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Rayleigh-Quotient Representation of the Real Parts, Imaginary Parts, and Moduli of the Eigenvalues of General Matrices

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Abstract

In the present paper, formulas for the Rayleigh-quotient representation of the real parts, imaginary parts, and moduli of the eigenvalues of general matrices are obtained that resemble corresponding formulas for the eigenvalues of self-adjoint and diagonalizable matrices. These formulas are of interest in Linear Algebra and in the theory of linear dynamical systems. The key point is that a weighted scalar product is used that is defined by means of a special positive definite matrix. As applications, one obtains convexity properties of newly-defined numerical ranges of a matrix. A numerical example underpins the theoretical findings.

1. Introduction

For self-adjoint matrices, there are formulas for the eigenvalues in the form of Rayleigh quotients; more precisely, max-, min-, min-max-, and max-min-formulas are known; for this, see, e.g., the book [1, Section 5.4]. Recently, the author has carried over these formulas to the real parts, imaginary parts, and moduli of diagonalizable matrices. The aim of the present paper is to extend these results to general matrices. We mention also that the presentation of this paper parallels that of [2]. So, similarities in the formulation do not happen by accident, but are intended in order to underline the similarities. As a consequence, many verbatim passages in the formulations are taken from there.

As it has already been said in [2], first, the obtained formulas are of interest on their own in Linear Algebra. Second, these are also of potential interest, for example, in the theory of linear dynamical systems. The reason for this is as follows. The real parts of the eigenvalues multiplied by the time are equal to the arguments of the exponential functions that describe the decay behavior of the solution (see, e.g., [3, Section 7.1, p.2011, Formulas (89), (90)]). Further, the system is asymptotically stable if the real parts of all eigenvalues are negative. Moreover, when the eigenvalues are pairwise conjugate-complex, then the moduli of the imaginary parts are the circular damped eigenfrequencies of the system (see, e.g., [3, Section 7.1, p. 2011, (89)]). Third, the paper could be of interest in graduate/undergraduate teaching or research at college level since its style is expository and since its subject can be seen as a supplement of the curriculum in Linear Algebra and Numerical Analysis.

The paper is structured as follows. In Section 2, preliminary materials are assembled on biorthogonality relations for the principal vectors of a general matrix A and the principal vectors of A^* that will be useful in the sequel. Moreover, the construction of positive semi-definite matrices R_j and of the positive definite matrix $R=\sum_{j=1}^{n} R_j$ is reviewed where the last one is employed to define a weighted scalar product $(\cdot,\cdot)_R$ that plays a key role in deriving the new results. In Sections 3, 4, and 5, formulas for the Rayleigh-quotient representation of the real parts, imaginary parts, and moduli of the eigenvalues of a general matrix are given, as the case may be. In Section 6, a connection between the matrices $R^{-1}\frac{A^*R+RA}{2}$, $R^{-1}\frac{RA-A^*R}{2i}$, and $R^{-1}A^*RA$ is established that play a key role in the study of the real parts, imaginary parts, and moduli of the eigenvalues of A, respectively. Section 7 describes the applications and, in Section 8, we give a numerical example. Finally, Section 9 contains the conclusions. The non-cited references [4], [5], [6], [7], [8], [9], [10], and [11] are given because they may be useful to the reader in the context of the present paper.

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2. Preliminaries

As a preparation to Theorem 2.1, we formulate the following *conditions*:

($C1'$) $A \in C^{n\times n}$
($C2'$) $\lambda_i,\ i=1,\cdots,r$ are the eigenvalues of A corresponding to the Jordan blocks $J_i(\lambda_i)\in C^{m_i\times m_i},\ i=1,\cdots,r$ with the chains of principal vectors $p_1^{(i)},\cdots,p_{m_i}^{(i)},\ i=1,\cdots,r$
($C3'$) ${u_1^{(i)}}^*,\cdots,{u_{m_i}^{(i)}}^*,\ i=1,\cdots,r$ are the principal vectors of A^* corresponding to the eigenvalues $\overline{\lambda_i},\ i=1,\cdots,r$ of the Jordan blocks $J_i(\overline{\lambda_i})\in C^{m_i\times m_i},\ i=1,\cdots,r$
($C4'$) $\lambda_i\neq\lambda_j,\ i\neq j,\ i,j=1,\cdots,r$

One has the following theorem.

Theorem 2.1. (Biorthogonality relations for principal vectors)
Let the conditions ($C1'$)-($C4'$) be fulfilled. Then, the systems $\{p_1^{(1)},\cdots,p_{m_1}^{(1)};\cdots;p_1^{(r)},\cdots,p_{m_r}^{(r)}\}$ and $\{{u_1^{(1)}}^*,\cdots,{u_{m_1}^{(1)}}^*;\cdots;{u_1^{(r)}}^*,\cdots,{u_{m_r}^{(r)}}^*\}$ can be constructed such that the following biorthogonality relations hold:

$$(p_k^{(i)},{u_l^{(i)}}^*)=\begin{cases}1, & l=m_i-k+1\\ 0, & l\neq m_i-k+1\end{cases}$$

$k=1,\cdots,m_i,\ i=1,\cdots,r$ and

$$(p_k^{(i)},{u_l^{(j)}}^*)=0,\ i\neq j,$$

$k=1,\cdots,m_i,\ l=1,\cdots,m_j,\ i,j=1,\cdots,r$.
So, with

$${v_l^{(i)}}^*:={u_{m_i-l+1}^{(i)}}^*,$$

$l=1,\cdots,m_i,\ i=1,\cdots,r$ one has the biorthogonality relations

$$(p_k^{(i)},{v_l^{(i)}}^*)=\delta_{kl},$$

$k,l=1,\cdots,m_i,\ i=1,\cdots,r$, and

$$(p_k^{(i)},{v_l^{(j)}}^*)=0,\ i\neq j,$$

$k=1,\cdots,m_i,\ l=1,\cdots,m_j,\ i,j=1,\cdots,r$.

Proof. See proof of [12, Theorem 2]. □

Remark 2.2. *The hypothesis $\lambda_i\neq\lambda_j,\ i\neq j,\ i,j=1,\cdots,r$ can be omitted, see [12, Theorem 4]. But, since in our example this condition is fulfilled, we preserve it.*

Theorem 2.3. (Construction of positive definite matrix R)
Let the conditions ($C1'$)-($C4'$) be fulfilled. Let $\alpha_j=\lambda_j(A)$ be the eigenvalues and $u_1^{(j)},\ldots,u_{m_j}^{(j)}$ be a chain of associated left principal vectors for $j=1,\ldots,r$. Further, let $A^*\in C^{n\times n}$ be the adjoint matrix of A so that ${u_1^{(j)}}^*,\ldots,{u_{m_j}^{(j)}}^*$ is a chain of right principal vectors corresponding to the eigenvalues $\overline{\alpha_j}=\lambda_j(A^*)$ for $j=1,\cdots,r$, i.e.,

$$u_k^{(j)}A=\alpha_j u_k^{(j)}+u_{k-1}^{(j)}$$

with $u_0^{(j)}=0,\ k=1,\ldots,m_j;j=1,\ldots,r$ and

$$A^*{u_k^{(j)}}^*=\overline{\alpha_j}{u_k^{(j)}}^*+{u_{k-1}^{(j)}}^*$$

with ${u_0^{(j)}}^*=0,\ j=1,\ldots,r$.
Let

$$\rho_j=\overline{\alpha_j}+\alpha_j=2Re\,\alpha_j=2Re\,\overline{\alpha_j},\ j=1,\cdots,r,$$

$$\sigma_j=\alpha_j-\overline{\alpha_j}=2\,i\,Im\,\alpha_j,\ j=1,\cdots,r,$$

and

$$R_j^{(k,k)}:={u_k^{(j)}}^*u_k^{(j)},\ k=1,\cdots,m_j,\ j=1,\cdots,r.$$

Then,

$$A^*R_j^{(1,1)}+R_j^{(1,1)}A=\rho_jR_j^{(1,1)},\ j=1,\cdots,r,$$

$$R_j^{(1,1)}A - A^* R_j^{(1,1)} = \sigma_j R_j^{(1,1)},\ j = 1, \cdots, r.$$

In other word, the matrix eigenvalue problem

$$A^*V + VA = \mu V$$

has the r solution pairs

$$(\mu, V) = (\rho_j, R_j^{(1,1)})$$

with real ρ_j, and the matrix eigenvalue problem

$$VA - A^*V = \mu V$$

has the r solution pairs

$$(\mu, V) = (\sigma_j, R_j^{(1,1)})$$

with purely imaginary σ_j.
The matrices $R_j^{(k,k)} \in \mathbb{C}^{n\times n},\ k = 1, \ldots, m_j,\ j = 1, \ldots, r$ are positive semi-definite. Further,

$$R := \sum_{j=1}^{r} R_j = \sum_{j=1}^{r}\sum_{k=1}^{m_j} R_j^{(k,k)} \tag{2.1}$$

is positive definite.

Proof. See [13, Theorem 2]. □

Remark 2.4. *Since R in (2.1) is positive definite, by*

$$(u,v)_R := (Ru, v),\ u,\ v \in \mathbb{C}^n,$$

a weighted scalar product $(\cdot,\cdot)_R$ *is defined and by*

$$\|u\|_R := (Ru, u)^{\frac{1}{2}},\ u \in \mathbb{C}^n,$$

a weighted norm $\|\cdot\|_R$.

3. Formulas for the representation of the real parts of the eigenvalues of a general matrix

In this section, we want to derive formulas for the representation of the real parts of the eigenvalues of a general matrix A by Rayleigh quotients. More precisely, max-, min-, min-max-, and max-min-representations are obtained corresponding to associated formulas for the eigenvalues of diagonalizable matrices in [2] or to the eigenvalues of self-adjoint matrices, assembled, for instance, in the book of [1, Section 5.4].
First, we derive a result similar to that of [2, Lemma 3.1]. For this, with the identity matrix E, we introduce the abbreviation

$$N_{\lambda_j(A)} := \{u \in \mathbb{C}^n \,|\, (A - \lambda_j(A)E)u = 0\},\ j = 1, \ldots, r$$

for the geometric eigenspaces so that

$$N_{\lambda_j(A)} = [p_1^{(j)}] = [p_j],\ j = 1, \ldots, r.$$

Herewith, we define

$$N_{\sigma(A)} := \bigoplus_{j=1}^{r} N_{\lambda_j(A)}.$$

We have the following lemma.

Lemma 3.1. *Let the conditions (C1′)-(C4′) be fulfilled and R be defined by (2.1).*
Then, with the denotations of Theorem 2.3,

$$(Au, u)_R = \sum_{j=1}^{r} \lambda_j(A)\,(u,u)_{R_j} + \sum_{j=1}^{r}\sum_{k=2}^{m_j} (u, v_k^{(j)*})\overline{(u, v_{k-1}^{(j)*})},\ u \in \mathbb{C}^n \tag{3.1}$$

leading to

$$(Au, u)_R = \sum_{j=1}^{r} \lambda_j(A)\,(u,u)_{R_j},\ u \in N_{\sigma(A)} \tag{3.2}$$

and thus to

$$Re(Au, u)_R = \sum_{j=1}^{r} Re\lambda_j(A)\,(u,u)_{R_j},\ u \in N_{\sigma(A)} \tag{3.3}$$

where

$$R_j=\sum_{k=1}^{m_j}R_j^{(k,k)}=\sum_{k=1}^{m_j}u_k^{(j)^*}u_k^{(j)}=\sum_{k=1}^{m_j}v_k^{(j)^*}v_k^{(j)},$$

$j=1,\ldots,r.$
If matrix A is, beyond this, asymptotically stable, i.e., if

$$Re\,\lambda_j(A)\,<0,\; j=1,\cdots,r,$$

then

$$Re\,(A\,u,u)_R=-\sum_{j=1}^{r}|Re\,\lambda_j(A)|\,(u,u)_{R_j},\; u\in N_{\sigma(A)},$$

so that, in this case,

$$Re\,(A\,u,u)_R<0,\; 0\neq u\in N_{\sigma(A)} \tag{3.4}$$

and

$$|Re\,(A\,u,u)_R|=\sum_{j=1}^{r}|Re\,\lambda_j(A)|\,(u,u)_{R_j},\; u\in N_{\sigma(A)}. \tag{3.5}$$

Proof. First, we prove (3.1). For this, let $u\in C^n$. Then with the denotations of Theorem 2.1,

$$u=\sum_{j=1}^{r}\sum_{k=1}^{m_j}(u,v_k^{(j)^*})\,p_k^{(j)}$$

leading to

$$\begin{aligned}Au &= \sum_{j=1}^{r}\sum_{k=1}^{m_j}(u,v_k^{(j)^*})Ap_k^{(j)}\\ &= \sum_{j=1}^{r}\sum_{k=1}^{m_j}(u,v_k^{(j)^*})\,[\lambda_j p_k^{(j)}+p_{k-1}^{(j)}]\\ &= \sum_{j=1}^{r}\lambda_j\sum_{k=1}^{m_j}(u,v_k^{(j)^*})\,p_k^{(j)}+\sum_{j=1}^{r}\sum_{k=2}^{m_j}(u,v_k^{(j)^*})\,p_{k-1}^{(j)}\end{aligned}$$

since $p_0^{(j)}=0,\; j=1,\ldots,r.$ This implies

$$\begin{aligned}(Au,Ru) &= \sum_{j=1}^{r}\lambda_j\sum_{k=1}^{m_j}(u,v_k^{(j)^*})\,(p_k^{(j)},Ru)+\sum_{j=1}^{r}\sum_{k=2}^{m_j}(u,v_k^{(j)^*})\,(p_{k-1}^{(j)},Ru)\\ &= \sum_{j=1}^{r}\lambda_j\sum_{k=1}^{m_j}(u,v_k^{(j)^*})\,(Rp_k^{(j)},u)+\sum_{j=1}^{r}\sum_{k=2}^{m_j}(u,v_k^{(j)^*})\,(Rp_{k-1}^{(j)},u).\end{aligned}$$

Now,

$$\begin{aligned}Rp_{k-1}^{(j)} &= \sum_{l=1}^{r}\sum_{s=1}^{m_l}R_l^{(s,s)}p_{k-1}^{(j)}=\sum_{l=1}^{r}\sum_{s=1}^{m_l}v_s^{(l)^*}v_s^{(l)}p_{k-1}^{(j)}\\ &= \sum_{l=1}^{r}\sum_{s=1}^{m_l}v_s^{(l)^*}\,(p_{k-1}^{(j)},v_s^{(l)^*})\\ &= \sum_{l=1}^{r}\sum_{s=1}^{m_l}v_s^{(l)^*}\,\delta_{jl}\,\delta_{s,k-1}=\sum_{s=1}^{m_j}v_s^{(j)^*}\,\delta_{s,k-1}=v_{k-1}^{(j)^*}.\end{aligned}$$

This leads to

$$\begin{aligned}(Au,Ru) &= \sum_{j=1}^{r}\lambda_j\sum_{k=1}^{m_j}(u,v_k^{(j)^*})\,(p_k^{(j)},Ru)+\sum_{j=1}^{r}\sum_{k=2}^{m_j}(u,v_k^{(j)^*})\,(v_{k-1}^{(j)^*},u)\\ &= \sum_{j=1}^{r}\lambda_j\sum_{k=1}^{m_j}(u,v_k^{(j)^*})\,(p_k^{(j)},Ru)+\sum_{j=1}^{r}\sum_{k=2}^{m_j}(u,v_k^{(j)^*})\,\overline{(u,v_{k-1}^{(j)^*})}.\end{aligned}$$

Further,

$$(p_{k-1}^{(j)}, Ru) = (Rp_{k-1}^{(j)}, u)$$

and, as before,

$$Rp_{k-1}^{(j)} = v_{k-1}^{(j)\,*} = R_j p_{k-1}^{(j)}.$$

Therefore,

$$(p_k^{(j)}, Ru) = (Rp_k^{(j)}, u) = (v_k^{(j)^*}, u)$$

and thus

$$\begin{aligned}\sum_{j=1}^{r} \lambda_j \sum_{k=1}^{m_j} (u, v_k^{(j)^*})\,(p_k^{(j)}, Ru) &= \sum_{j=1}^{r} \lambda_j \sum_{k=1}^{m_j} (u, v_k^{(j)^*})\,(v_k^{(j)^*}, u) \\ &= \sum_{j=1}^{r} \lambda_j \sum_{k=1}^{m_j} |(u, v_k^{(j)^*})| = \sum_{j=1}^{r} \lambda_j (u,u)_{R_j}\end{aligned}$$

since

$$(u,u)_{R_j} = (R_j u, u) = \sum_{k=1}^{m_j} (R_j^{(k,k)} u, u) = \sum_{k=1}^{m_j} (v_k^{(j)^*} v_k u, u) = \sum_{k=1}^{m_j} (v_k^{(j)} u, v_k^{(j)} u) = \sum_{k=1}^{m_j} |(u, v_k^{(j)^*})|.$$

So, we obtain (3.1).
In order to get (3.2), i.e.,

$$(Au, Ru) = \sum_{j=1}^{r} \lambda_j(A)(u,u)_{R_j},$$

we must have

$$\sum_{j=1}^{r} \sum_{k=2}^{m_j} (u, v_k^{(j)^*}) \overline{(u, v_{k-1}^{(j)\,*})} = 0.$$

Sufficient for this is

$$(u, v_k^{(j)^*}) = 0, \quad k = 2, \ldots, m_j,\; j = 1, \ldots, r,$$

for example,

$$u \in \bigoplus_{j=1}^{r} [p_1^{(j)}] = \bigoplus_{j=1}^{r} N_{\lambda_j(A)} = N_{\sigma(A)}.$$

The rest of the proof is clear. □

Remark 3.2. *We have shown that*

$$N_{\sigma(A)} \subset \{u \in C^n \mid \sum_{j=1}^{r} \sum_{k=2}^{m_j} (u, v_k^{(j)^*}) \overline{(u, v_{k-1}^{(j)\,*})} = 0\} =: N.$$

But, the set on the right-hand side can be larger than the set $N_{\sigma(A)}$. For, if $m_s > 1$ for some $s \in \{1, \ldots, r\}$, then

$$p_{m_s}^{(s)} \in N,$$

even though $p_{m_s}^{(s)} \notin N_{\sigma(A)}$. Moreover, we have even for all single $p_j^{(k)}$ $j = 1, \cdots, m_k, k = 1, \cdots, r$ the relations

$$p_j^{(k)} \in N.$$

Remark 3.3. *If condition (C4′) is not fulfilled, then the results of Lemma 3.1 remain valid if its formulation is adapted to [14, Theorem 4]. The details are left to the reader.*

Next, we have the following lemma.

Lemma 3.4. *Let the conditions (C1′)-(C4′) be fulfilled and R be defined by (2.1). Further, let matrix A be asymptotically stable. Then, $A^*R + RA$ is negative definite on $N_{\sigma(A)}$.*

Proof. With Lemma 3.1, Formula (3.3), and $\rho_j = 2Re\,\lambda_j(A),\ j = 1,\cdots,n$, we obtain

$$\begin{aligned}
(-[A^*R+RA]u,u) &= \sum_{j=1}^{n}(-\rho_j)\,(u,u)_{R_j} = 2\sum_{j=1}^{r} Re\,(-\lambda_j(A))\,(u,u)_{R_j} \\
&= 2\sum_{j=1}^{r} |Re\,\lambda_j(A)|\,(u,u)_{R_j} \\
&\geq 2\min_{j=1,\cdots,r}|Re\,\lambda_j(A)|\sum_{j=1}^{n}(u,u)_{R_j} \\
&= c_0(Ru,u) > 0,\ 0 \neq u \in N_{\sigma(A)}
\end{aligned}$$

with $c_0 = 2\min_{j=1,\cdots,r}|Re\,\lambda_j(A)| > 0.$ □

Similarly as in [2, Formula (3.6)], we define the following vector spaces.

$$\begin{aligned}
&M_{1,N_{\sigma(A)}} := N_{\sigma(A)}, \\
&M_{k,N_{\sigma(A)}} := \{u \in N_{\sigma(A)}\,|\,(u,u)_{R_i} = 0,\ i = 1,2,\cdots,k-1\},\ k = 2,\cdots,r.
\end{aligned} \tag{3.6}$$

The next lemma characterizes these spaces.

Lemma 3.5. *Let the conditions (C1′)-(C4′) be fulfilled.*
Then,

$$M_{k,N_{\sigma(A)}} = [p_k,\ldots,p_r],\ k = 1,2,\ldots,r. \tag{3.7}$$

Proof. The proof is done for $k = 3$. The general case can be made by induction. So, we have to prove

$$M_{3,N_{\sigma(A)}} = \{u \in N_{\sigma(A)}\,|\,(u,u)_{R_1} = 0,\ (u,u)_{R_2} = 0\} = [p_3,\ldots,p_r].$$

(i) $\underline{[p_3,p_4,\cdots,p_r] \subset M_{3,N_{\sigma(A)}}}$:

Let $u \in [p_3,p_4,\cdots,p_r]$. Then, $u = \sum_{j=3}^{r}\beta_j p_j$ with elements $\beta_j \in C$, $j = 3,\cdots,r$. Let $s \in \{1,2\}$. This entails, due to Theorem 2.3, Lemma 3.1, and $(p_j,p_k)_{R_s} = (p_j,p_k)_{R_s^{(1,1)}}$,

$$\begin{aligned}
(u,u)_{R_s} &= \sum_{j,k=3}^{r}\beta_j\overline{\beta_k}(p_j,p_k)_{R_s} = \sum_{j,k=3}^{r}\beta_j\overline{\beta_k}(p_j,p_k)_{R_s^{(1,1)}} \\
&= \sum_{j,k=3}^{r}\beta_j\overline{\beta_k}(R_s^{(1,1)}p_1^{(j)},p_1^{(k)}) = \sum_{j,k=3}^{r}\beta_j\overline{\beta_k}(v_1^{(s)^*}v_1^{(s)}p_1^{(j)},p_1^{(k)}) \\
&= \sum_{j,k=3}^{r}\beta_j\overline{\beta_k}(v_1^{(s)^*}\underbrace{(p_1^{(j)},v_1^{(s)^*})}_{\delta_{js}=0},p_1^{(k)}) = 0,
\end{aligned}$$

$j \in \{3,\ldots,r\}$, $s \in \{1,2\}$. Therefore, $(u,u)_{R_s} = 0$, $s = 1,2$ and thus $u \in M_{3,N_{\sigma(A)}}$ so that $[p_3,p_4,\cdots,p_r] \subset M_{3,N_{\sigma(A)}}$ is proven.

(ii) $\underline{M_{3,N_{\sigma(A)}} \subset [p_3,p_4,\cdots,p_r]}$:

Let $u \in M_{3,N_{\sigma(A)}}$. This implies $u \in N_{\sigma(A)}$ with $(u,u)_{R_s} = 0$, $s = 1,2$ or

$$\begin{aligned}
(R_s u,u) &= \textstyle\sum_{l=1}^{m_s}(R_s^{(l,l)}u,u) = \sum_{l=1}^{m_s}(v_l^{(s)^*}v_l^{(s)}u,u) = \sum_{l=1}^{m_s}(v_l^{(s)^*}(u,v_l^{(s)^*}),u) \\
&= \textstyle\sum_{l=1}^{m_s}(v_l^{(s)^*},u)\overline{(v_l^{(s)^*},u)} = \sum_{l=1}^{m_s}|(v_l^{(s)^*},u)|^2 = 0,\ s = 1,2,
\end{aligned}$$

that is, in particular,

$$u \in N_{\sigma(A)} \quad \text{with} \quad (u,v_1^{(s)^*}) = (u,v_s^*) = 0,\ s = 1,2. \tag{3.8}$$

Since $u \in N_{\sigma(A)}$, we have,

$$u = \sum_{j=1}^{r}(u,v_j^*)p_j$$

so that with (3.8),

$$u = \sum_{j=1}^{r}(u,v_j^*)p_j = \sum_{j=3}^{r}(u,v_j^*)p_j \in [p_3,\ldots,p_r].$$

This completes the proof of the assertion. □

Similarly to [2], we suppose that the eigenvalues $\lambda_1(A),\cdots,\lambda_r(A)$ of matrix A are arranged such that

$$Re\,\lambda_1(A)\geq Re\,\lambda_2(A)\geq\cdots\geq Re\,\lambda_r(A). \tag{3.9}$$

If A is asymptotically stable, (3.9) is replaced by

$$|Re\,\lambda_1(A)|\geq|Re\,\lambda_2(A)|\geq\cdots\geq|Re\,\lambda_r(A)|. \tag{3.10}$$

One has the following theorem.

Theorem 3.6. *Let the conditions (C1′)-(C4′) be fulfilled. Further, let the eigenvalues of A be arranged according to (3.9). Moreover, let the vector spaces $M_{k,N_{\sigma(A)}}$, $k=1,\cdots,r$ be defined by (3.6) or (3.7).*
Then,

$$Re\,\lambda_k(A)=\max_{0\neq u\in M_{k,N_{\sigma(A)}}}\frac{Re(Au,u)_R}{(u,u)_R},\ k=1,2,\cdots,r. \tag{3.11}$$

If matrix A is, beyond this, asymptotically stable, and if the eigenvalues are arranged according to (3.10), then also

$$|Re\,\lambda_k(A)|=\max_{0\neq u\in M_{k,N_{\sigma(A)}}}\frac{|Re(Au,u)_R|}{(u,u)_R},\ k=1,2,\cdots,r. \tag{3.12}$$

The maximum is attained for $u=p_k$.

Proof. According to (3.3), one has

$$Re(Au,u)_R=\sum_{j=1}^{r}Re\,\lambda_j(A)(R_ju,u),\ u\in N_{\sigma(A)}.$$

Choosing $k\in\{1,\cdots,r\}$ fixed and $u\in M_{k,N_{\sigma(A)}}$, using (3.6), one obtains

$$\begin{aligned}Re(Au,u)_R &= \sum_{j=k}^{r}Re\,\lambda_j(A)(R_ju,u)\leq\max_{j=k,\cdots,r}Re\,\lambda_j(A)\sum_{j=k}^{r}(R_ju,u)\\ &= Re\,\lambda_k(A)\sum_{j=1}^{r}(R_ju,u)=Re\,\lambda_k(A)\,(u,u)_R,\end{aligned}$$

that is,

$$\frac{Re(Au,u)_R}{(u,u)_R}\leq Re\,\lambda_k(A),\ 0\neq u\in M_{k,N_{\sigma(A)}}$$

and thus

$$\max_{0\neq u\in M_{k,N_{\sigma(A)}}}\frac{Re(Au,u)_R}{(u,u)_R}\leq Re\,\lambda_k(A).$$

Now, the maximum is attained for $u=p_k\in M_{k,N_{\sigma(A)}}$, that is,

$$Re\,\lambda_k(A)=\frac{Re(Ap_k,p_k)_R}{(p_k,p_k)_R}\leq\max_{0\neq u\in M_{k,N_{\sigma(A)}}}\frac{Re(Au,u)_R}{(u,u)_R}\leq Re\,\lambda_k(A)$$

so that the assertion (3.11) is proven.
Relation (3.12) is proven in the same way as (3.11), but based on (3.5) instead of (3.3) and (3.10) instead of (3.9). □

Remark 3.7. *As we have seen, the proof is similar to that of [2, Theorem 3.4]. The essential difference is that the full space $\mathbb{C}^n$ is replaced by the geometric eigenspace $N_{N_{\sigma(A)}}\subset\mathbb{C}^n$. Therefore, in the sequel, we state the results without proofs.*

Next, we want to state a min-max characterization for the real parts of eigenvalues similar to results for diagonalizable matrices in [2, Theorem 3.5].
One has the following theorem.

Theorem 3.8. *Let the conditions (C1′)-(C4′) be fulfilled. Further, let the eigenvalues of A be arranged according to (3.9).*
Then, for every $j=1,\cdots,r$ and every subspace $M\subset N_{\sigma(A)}$ with $\dim M=m=r+1-j$, the following inequalities are valid:

$$Re\,\lambda_j(A)\leq\max_{0\neq u\in M}\frac{Re(Au,u)_R}{(u,u)_R}\leq Re\,\lambda_1(A), \tag{3.13}$$

and the following representation formulas hold:

$$Re\,\lambda_j(A)=\min_{\dim M=m}\max_{0\neq u\in M\subset N_{\sigma(A)}}\frac{Re(Au,u)_R}{(u,u)_R}.$$

If matrix A is, beyond this, asymptotically stable and the eigenvalues are arranged according to (3.10), then also

$$|Re\,\lambda_j(A)|=\min_{\dim M=m}\max_{0\neq u\in M\subset N_{\sigma(A)}}\frac{|Re(Au,u)_R|}{(u,u)_R}.$$

Remark 3.9. *From* (3.13)*, it follows*

$$\frac{Re\,(Au,u)_R}{(u,u)_R} \le \nu[A] = \max_{j=1,\cdots,r} Re\,\lambda_j(A),\ 0 \ne u \in N_{\sigma(A)}.$$

For the following theorem, we need the vector spaces N_k defined by

$$N_k := [p_1, p_2, \cdots, p_k],\ k = 1,2,\cdots,r. \tag{3.14}$$

Then, we have a result similar to that of Theorem 3.6.

Theorem 3.10. *Let the conditions (C1′)-(C4′) be fulfilled. Further, let the eigenvalues of A be arranged according to* (3.9)*. Moreover, let the vector spaces N_k, $k = 1,\cdots,r$ be defined by* (3.14)*.*
Then,

$$Re\,\lambda_k(A) = \min_{0\ne u\in N_k} \frac{Re(Au,u)_R}{(u,u)_R},\ k = 1,2,\cdots,r.$$

If matrix A is, beyond this, asymptotically stable and if the eigenvalues are arranged according to (3.10)*, then also*

$$|Re\,\lambda_k(A)| = \min_{0\ne u\in N_k} \frac{|Re(Au,u)_R|}{(u,u)_R},\ k = 1,2,\cdots,r.$$

The minimum is attained for $u = p_k$.

Next, we want to derive a max-min characterization for the real parts of eigenvalues similar to results for diagonalizable matrices in [2]. One has the following theorem.

Theorem 3.11. *Let the conditions (C1′)-(C4′) be fulfilled. Further, let the eigenvalues of A be arranged according to* (3.9)*.*
Then, for every $j = 1,\cdots,r$ and every subspace $N \subset N_{\sigma(A)}$ with $\dim N = j$, the following inequalities are valid:

$$Re\,\lambda_r(A) \le \min_{0\ne u\in N} \frac{Re(Au,u)_R}{(u,u)_R} \le Re\,\lambda_j(A), \tag{3.15}$$

and the following representation formulas hold:

$$Re\,\lambda_j(A) = \max_{dimN=j}\ \min_{0\ne u\in N} \frac{Re(Au,u)_R}{(u,u)_R}.$$

If matrix A is, beyond this, asymptotically stable and the eigenvalues are arranged according to (3.10)*, then also*

$$|Re\,\lambda_j(A)| = \max_{dimN=j}\ \min_{0\ne u\in N} \frac{|Re(Au,u)_R|}{(u,u)_R}.$$

Remark 3.12. *From* (3.15)*, it follows*

$$\frac{Re\,(Au,u)_R}{(u,u)_R} \ge -\nu[-A] = \min_{j=1,\cdots,r} Re\,\lambda_j(A),\ 0 \ne u \in N_{\sigma(A)}.$$

4. Formulas for the representation of the imaginary parts of the eigenvalues of a general matrix

In this section, we want to state formulas for the representation of the imaginary parts of the eigenvalues of a general matrix A by Rayleigh quotients. More precisely, max-, min-, min-max-, and max-min-representations are obtained corresponding to associated formulas for the eigenvalues of self-adjoint matrices in the textbook [1, Section 5.4] resp. corresponding to those for the imaginary parts of eigenvalues of diagonalizable matrices in [2].
First, we state a result similar to that of [1, Section 5.4 (18)]. This is done in the following Formula (4.1).

Lemma 4.1. *Let the conditions (C1′)-(C4′) be fulfilled and R be defined by (2.1).*
Then, with the denotations of Theorem 2.3,

$$Im(Au,u)_R = \sum_{j=1}^{r} Im\lambda_j(A)\,(u,u)_{R_j},\ u \in N_{\sigma(A)}. \tag{4.1}$$

Proof. The proof follows immediately from (3.2). □

Similarly to [1, Section 5.4 (22)] or (3.9), we suppose that the eigenvalues $\lambda_1(A),\cdots,\lambda_r(A)$ of matrix A are arranged such that

$$Im\,\lambda_1(A) \ge Im\,\lambda_2(A) \ge \cdots \ge Im\,\lambda_r(A). \tag{4.2}$$

One has the following theorem.

Theorem 4.2. *Let the conditions (C1′)-(C4′) be fulfilled. Further, let the eigenvalues of A be arranged according to* (4.2)*. Moreover, let the vector spaces $M_{k,N_{\sigma(A)}}$, $k = 1,\cdots,r$ be defined by* (3.6) *or* (3.7)*.*
Then,

$$Im\,\lambda_k(A) = \max_{0\ne u\in M_{k,N_{\sigma(A)}}} \frac{Im(Au,u)_R}{(u,u)_R},\ k = 1,2,\cdots,r.$$

The maximum is attained for $u = p_k$.

Next, we want to state a min-max characterization for the imaginary parts of eigenvalues that corresponds to results for diagonalizable matrices in [2] or that corresponds to results for the real parts of eigenvalues of general matrices in Section 3.
One has the following theorem.

Theorem 4.3. *Let the conditions (C1′)-(C4′) be fulfilled. Further, let the eigenvalues of A be arranged according to* (4.2).
Then, for every $j = 1,\cdots,r$ *and every subspace* $M \subset N_{\sigma(A)}$ *with* $dim M = m = r+1-j$*, the following inequalities are valid:*

$$Im\,\lambda_j(A) \le \max_{0\neq u\in M} \frac{Im(Au,u)_R}{(u,u)_R} \le Im\,\lambda_1(A), \tag{4.3}$$

and the following representation formulas hold:

$$Im\,\lambda_j(A) = \min_{dimM=m}\max_{0\neq u\in M\subset N_{\sigma(A)}} \frac{Im(Au,u)_R}{(u,u)_R}.$$

Remark 4.4. *From* (4.3)*, it follows*

$$\frac{Im\,(Au,u)_R}{(u,u)_R} \le \max_{j=1,\cdots,r} Im\,\lambda_j(A),\; 0\neq u\in N_{\sigma(A)}.$$

With the vector spaces N_k, we have the following theorem.

Theorem 4.5. *Let the conditions (C1′)-(C4′) be fulfilled. Further, let the eigenvalues of A be arranged according to* (4.2)*. Moreover, let the vector spaces* N_k, $k = 1,\cdots,r$ *be defined by* (3.14)*.*
Then,

$$Im\,\lambda_k(A) = \min_{0\neq u\in N_k} \frac{Im(Au,u)_R}{(u,u)_R},\; k = 1,2,\cdots,r.$$

The minimum is attained for $u = p_k$*.*

Next, we state a max-min characterization for the imaginary parts of eigenvalues for general matrices similar to results for the real parts in Section 3.
One has the following theorem.

Theorem 4.6. *Let the conditions (C1′)-(C4′) be fulfilled. Further, let the eigenvalues of A be arranged according to* (4.2)*.*
Then, for every $j = 1,\cdots,r$ *and every subspace* $N \subset N_{\sigma(A)}$ *with* $dim N = j$*, the following inequalities are valid:*

$$Im\,\lambda_r(A) \le \min_{0\neq u\in N} \frac{Im(Au,u)_R}{(u,u)_R} \le Im\,\lambda_j(A), \tag{4.4}$$

and the following representation formulas hold:

$$Im\,\lambda_j(A) = \max_{dimN=j}\min_{0\neq u\in N} \frac{Im(Au,u)_R}{(u,u)_R}.$$

Remark 4.7. *From* (4.4)*, it follows*

$$\frac{Im\,(Au,u)_R}{(u,u)_R} \ge \min_{j=1,\cdots,r} Im\,\lambda_j(A),\; 0\neq u\in N_{\sigma(A)}.$$

5. Formulas for the representation of the moduli of the eigenvalues of a general matrix

In this section, we want to derive formulas for the representation of the moduli of the eigenvalues of a general matrix A by Rayleigh quotients. More precisely, max-, min-, min-max-, and max-min-representations are obtained corresponding to associated formulas for the eigenvalues of diagonalizable matrices in [2] and for the real and imaginary parts of eigenvalues of general matrices in Sections 3 and 4.
First, we derive a result similar to that of [2, Lemma 5.1].

Lemma 5.1. *Let the conditions (C1′)-(C4′) be fulfilled and R be defined by* (2.1)*.*
Then, with the denotations of Theorem 2.1,

$$\begin{aligned} \|Au\|_R^2 &= (RAu,Au) = (A^*RAu,u) = (R^{-1}A^*RAu,u)_R \\ &= \sum_{j=1}^{r} |\lambda_j(A)|^2\,(u,u)_{R_j} \\ &+ \sum_{j=1}^{r} \lambda_j(A) \sum_{k=1}^{m_j-1} (u,v_k^{(j)*})\overline{(u,v_{k+1}^{(j)*})} \\ &+ \sum_{j=1}^{r} \overline{\lambda_j(A)} \sum_{k=2}^{m_j} (u,v_k^{(j)*})\overline{(u,v_{k-1}^{(j)*})} \\ &+ \sum_{j=1}^{r}\sum_{k=2}^{m_j} |(u,v_k^{(j)*})|^2,\; u\in \mathbb{C}^n \end{aligned} \tag{5.1}$$

leading to

$$\begin{aligned} \|Au\|_R^2 &= (RAu, Au) = (A^*RAu, u) = (R^{-1}A^*RAu, u)_R \\ &= \sum_{j=1}^{r} |\lambda_j(A)|^2 \, (u,u)_{R_j}, \; u \in N_{\sigma(A)} \end{aligned} \tag{5.2}$$

where

$$(u,u)_{R_j} = (u,u)_{R_j^{(1,1)}}, \; u \in N_{\sigma(A)},$$

$j = 1, \dots, r.$

Proof. First, we prove (5.1). For this, let $u \in C^n$. Then with the denotations of Theorem 2.1,

$$u = \sum_{j=1}^{r} \sum_{k=1}^{m_j} (u, v_k^{(j)^*}) \, p_k^{(j)} \tag{5.3}$$

leading to

$$\begin{aligned} Au &= \sum_{j=1}^{r} \sum_{k=1}^{m_j} (u, v_k^{(j)^*}) \, A p_k^{(j)} \\ &= \sum_{j=1}^{r} \sum_{k=1}^{m_j} (u, v_k^{(j)^*}) \, [\lambda_j p_k^{(j)} + p_{k-1}^{(j)}] \\ &= \sum_{j=1}^{r} \lambda_j \sum_{k=1}^{m_j} (u, v_k^{(j)^*}) \, p_k^{(j)} + \sum_{j=1}^{r} \sum_{k=2}^{m_j} (u, v_k^{(j)^*}) \, p_{k-1}^{(j)} \end{aligned}$$

since $p_0^{(j)} = 0$, $j = 1, \dots, r$. This implies

$$RAu = \sum_{j=1}^{r} \lambda_j \sum_{k=1}^{m_j} (u, v_k^{(j)^*}) \, R p_k^{(j)} + \sum_{j=1}^{r} \sum_{k=2}^{m_j} (u, v_k^{(j)^*}) \, R p_{k-1}^{(j)}.$$

Now,

$$\begin{aligned} R p_k^{(j)} &= \sum_{l=1}^{r} \sum_{s=1}^{m_l} R_l^{(s,s)} p_k^{(j)} = \sum_{l=1}^{r} \sum_{s=1}^{m_l} v_s^{(l)^*} v_s^{(l)} p_k^{(j)} \\ &= \sum_{l=1}^{r} \sum_{s=1}^{m_l} v_s^{(l)^*} \, (p_k^{(j)}, v_s^{(l)^*}) \\ &= \sum_{l=1}^{r} \sum_{s=1}^{m_l} v_s^{(l)^*} \, \delta_{jl} \, \delta_{s,k} = \sum_{s=1}^{m_j} v_s^{(j)^*} \, \delta_{s,k} = v_k^{(j)^*} \end{aligned}$$

and correspondingly

$$R p_{k-1}^{(j)} = v_{k-1}^{(j)\,*}.$$

This leads to

$$RAu = \sum_{j=1}^{r} \lambda_j \sum_{k=1}^{m_j} (u, v_k^{(j)^*}) \, v_k^{(j)^*} + \sum_{j=1}^{r} \sum_{k=2}^{m_j} (u, v_k^{(j)^*}) \, v_{k-1}^{(j)\,*}. \tag{5.4}$$

Thus,

$$\begin{aligned} A^*RAu &= \sum_{j=1}^{r} \lambda_j \sum_{k=1}^{m_j} (u, v_k^{(j)^*}) A^* v_k^{(j)^*} + \sum_{j=1}^{r} \sum_{k=2}^{m_j} (u, v_k^{(j)^*}) A^* v_{k-1}^{(j)\,*} \\ &= \sum_{j=1}^{r} \lambda_j \sum_{k=1}^{m_j} (u, v_k^{(j)^*}) A^* u_{m_j-k+1}^{(j)\,*} + \sum_{j=1}^{r} \sum_{k=2}^{m_j} (u, v_k^{(j)^*}) A^* u_{m_j-(k-1)+1}^{(j)\,*} \\ &= \sum_{j=1}^{r} \lambda_j \sum_{k=1}^{m_j} (u, v_k^{(j)^*}) \, [\overline{\lambda_j} u_{m_j-k+1}^{(j)\,*} + u_{m_j-k}^{(j)\,*}] \\ &+ \sum_{j=1}^{r} \sum_{k=2}^{m_j} (u, v_k^{(j)^*}) \, [\overline{\lambda_j} u_{m_j-k+2}^{(j)\,*} + u_{m_j-k+1}^{(j)\,*}] \\ &= \sum_{j=1}^{r} \lambda_j \sum_{k=1}^{m_j} (u, v_k^{(j)^*}) \, [\overline{\lambda_j} v_k^{(j)^*} + v_{k+1}^{(j)\,*}] \\ &+ \sum_{j=1}^{r} \sum_{k=2}^{m_j} (u, v_k^{(j)^*}) \, [\overline{\lambda_j} v_{k-1}^{(j)\,*} + v_k^{(j)^*}] \end{aligned}$$

and thus

$$\begin{aligned} A^*RAu &= \sum_{j=1}^{r}|\lambda_j|^2\sum_{k=1}^{m_j}(u,v_k^{(j)^*})\,v_k^{(j)^*}+\sum_{j=1}^{r}\lambda_j\sum_{k=1}^{m_j}(u,v_k^{(j)^*})v_{k+1}^{(j)}{}^{*}\\ &+\sum_{j=1}^{r}\overline{\lambda_j}\sum_{k=2}^{m_j}(u,v_k^{(j)^*})\,v_{k-1}^{(j)}{}^{*}+\sum_{j=1}^{r}\sum_{k=2}^{m_j}(u,v_k^{(j)^*})\,v_k^{(j)^*}. \end{aligned} \tag{5.5}$$

From (5.3) and (5.5), we conclude that the following chain of equations is valid

$$\begin{aligned} (A^*RAu,u) &= (\sum_{j=1}^{r}|\lambda_j|^2\sum_{k=1}^{m_j}(u,v_k^{(j)^*})\,v_k^{(j)^*},\sum_{l=1}^{r}\sum_{s=1}^{m_l}(u,v_s^{(l)^*})\,p_s^{(l)})\\ &+(\sum_{j=1}^{r}\lambda_j\sum_{k=1}^{m_j}(u,v_k^{(j)^*})v_{k+1}^{(j)}{}^{*},\sum_{l=1}^{r}\sum_{s=1}^{m_l}(u,v_s^{(l)^*})\,p_s^{(l)})\\ &+(\sum_{j=1}^{r}\overline{\lambda_j}\sum_{k=2}^{m_j}(u,v_k^{(j)^*})\,v_{k-1}^{(j)}{}^{*},\sum_{l=1}^{r}\sum_{s=1}^{m_l}(u,v_s^{(l)^*})\,p_s^{(l)})\\ &+(\sum_{j=1}^{r}\sum_{k=2}^{m_j}(u,v_k^{(j)^*})\,v_k^{(j)^*},\sum_{l=1}^{r}\sum_{s=1}^{m_l}(u,v_s^{(l)^*})\,p_s^{(l)})\\ &= \sum_{j=1}^{r}|\lambda_j|^2\sum_{k=1}^{m_j}(u,v_k^{(j)^*})\sum_{l=1}^{r}\sum_{s=1}^{m_l}\overline{(u,v_s^{(l)^*})}\,\underbrace{(v_k^{(j)^*},p_s^{(l)})}_{\delta_{lj}\delta_{sk}}\\ &+\sum_{j=1}^{r}\lambda_j\sum_{k=1}^{m_j}(u,v_k^{(j)^*})\sum_{l=1}^{r}\sum_{s=1}^{m_l}\overline{(u,v_s^{(l)^*})}\,\underbrace{(v_{k+1}^{(j)}{}^{*},p_s^{(l)})}_{\delta_{lj}\delta_{s,k+1}}\\ &+\sum_{j=1}^{r}\overline{\lambda_j}\sum_{k=2}^{m_j}(u,v_k^{(j)^*})\sum_{l=1}^{r}\sum_{s=1}^{m_l}\overline{(u,v_s^{(l)^*})}\,\underbrace{(v_{k-1}^{(j)}{}^{*},p_s^{(l)})}_{\delta_{lj}\delta_{s,k-1}}\\ &+\sum_{j=1}^{r}\sum_{k=2}^{m_j}(u,v_k^{(j)^*})\sum_{l=1}^{r}\sum_{s=1}^{m_l}\overline{(u,v_s^{(l)^*})}\,\underbrace{(v_k^{(j)^*},p_s^{(l)})}_{\delta_{lj}\delta_{s,k}} \end{aligned}$$

so that

$$\begin{aligned} (A^*RAu,u) &= \sum_{j=1}^{r}|\lambda_j|^2\sum_{k=1}^{m_j}(u,v_k^{(j)^*})\,\overline{(u,v_k^{(j)^*})}\\ &+\sum_{j=1}^{r}\lambda_j\sum_{k=1}^{m_j}(u,v_k^{(j)^*})\overline{(u,v_{k+1}^{(j)}{}^{*})}\\ &+\sum_{j=1}^{r}\overline{\lambda_j}\sum_{k=2}^{m_j}(u,v_k^{(j)^*})\,\overline{(u,v_{k-1}^{(j)}{}^{*})}\\ &+\sum_{j=1}^{r}\sum_{k=2}^{m_j}(u,v_k^{(j)^*})\,\overline{(u,v_k^{(j)^*})}. \end{aligned} \tag{5.6}$$

Now,

$$\begin{aligned} \sum_{k=1}^{m_j}(u,v_k^{(j)^*})\,\overline{(u,v_k^{(j)^*})} &= (u,\sum_{k=1}^{m_j}(u,v_k^{(j)^*})\,v_k^{(j)^*})=(u,\sum_{k=1}^{m_j}v_k^{(j)^*}v_k^{(j)}u)\\ &= (u,R_ju)=(u,u)_{R_j}. \end{aligned} \tag{5.7}$$

Further, for $k=m_j$,

$$v_{k+1}^{(j)}{}^{*}=u_{m_j-(k+1)+1}^{(j)}{}^{*}=v_{m_j-(m_j+1)+1}^{(j)}{}^{*}=u_0^{(j)^*}=0. \tag{5.8}$$

With (5.4)-(5.8), relation (5.1) follows. The rest is clear. □

Similarly to [2], we suppose that the eigenvalues $\lambda_1(A),\cdots,\lambda_r(A)$ of matrix A are arranged such that

$$|\lambda_1(A)|\geq|\lambda_2(A)|\geq\cdots\geq|\lambda_r(A)|. \tag{5.9}$$

One has the following theorem.

Theorem 5.2. *Let the conditions (C1′)-(C4′) be fulfilled. Further, let the eigenvalues of A be arranged according to (5.9). Moreover, let the vector spaces $M_{k,N_{\sigma(A)}}$, $k=1,\cdots,r$ be defined by (3.6) or (3.7).*
Then,

$$|\lambda_k(A)|=\max_{0\neq u\in M_{k,N_{\sigma(A)}}}\frac{\|Au\|_R}{\|u\|_R},\ k=1,2,\cdots,r.$$

The maximum is attained for $u=p_k$.

Proof. For the proof, one uses (5.2) and proceeds as in the proof of [2] with the only difference that the full space $\mathbb{C}^n$ is replaced by the subspace $N_{\sigma(A)}$. □

In the same way, one obtains the following results.

Theorem 5.3. *Let the conditions (C1′)-(C4′) be fulfilled. Further, let the eigenvalues of A be arranged according to (5.9). Then, for every $j = 1,\cdots,r$ and every subspace $M \subset N_{\sigma(A)}$ with $\dim M = m = r+1-j$, the following inequalities are valid:*

$$|\lambda_j(A)| \leq \max_{0\neq u\in M} \frac{\|Au\|_R}{\|u\|_R} \leq |\lambda_1(A)|, \tag{5.10}$$

and the following representation formulas hold:

$$|\lambda_j(A)| = \min_{\dim M=m} \max_{0\neq u\in M\subset N_{\sigma(A)}} \frac{\|Au\|_R}{\|u\|_R}.$$

Remark 5.4. *From (5.9), it follows*

$$\frac{\|Au\|_R}{\|u\|_R} \leq \max_{j=1,\cdots,r} |\lambda_j(A)| = \rho(A),\ 0\neq u\in N_{\sigma(A)},$$

where $\rho(A)$ is the spectral radius *of matrix A.*

Further, we have the following theorem.

Theorem 5.5. *Let the conditions (C1′)-(C4′) be fulfilled. Further, let the eigenvalues of A be arranged according to (5.9). Moreover, let the vector spaces N_k, $k = 1,\cdots,r$ be defined by (3.14). Then,*

$$|\lambda_k(A)| = \min_{0\neq u\in N_k} \frac{\|Au\|_R}{\|u\|_R},\ k = 1,2,\cdots,r.$$

The minimum is attained for $u = p_k$.

Moreover, the following theorem holds.

Theorem 5.6. *Let the conditions (C1′)-(C4′) be fulfilled. Further, let the eigenvalues of A be arranged according to (5.9). Then, for every $j = 1,\cdots,r$ and every subspace $N \subset N_{\sigma(A)}$ with $\dim N = j$, the following inequalities are valid:*

$$|\lambda_r(A)| \leq \min_{0\neq u\in N} \frac{\|Au\|_R}{\|u\|_R} \leq |\lambda_j(A)|, \tag{5.11}$$

and the following representation formulas hold:

$$|\lambda_j(A)| = \max_{\dim N=j} \min_{0\neq u\in N} \frac{\|Au\|_R}{\|u\|_R}.$$

Remark 5.7. *From (5.11), it follows*

$$\begin{aligned}\frac{\|Au\|_R}{\|u\|_R} \geq |\lambda_r(A)| &= \min_{j=1,\cdots,r} |\lambda_j(A)| = \min_{j=1,\cdots,r} \frac{1}{|\lambda_j(A^{-1})|}\\ &= \frac{1}{\max\limits_{j=1,\cdots,r} |\lambda_j(A^{-1})|} = \frac{1}{\rho(A^{-1})} = (\rho(A^{-1}))^{-1},\ 0\neq u\in N_{\sigma(A)}.\end{aligned}$$

6. Connection between the matrices $\mathbf{R^{-1}\frac{A^*R+RA}{2}}$, $\mathbf{R^{-1}\frac{RA-A^*R}{2i}}$, and $\mathbf{R^{-1}A^*RA}$

In [2, Section 6], for diagonalizable matrices $A \in \mathbb{C}^{n\times n}$, we have shown that the equation

$$\left(R^{-1}\tfrac{A^*R+RA}{2}\right)^2 + \left(R^{-1}\tfrac{RA-A^*R}{2i}\right)^2 = R^{-1}A^*RA \tag{6.1}$$

is valid. In this section, we prove that this equation remains valid in the subspace

$$N'_{\sigma(A)} := \bigoplus_{\substack{m_j=1\\ j=1}}^{r} N_{\lambda_j(A)} \subset \bigoplus_{j=1}^{r} N_{\lambda_j(A)} = N_{\sigma(A)}. \tag{6.2}$$

One has the following theorem.

Theorem 6.1. *Let the conditions (C1′)-(C4′) be fulfilled, and R be defined by (2.1). Then,*

$$\left[\left(R^{-1}\tfrac{A^*R+RA}{2}\right)^2 + \left(R^{-1}\tfrac{RA-A^*R}{2i}\right)^2\right] u = R^{-1}A^*RAu,\ u\in N'_{\sigma(A)}. \tag{6.3}$$

Proof. Let $j \in \{1, \cdots, r\}$ with $m_j = 1$. According to [13, Theorems 6 and 7], one has

$$\begin{aligned}
&\left[\left(R^{-1}\frac{A^*R+RA}{2}\right)^2 + \left(R^{-1}\frac{RA-A^*R}{2i}\right)^2\right] p_j \\
&= \left(R^{-1}\frac{A^*R+RA}{2}\right)^2 p_j + \left(R^{-1}\frac{RA-A^*R}{2i}\right)^2 p_j \\
&= [Re\lambda_j(A)]^2 p_j + [Im\lambda_j(A)]^2 p_j \\
&= |\lambda_j(A)|^2 p_j = \lambda_j(R^{-1}A^*RA) p_j = R^{-1}A^*RAp_j.
\end{aligned}$$

This implies (6.3). □

Remark 6.2. *For diagonalizable matrices $A \in \mathbf{C}^{n\times n}$, the equations (6.3) deliver (6.1) since then $N'_{\sigma(A)} = N_{\sigma(A)} = \mathbf{C}^{n\times n}$.*

7. Applications

In this section, we apply the results of Sections 3, 4, and 5 to obtain the convexity of newly-defined numerical ranges of a general matrix A.

7.1. Applications pertinent to Section 3

Let the conditions $(C1')$-$(C4')$ be fulfilled.
The *numerical range of A restricted to $N_{\sigma(A)}$ with respect to the weighted scalar product $(\cdot,\cdot)_R$* is defined by

$$W_{N_{\sigma(A)},(\cdot,\cdot)_R}(A) = \left\{ z \in \mathbf{C} \mid z = \frac{(Au,u)_R}{(u,u)_R},\ 0 \neq u \in N_{\sigma(A)} \right\}.$$

Further, let

$$Re[W_{N_{\sigma(A)},(\cdot,\cdot)_R}(A)] := \left\{ x \in \mathbf{R} \,|\, x = \frac{Re(Au,u)_R}{(u,u)_R},\ 0 \neq u \in N_{\sigma(A)} \right\};$$

we call it *real part of the numerical range $W_{N_{\sigma(A)},(\cdot,\cdot)_R}(A)$*.
Let $\sigma(A) = \{\lambda_j(A),\ j = 1, \cdots, r\}$ be the *spectrum of A*, i.e., the set of all eigenvalues of A.
Similarly as before, we define

$$Re[\sigma(A)] := \{Re\lambda_j(A),\ j = 1, \cdots, r\}$$

and call it the *real part of the spectrum of A*.
Finally, let $co\{Re[\sigma(A)]\}$ be the *convex hull of $Re[\sigma(A)]$*.
Next, we show the following corollary as an application of Theorem 3.8, Formula (3.13), and Theorem 3.11, Formula (3.15).

Corollary 7.1. (Application 1)
Let the conditions $(C1')$-$(C4')$ be fulfilled.
Then, the set $Re[W_{N_{\sigma(A)},(\cdot,\cdot)_R}(A)]$ is convex, and one has the chain of equations

$$\begin{aligned}
Re[W_{N_{\sigma(A)},(\cdot,\cdot)_R}(A)] &= \left\{ x \in \mathbf{R} \,|\, x = \frac{Re(Au,u)_R}{(u,u)_R}, 0 \neq u \in N_{\sigma(A)} \right\} \\
&= \left\{ x \in \mathbf{R} \,|\, x = \frac{(R^{-1}\frac{A^*R+RA}{2}u,u)_R}{(u,u)_R}, 0 \neq u \in N_{\sigma(A)} \right\} \\
&= W_{N_{\sigma(A)},(\cdot,\cdot)_R}(R^{-1}\tfrac{A^*R+RA}{2}) \\
&= co\{Re[\sigma(A)]\}.
\end{aligned}$$

If the eigenvalues of A are arranged according to (3.9), then

$$Re[W_{N_{\sigma(A)},(\cdot,\cdot)_R}(A)] = [Re\lambda_r(A), Re\lambda_1(A)].$$

Proof. Let $0 \neq u \in N_{\sigma(A)}$. Then,

$$2\frac{Re(Au,u)_R}{(u,u)_R} = \frac{([A^*R+RA]u,u)}{(u,u)_R} = \frac{(R^{-1}[A^*R+RA]u,u)_R}{(u,u)_R}.$$

The convexity follows from the last form with $R^{-1}[A^*R+RA]$ and the scalar product $(\cdot,\cdot)_R$, see the convexity of the numerical range of a matrix due to Hausdorff in [1, Section 5.4]. Since, with (3.9), one has

$$co\{Re[\sigma(A)]\} = [Re\lambda_r(A), Re\lambda_1(A)],$$

it remains to show that

$$Re[W_{N_{\sigma(A)},(\cdot,\cdot)_R}(A)] = [Re\,\lambda_r(A), Re\,\lambda_1(A)].$$

The proof of this relation is as follows.
(i) $Re[W_{N_{\sigma(A)},(\cdot,\cdot)_R}(A)] \subset [Re\,\lambda_r(A), Re\,\lambda_1(A)]$
This inclusion can be deduced from (3.13) with $dim M = m = r - j + 1$ for $j = 1$ and (3.15) with $dim N = r$. Namely, from (3.13), for $j = 1$ and $dim M = r$, i.e., $M = N_{\sigma(A)}$, one has

$$\max_{0\neq u\in N_{\sigma(A)}} \frac{Re(Au,u)_R}{(u,u)_R} \leq Re\,\lambda_1(A)$$

and from (3.15), for $j = r$ and $dim N = r$, i.e., $N = N_{\sigma(A)}$,

$$\min_{0\neq u\in N_{\sigma(A)}} \frac{Re(Au,u)_R}{(u,u)_R} \geq Re\,\lambda_r(A).$$

(ii) $[Re\,\lambda_r(A), Re\,\lambda_1(A)] \subset Re[W_{N_{\sigma(A)},(\cdot,\cdot)_R}(A)]$
Let $\beta \in [Re\,\lambda_r(A), Re\,\lambda_1(A)]$. Then, there exists an α in $0 \leq \alpha \leq 1$ with

$$\beta = \alpha\, Re\lambda_r(A) + (1-\alpha)\, Re\lambda_1(A).$$

Now, due to Theorem 2.3, with the eigenvectors p_r and p_1,

$$Re\lambda_r(A) = \frac{Re(Ap_r,p_r)_R}{(p_r,p_r)_R} \in Re[W_{N_{\sigma(A)},(\cdot,\cdot)_R}(A)]$$

and

$$Re\lambda_1(A) = \frac{Re(Ap_1,p_1)_R}{(p_1,p_1)_R} \in Re[W_{N_{\sigma(A)},(\cdot,\cdot)_R}(A)].$$

Thus, due to the convexity of $Re[W_{N_{\sigma(A)},(\cdot,\cdot)_R}(A)]$, it follows that $\beta \in Re[W_{N_{\sigma(A)},(\cdot,\cdot)_R}(A)]$.
In other words, the proof is done along the same line as for [2, Section 7.1] with (3.13) for $j = 1$ and (3.15) for $j = r$. □

Remark 7.2. *One has the relations*

$$Re\lambda_1(A) = \max_{j=1,\cdots,r} Re\lambda_j(A) = \nu[A]$$

and

$$Re\lambda_r(A) = \min_{j=1,\cdots,r} Re\lambda_j(A) = -\nu[-A].$$

Corollary 7.3. (Application 2)
Let the conditions (C1′)-(C4′) be fulfilled. Further, let A be asymptotically stable.
Then,

$$Re[W_{N_{\sigma(A)},(\cdot,\cdot)_R}(A)] \subset \boldsymbol{R}^- = \{x \in \boldsymbol{R}\,|\,x < 0\}.$$

If A is only stable, then

$$Re[W_{N_{\sigma(A)},(\cdot,\cdot)_R}(A)] \subset \boldsymbol{R}_0^- = \{x \in \boldsymbol{R}\,|\,x \leq 0\}.$$

Proof. The first assertion follows from (3.4). The second assertion follows in a similar way. □

7.2. Applications pertinent to Section 4

In this section, we proceed in a similar way as in 7.1. So, let

$$Im[W_{N_{\sigma(A)},(\cdot,\cdot)_R}(A)] := \left\{ x \in \boldsymbol{R}\,|\,x = \frac{Im(Au,u)_R}{(u,u)_R},\ 0 \neq u \in N_{\sigma(A)} \right\};$$

we call it *imaginary part of the numerical range* $W_{N_{\sigma(A)},(\cdot,\cdot)_R}(A)$.
Further, we define

$$Im[\sigma(A)] := \{Im\,\lambda_j(A),\ j = 1,\cdots,r\}$$

and call it the *imaginary part of the spectrum of A*.
Finally, let $co\{Im[\sigma(A)]\}$ be the *convex hull of* $Im[\sigma(A)]$.

Herewith, we obtain the following corollary.

Corollary 7.4. (Application 3)
Let the conditions $(C1')$-$(C4')$ be fulfilled.
Then, the set $Im[W_{N_{\sigma(A)},(\cdot,\cdot)_R}(A)]$ is convex, and one has the chain of equations

$$\begin{aligned} Im[W_{N_{\sigma(A)},(\cdot,\cdot)_R}(A)] &= \left\{x \in \mathbf{R} \,|\, x = \frac{Im(Au,u)_R}{(u,u)_R}, 0 \neq u \in N_{\sigma(A)}\right\} \\ &= \left\{x \in \mathbf{R} \,|\, x = \frac{(R^{-1}\frac{RA-A^*R}{2i}u,u)_R}{(u,u)_R}, 0 \neq u \in N_{\sigma(A)}\right\} \\ &= W_{N_{\sigma(A)},(\cdot,\cdot)_R}(R^{-1}\tfrac{RA-A^*R}{2i}) \\ &= co\{Im[\sigma(A)]\}. \end{aligned}$$

Proof. The assertion follows as in [2, Section 7.2] along with (4.3) for $j=1$ and (4.4) for $j=r$. □

7.3. Applications pertinent to Section 5

In this subsection, we continue along the same lines as in 7.1 and 7.2.
Let

$$W_{N_{\sigma(A)},\|\cdot\|_R}(A) := \left\{x \in \mathbf{R}_0^+ \,|\, x = \frac{\|Au\|_R}{\|u\|_R},\ 0 \neq u \in N_{\sigma(A)}\right\}.$$

Further,

$$|\sigma(A)| := \{|\lambda_j(A)|,\ j = 1,\cdots,r\}$$

is called the *modulus of the spectrum of* A.
Moreover, let $co\{|\sigma(A)|\}$ be the *convex hull of* $|\sigma(A)|$.
Finally, let $S \subset \mathbf{R}_0^+$ be any subset. We define

$$S^2 := \{y \,|\, y = s^2, s \in S\}.$$

Next, we show the following corollary.

Corollary 7.5. (Application 4)
Let the conditions $(C1')$-$(C4')$ be fulfilled. Further, let the eigenvalues of A be arranged according to (5.9).
Then, the set $[W_{N_{\sigma(A)},\|\cdot\|_R}(A)]^2$ is convex, and one has the chain of relations

$$\begin{aligned} [W_{N_{\sigma(A)},\|\cdot\|_R}(A)]^2 &= \left\{x \in \mathbf{R}_0^+ \,|\, x = \left[\frac{\|Au\|_R}{\|u\|_R}\right]^2 = \frac{\|Au\|_R^2}{\|u\|_R^2}, 0 \neq u \in N_{\sigma(A)}\right\} \\ &= \left\{x \in \mathbf{R}_0^+ \,|\, x = \frac{([R^{-1}A^*RA]u,u)_R}{(u,u)_R}, 0 \neq u \in N_{\sigma(A)}\right\} \\ &= [|\lambda_r(A)|^2, |\lambda_1(A)|^2] \\ &= co\{|\sigma(A)|^2\}. \end{aligned}$$

If A is regular, then $R^{-1}A^*RA$ is positive definite.

Proof. The assertion follows as for [2, Corollary 7.4] along with (5.10) for $j=1$ and (5.11) for $j=r$. Further, $R^{-1}A^*RA$ is apparently regular if A is so as well as self-adjoint and positive definite in the weighted scalar product $(\cdot,\cdot)_R$. □

Next, for $S \subset \mathbf{R}_0^+$, we define

$$\sqrt{S} := \{y \,|\, y = \sqrt{s},\ s \in S\}.$$

Herewith, one can rewrite Corollary 7.5 in the following form.

Corollary 7.6. (Application 5)
Let the conditions $(C1')$-$(C4')$ be fulfilled. Further, let the eigenvalues of A be arranged according to (5.9).
Then, the set $W_{N_{\sigma(A)},\|\cdot\|_R}(A)$ is convex, and one has the chain of relations

$$\begin{aligned} W_{N_{\sigma(A)},\|\cdot\|_R}(A) &= \left\{x \in \mathbf{R}_0^+ \,|\, x = \frac{\|Au\|_R}{\|u\|_R}, 0 \neq u \in N_{\sigma(A)}\right\} \\ &= \left\{x \in \mathbf{R}_0^+ \,|\, x = \sqrt{\frac{([R^{-1}A^*RA]u,u)_R}{(u,u)_R}}, 0 \neq u \in N_{\sigma(A)}\right\} \\ &= [|\lambda_r(A)|, |\lambda_1(A)|] \\ &= co\{|\sigma(A)|\}. \end{aligned}$$

Proof. For any subset $S \subset \boldsymbol{R}_0^+$, one has

$$\sqrt{S^2} = (\sqrt{S})^2 = S.$$

Thus, from Corollary 7.5, the equations of Corollary 7.6 follow. □

8. Numerical example

In this section, we check the results of Subsection 7.1 as well as of Theorem 6.1, Formula (6.3) numerically. The numerical check of the results of Subsections 7.2 and 7.3 is left to the reader.

8.1. A two-mass vibration model

We take up the multi-mass vibration model of [12], shown in Figure 8.1.

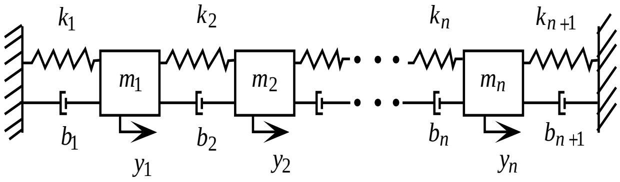

Figure 8.1: Multi-mass vibration model

and study the case $\mathbf{n} = \mathbf{2}$ as in [13]. For the sake of completeness, we give again the details. The associated initial value problem is given by

$$M\ddot{y} + B\dot{y} + Ky = 0,\ y(0) = y_0,\ \dot{y}(0) = \dot{y}_0,$$

where $y = [y_1, y_2]^T$ and

$$M = \left[\begin{array}{c|c} m_1 & 0 \\ \hline 0 & m_2 \end{array}\right],$$

$$B = \left[\begin{array}{c|c} b_1 + b_2 & -b_2 \\ \hline -b_2 & b_2 + b_3 \end{array}\right],$$

$$K = \left[\begin{array}{c|c} k_1 + k_2 & -k_2 \\ \hline -k_2 & k_2 + k_3 \end{array}\right],$$

with the *mass, damping,* and *stiffness matrices* M, B, and K, as the case may be, and the *displacement vector* y. In *state-space description*, this problem takes the form

$$\dot{x} = Ax,\ t \geq 0,\ x(0) = x_0,$$

where $x = [y^T, z^T]^T$, $z = \dot{y}$, and where the *system matrix* A is given by

$$A = \left[\begin{array}{c|c} 0 & E \\ \hline -M^{-1}K & -M^{-1}B \end{array}\right].$$

(i) *Construction of a non-diagonalizable matrix A:*
The pertinent characteristic equation reads

$$|\lambda^2 M + \lambda B + K| = \left|\begin{array}{c|c} \lambda^2 m_1 + \lambda(b_1 + b_2) + (k_1 + k_2) & \lambda(-b_2) - k_2 \\ \hline \lambda(-b_2) - k_2 & \lambda^2 m_2 + \lambda(b_2 + b_3) + (k_2 + k_3) \end{array}\right| = 0.$$

As in [13], for the construction of a case with non-diagonalizable matrix A, we choose

$$b_2 = 0,\ m_2 = m_1 = 1,\ b_3 = b_1,\ k_3 = k_1.$$

Then,

$$\lambda^2 m_1 + \lambda b_1 + (k_1 + k_2) = sk_2 \quad \text{with} \quad s \in \{+1, -1\}.$$

Hence, with $m_1 = 1$,

$$\lambda = -\frac{b_1}{2} \pm \sqrt{(\frac{b_1}{2})^2 - k_1 - k_2 + sk_2}.$$

Now, in order to get one real solution, we set

$$k_1 := (\frac{b_1}{2})^2.$$

This implies

$$\lambda = \begin{cases} -\frac{b_1}{2}, & s=+1, \\ -\frac{b_1}{2} \pm i\sqrt{2k_2}, & s=-1. \end{cases}$$

(ii) *Data:*
Like in [13], as numerical values for the quantities not yet specified, we choose $b_1 = 1/4$, $k_2 = 2^3 = 8$. On the whole, this delivers the following data:

$$m_1 = m_2 = 1;\ b_1 = 1/4, b_2 = 0, b_3 = 1/4;\ k_1 = 1/64 = 1/2^4, k_2 = 8, k_3 = 1/64 = 1/2^4,$$

which leads to

$$\begin{aligned} M &= \left[\begin{array}{c|c} m_1 & 0 \\ \hline 0 & m_2 \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right], \\ B &= \left[\begin{array}{c|c} b_1+b_2 & -b_2 \\ \hline -b_2 & b_2+b_3 \end{array}\right] = \left[\begin{array}{c|c} 0.25 & 0 \\ \hline 0 & 0.25 \end{array}\right], \\ K &= \left[\begin{array}{c|c} k_1+k_2 & -k_2 \\ \hline -k_2 & k_2+k_3 \end{array}\right] = \left[\begin{array}{c|c} 1/64+8 & -1/2 \\ \hline -1/2 & 8+1/64 \end{array}\right] = \left[\begin{array}{c|c} 8.015625 & -0.5 \\ \hline -0.5 & 8.015625 \end{array}\right]. \end{aligned}$$

Further, we choose

$$t_0 = 0$$

as well as

$$y_0 = [-1, 1]^T$$

and

$$\dot{y}_0 = [-1, -1]^T,$$

but y_0 and $\dot{y}_0$ are not needed here.

8.2. Computation of important quantities

Using the Matlab routine *jordan*, one obtains

$$\begin{aligned} \lambda_1(A) &= -0.1250 + 4.0000i, \\ \lambda_2(A) &= -0.1250 - 4.0000i, \\ \lambda_3(A) &= -0.1250, \\ \lambda_4(A) &= \lambda_3(A). \end{aligned}$$

The pertinent eigenvectors and principal vectors are

$$\left[p_1^{(1)}, p_1^{(2)}, p_1^{(3)}, p_2^{(3)}\right] = [p_1, p_2, p_3, p_4]$$

with

$$\begin{aligned} \left[p_1^{(1)}, p_1^{(2)}\right] &= [p_1, p_2] \\ &= \begin{bmatrix} 0.250000000000000 - 0.007812500000000i & 0.250000000000000 + 0.007812500000000i \\ -0.250000000000000 + 0.007812500000000i & -0.250000000000000 - 0.007812500000000i \\ 0 + 1.000976562500000i & 0 - 1.000976562500000i \\ 0 - 1.000976562500000i & 0 + 1.000976562500000i \end{bmatrix}. \end{aligned}$$

and

$$\begin{aligned} \left[p_1^{(3)}, p_2^{(3)}\right] &= [p_3, p_4] \\ &= \begin{bmatrix} 0.062500000000000 & 0.500000000000000 \\ 0.062500000000000 & 0.500000000000000 \\ -0.007812500000000 & 0 \\ -0.007812500000000 & 0 \end{bmatrix}. \end{aligned}$$

They are apparently unnormed. The algebraic multiplicities are thus $m_1 = m_2 = 1$ and $m_3 = 2$.

For the adjoint matrix A^*, we obtain

$$\begin{array}{rcl}\lambda_1(A^*) & = & -0.1250-4.0000i,\\ \lambda_2(A^*) & = & -0.1250+4.0000i,\\ \lambda_3(A^*) & = & -0.1250,\\ \lambda_4(A^*) & = & \lambda_3(A^*).\end{array}$$

The associated eigenvectors and principal vectors are

$$\left[u_1^{(1)^*}, u_1^{(2)^*}, u_1^{(3)^*}, u_2^{(3)^*}\right] = [u_1^*, u_2^*, u_3^*, u_4^*]$$

with

$$\begin{array}{rcl}\left[u_1^{(1)^*}, u_1^{(2)^*}\right] & = & [u_1^*, u_2^*]\\ & = & \left[\begin{array}{rr} 0.250000000000000+0.007812500000000i & 0.250000000000000-0.007812500000000i\\ -0.250000000000000+0.007812500000000i & -0.250000000000000-0.007812500000000i\\ 0-0.062500000000000i & 0+0.062500000000000i\\ 0+0.062500000000000i & 0-0.062500000000000i\end{array}\right].\end{array}$$

and

$$\begin{array}{rcl}\left[u_1^{(3)^*}, u_2^{(3)^*}\right] & = & [u_3^*, u_4^*]\\ & = & \left[\begin{array}{lr} 0.062500000000000 & 0.500000000000000\\ 0.062500000000000 & 0.500000000000000\\ 0.500000000000000 & 0\\ 0.500000000000000 & 0\end{array}\right].\end{array}$$

They are also unnormed.
Now, we biorthogonalize these vectors based on Theorem 2.1 such that the relations

$$(p_k^{(i)}, u_l^{(i)^*}) = \begin{cases} 1, & l = m_i - k + 1\\ 0, & l \neq m_i - k + 1\end{cases}$$

and

$$(p_k^{(i)}, u_l^{(j)^*}) = 0,\ i \neq j.$$

So, with

$$v_l^{(i)^*} = u_{m_i-l+1}^{(i)}{}^*,$$

one has the biorthogonality relations

$$(p_k^{(i)}, v_l^{(j)^*}) = \delta_{kl}\delta_{ij}. \tag{8.1}$$

We give the details. Define

$$\begin{array}{l} v_1^* = v_1^{(1)^*} = u_1^{(1)^*} = u_1^*,\\ v_2^* = v_1^{(2)^*} = u_1^{(2)^*} = u_2^*,\\ v_3^* = v_1^{(3)^*} = u_2^{(3)^*} = u_4^*,\\ v_4^* = v_2^{(3)^*} = u_1^{(3)^*} = u_3^*.\end{array}$$

Then,

$$\alpha_3 := -\frac{(p_4\,, v_3^*)}{(p_3\,, v_3^*)} = -8.$$

Define

$$w_4 = p_4 + \alpha_3 p_3$$

and replace p_4 by w_4, i.e., in Matlab set $p_4 = w_4$.
Normalize $v_i^*,\ i = 1,\dots,4$ by the substitutions

$$v_i^* \to \frac{v_i^*}{\|v_i^*\|_2},\ i = 1,\dots,4$$

and $p_i,\ i = 1,\dots,4$ by

$$p_i \to \frac{p_i}{(p_i\,, v_i^*)}\ i = 1,\dots,4.$$

Then, we obtain

$$\left[p_1^{(1)}, p_1^{(2)}, p_1^{(3)}, p_2^{(3)}\right] = [p_1, p_2, p_3, p_4]$$

with

$$\begin{aligned}\left[p_1^{(1)}, p_1^{(2)}\right] &= [p_1, p_2] \\ &= \begin{bmatrix} 0.364601934049314 & 0.364601934049314 \\ -0.364601934049314 & -0.364601934049314 \\ -0.045575241756164 + 1.458407736197255i & -0.045575241756164 - 1.458407736197255i \\ 0.045575241756164 - 1.458407736197255i & 0.045575241756164 + 1.458407736197255i \end{bmatrix}.\end{aligned}$$

and

$$\begin{aligned}\left[p_1^{(3)}, p_2^{(3)}\right] &= [p_3, p_4] \\ &= \begin{bmatrix} 0.707106781186548 & 0 \\ 0.707106781186548 & 0 \\ -0.088388347648318 & 0.712609640686961 \\ -0.088388347648318 & 0.712609640686961 \end{bmatrix}.\end{aligned}$$

as well as

$$\left[v_1^{(1)*}, v_1^{(2)*}, v_1^{(3)*}, v_2^{(3)*}\right] = [v_1^*, v_2^*, v_3^*, v_4^*]$$

with

$$\begin{aligned}\left[v_1^{(1)*}, v_1^{(2)*}\right] &= [v_1^*, v_2^*] \\ &= \begin{bmatrix} 0.685679302968773 + 0.021427478217774i & 0.685679302968773 - 0.021427478217774i \\ -0.685679302968773 - 0.021427478217774i & -0.685679302968773 + 0.021427478217774i \\ 0 + 0.171419825742193i & 0 - 0.171419825742193i \\ 0 - 0.171419825742193i & 0 + 0.171419825742193i \end{bmatrix}.\end{aligned}$$

and

$$\begin{aligned}\left[v_1^{(3)*}, v_2^{(3)*}\right] &= [v_3^*, v_4^*] \\ &= \begin{bmatrix} 0.707106781186547 & 0.087705801930703 \\ 0.707106781186547 & 0.087705801930703 \\ 0 & 0.701646415445623 \\ 0 & 0.701646415445623 \end{bmatrix}.\end{aligned}$$

With these normed vectors, relations (8.1) are computationally verified.
Further, R in (2.1) can be written as

$$\begin{aligned} R &= u_1^{(1)*} u_1^{(1)} + u_1^{(2)*} u_1^{(2)} + u_1^{(3)*} u_1^{(3)} + u_2^{(3)*} u_2^{(3)} \\ &= \textstyle\sum_{i=1}^4 u_i^* u_i = \sum_{i=1}^4 v_i^* v_i \\ &= v_1^{(1)*} v_1^{(1)} + v_1^{(2)*} v_1^{(2)} + v_1^{(3)*} v_1^{(3)} + v_2^{(3)*} v_2^{(3)}, \end{aligned}$$

where it goes without saying that the u_i^* and u_i are normed in a similar way as the v_i^* and v_i. Matlab delivers

$$R = \begin{bmatrix} 1.448922794377340 & -0.433538178992724 & 0.068884650702833 & 0.054192272374091 \\ -0.433538178992724 & 1.448922794377340 & 0.054192272374091 & 0.068884650702833 \\ 0.068884650702833 & 0.054192272374091 & 0.551077205622660 & 0.433538178992725 \\ 0.054192272374091 & 0.068884650702833 & 0.433538178992725 & 0.551077205622660 \end{bmatrix}.$$

The eigenvalues of R in (8.5) are given by

$$\begin{aligned} \lambda_1(R) &= 0.117416726023999, \\ \lambda_2(R) &= 0.875965265410791, \\ \lambda_3(R) &= 1.124034734589208, \\ \lambda_4(R) &= 1.882583273976000, \end{aligned}$$

so that R is positive definite.

Remark 8.1. *The vector $p_2^{(3)}$ is a principal vector of stage 2 for matrix A. But, since it is normed such that $(p_2^{(3)}, v_2^{(3)^*}) = 1$ instead of $\|p_2^{(3)}\|_2 = 1$, the equation $Ap_2^{(3)} = \lambda_3 p_2^{(3)} + p_1^{(3)}$ does* **not** *hold, but instead, the equation $Ap_2^{(3)} = \lambda_3 p_2^{(3)} + \gamma_1^{(3)} p_1^{(3)}$ is valid with a factor $\gamma_1^{(3)} \neq 0$, $\gamma_1^{(3)} \neq 1$. Similarly, $u_2^{(3)^*}$ is principal vector of stage 2 for A^*. Similarly, due to the biorthogonalization process, the equation $A^* u_2^{(3)^*} = \overline{\lambda_3} u_2^{(3)^*} + u_1^{(3)^*}$ does* **not** *hold, but instead, the equation $A^* u_2^{(3)^*} = \overline{\lambda_3} u_2^{(3)^*} + \delta_1^{(3)} u_1^{(3)^*}$ is valid with a factor $\delta_1^{(3)} \neq 0$, $\delta_1^{(3)} \neq 1$. We leave it to the reader to check this numerically on our example.*

Remark 8.2. *Due to the foregoing remark, Formula (3.1) looks somewhat different. But, Formula (3.2) remains valid which is the important point since the subsequent findings are based on Formula (3.2), not on Formula (3.1).*

8.3. Numerical check of the validity of Corollary 7.1 (Application 1)

Here, we check the validity of

$$\frac{Re(Au,u)_R}{(u,u)_R} \in Re[W_{N_{\sigma(A)},(\cdot,\cdot)_R}(A)] = [Re\,\lambda_3(A), Re\,\lambda_1(A)],\ 0 \neq u \in N_{\sigma(A)}.$$

or

$$\frac{Re(Au,u)_R}{(u,u)_R} = -0.125,\ 0 \neq u \in N_{\sigma(A)} \subset \mathbb{C}^4.$$

We choose $u \in \{p, q, w\}$ where

$$\begin{aligned} p &= p1 + p2, \\ q &= 2\,p1 - 3\,p3, \\ w &= -4\,p2 + 5\,p4. \end{aligned}$$

We obtain

$$\begin{aligned} \frac{Re(Ap,p)_R}{(p,p)_R} &= -0.125000000000000, \\ \frac{Re(Aq,q)_R}{(q,q)_R} &= -0.125000000000000, \\ \frac{Re(Aw,w)_R}{(w,w)_R} &= -0.616601082213326 \neq -0.125, \end{aligned}$$

where the last result is not surprising since $p_4 = p_2^{(3)} \notin N_{\sigma(A)}$ and thus $w \notin N_{\sigma(A)}$.

8.4. Computational verification of the validity of Theorem 6.1

Here, we check Formula (6.3) of Theorem 6.1 With (3.9), from (6.2) we obtain

$$N'_{\sigma(A)} = N_{\lambda_1(A)} \oplus N_{\lambda_2(A)}$$

and

$$D := \left(R^{-1}\tfrac{A^*R+RA}{2}\right)^2 + \left(R^{-1}\tfrac{RA-A^*R}{2i}\right)^2 - R^{-1}A^*RA =$$

$$\begin{bmatrix} 0.253906250000002 & 0.253906250000000 & 0.000000000000000 & 0.000000000000000 \\ 0.253906249999997 & 0.253906250000002 & 0.000000000000000 & 0.000000000000000 \\ -0.063476562500000 & -0.063476562500000 & -0.253906249999998 & -0.253906250000002 \\ -0.063476562500000 & -0.063476562500000 & -0.253906250000002 & -0.253906250000000 \end{bmatrix}.$$

For

$$p = 2p_1 - 3ip_2 \in N'_{\sigma(A)},$$

we obtain

$$p = \begin{bmatrix} 0.729203868098627 - 1.093805802147941i \\ -0.729203868098627 + 1.093805802147941i \\ -4.466373692104092 + 3.053541197663002i \\ 4.466373692104092 - 3.053541197663002i \end{bmatrix}$$

and

$$Dp = \begin{bmatrix} 0.010908063192107 - 0.018157611026833i \\ -0.030646883054894 + 0.047235291651105i \\ -0.153210777398272 + 0.103250741290140i \\ 0.073274719625260 - 0.045519144009631i \end{bmatrix} \times 10^{-13} \doteq 0$$

so that (6.3) is fulfilled for $u = p$. On the other hand, for

$$q = p_1 + p_3 \notin N'_{\sigma(A)},$$

we obtain

$$q = \begin{bmatrix} 1.071708715235861 \\ 0.342504847137234 \\ -0.133963589404483 + 1.458407736197255i \\ -0.042813105892154 - 1.458407736197255i \end{bmatrix}$$

and

$$Dq = \begin{bmatrix} 0.359077662321295 + 0.000000000000000i \\ 0.359077662321291 - 0.000000000000000i \\ -0.044884707790162 + 0.000000000000005i \\ -0.044884707790162 - 0.000000000000003i \end{bmatrix} \neq 0$$

which is not surprising since $q \notin N'_{\sigma(A)}$.

8.5. Computational aspects

In this subsection, we say something about the used computer equipment and the computation times.
(i) As to the *computer equipment*, the following hardware was available: an Intel Core2 Duo Processor at 3166 GHz, a 500 GB mass storage facility, and two 2048 MB high-speed memories. As software package for the computations, we used MATLAB, Version 7.11.
(ii) The *computation time t* of an operation was determined by the command sequence *t1=clock; operation; t=etime(clock,t1)*. It is put out in seconds, rounded to four decimal places. For the computation of the eigenvalues of matrix A in Subsection 5.3, we used the command [XA,DA]=eig(A); the pertinent computation time was less than 0.0001 s.

9. Conclusion

It has been shown that there exist Rayleigh-quotient representations of the real parts, imaginary parts, and moduli of the eigenvalues of general matrices that parallel those representations known for the eigenvalues of self-adjoint matrices and corresponding to the ones for diagonalizable matrices. The key idea is to use a weighted scalar product defined by a positive definite matrix that is constructed by means of the left principal vectors of the considered matrix and the right principal vectors of its adjoint. As Formulas (3.3), (4.1), and (5.2) show, one essentially obtains the results for general matrices in the same way as for diagonalizable matrices by replacing the full space $\mathbb{C}^n$ with the geometric eigenspace $N_{\sigma(A)}$. The results are of interest on their own in Linear Algebra. They are also of potential interest in applications. For example, in the theory of linear dynamical systems, in the study of stability of a vibration problem, the real parts of the eigenvalues of the system matrix are important. Moreover, in systems with conjugate-complex eigenvalues, the moduli of the imaginary parts of the eigenvalues are the circular damped eigenfrequencies of the system. Finally, it could also be of interest for college teaching or research. The relation $(R^{-1}\frac{A^*R+RA}{2})^2 + (R^{-1}\frac{RA-A^*R}{2i})^2 = R^{-1}A^*RA$ derived for diagonalizable matrices A in [2] turns out to be valid only on $N'_{\sigma(A)}$. One feature of the present paper is also that, in the special case of diagonalizable matrices, we get back the results of [2]. On the whole, the results should be of interest to mathematicians as well as engineers.

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