metric and Generalized Metric Spaces, Int. Journal of Math. Analysis, 5, No. 47 (2011) 2313–2320.

[13] S. Moradi and A. Beiranvand, Fixed Point of $T_F - contractive$ Single-valued Mappings, Iranian Journal of Mathematical Sciences and Informatics, 5, No. 2 (2010), 25–32.

[14] S. Moradi, Z. Fathi, E. Analouee, The common fixed point of single-valued generalized φ_f-weakly contractive mappings, Applied Mathematics Letters, 24 (2011) 771–776.

[15] S. Moradi, F. Khojasteh, Endpoints of multi-valued generalized weak contraction mappings, Nonlinear Analysis, 74 (2011) 2170–2174.

[16] A. Ozturk, Advances in the Theory of Nonlinear Analysis and its Application A fixed point theorem for mappings with an F-contractive iterate, 3 (4) (2019), 231 âĂŞ 235.

[17] B. E. Rhoades, Two fixed-point theorems for mappings satisfying a general contractive condition of integral type, Int. J. M. and M. since, 63 (2003) 4007–4013.

[18] B.E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Anal., 47 (2001) 2683–2693.

[19] B. Samet and C. Vetro, An Integral Version of Ćirić's Fixed Point Theorem, Mediterr. J. Math., (2011), doi: 10.1007/s00009-011-0120-1.

[20] Q. Zhang and Y.song, Fixed point theory for generalized φ-weak contractions, App. Math. Letters, 22 (2009) 75-78.