Turk. J. Math. Comput. Sci.
13(1)(2021) 89–93
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DOI : 10.47000/tjmcs.789131

Two Types of Prime UP-Filters in Meet-Commutative UP-Algebras

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Received: 01-09-2020 • Accepted: 22-04-2021

Abstract. The concept of meet-commutative UP-algebra was introduced by Sawika et al. in 2016 and the concept of prime filters in such algebras was introduced and analyzed by Muhiuddin, Romano and Jun. In this article, as a continuation of the previous one, we introduce one more type of prime UP-filters in meet-commutative UP-algebras and we analyze the interrelationships of these two types of prime UP-filters.

2010 AMS Classification: 03G25, 06F35

Keywords: UP-algebra, meet-commutative UP-algebra, prime UP-filter of the first kind, prime UP-filter of the second kind.

1. Introduction

The concept of KU-algebras was introduced in 2009 by Prabpayak and Leerawat in the article [9]. Iampan introduced the concept of UP-algebras as a generalization of KU-algebras ([2]). This class of logical algebras is the subject of many studies (See for example [3, 15–17]). In [16], Somjanta et al. introduced the notion of filters in this class of algebra. Proper UP-filter in a UP-algebra was introduced by Romano 2018 ([10, 11]). Jun and Iampan then introduced and analyzed several classes of filters in UP algebras such as implicative, comparative and shift UP-filters (see, for example, [4–6]). The concept of weak implicative UP-filters in a UP-algebra was introduced and analyzed by Romano and Jun ([12]).

The concept of meet-commutative UP-algebras was introduced in article [14]. In article [8], a number of important properties of meet-commutative UP-algebras are given. In addition, in such UP-algebras, the concept of prime UP-filters was introduced and analyzed. This seems to justify our interest in studying the properties of UP-algebras in which the property of meet-commutativity is present.

First, in this article it is shown that the semi-lattice $(A, \sqcup)$ is distributive for any meet-commutative UP-algebra with respect to the least upper bound '$\sqcup$' (Proposition 2.4). The term 'distributive semilattice' used here differs from the concept of distributive semi-lattice introduced in the book [1]. Then, it is quite justified to analyze whether this semi-lattice satisfies some of the properties of the general semi-lattices. The second, we introduce one more type of prime filters in meet-commutative UP-algebras (Definition 3.3) and we analyze the interrelationships of these two types of prime UP-filters. It is shown that every prime UP-filter of the second kind of a meet-commutative UP-algebra is a prime UP-filter of the first kind (Theorem 3.5) but that the reverse does not have to be (Example 3.4). In addition, some sufficient conditions have been found that a meet-commutative UP-algebra must satisfy in order for these two classes of prime UP-filters of it to coincide (Theorem 3.7 and Theorem 3.8).

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2. Preliminaries

In this section, taking from the literature, we will repeat some concepts and statements of interest for this research. The way of writing formulas in this text is a strictly logical way of writing. So we tried to get the formulas written in the standard way of writing formulas in logic. For example, the signs '$\wedge$' and '$\vee$' are logical conjunction and disjunction. In addition, we use the symbol ':=' to indicate abbreviations in a sense such as $S := \{u \in A : P(u)\}$ where the symbol S denotes the set determined by predicate P on the right side of the notation ':='.

An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* (see [2]) if it satisfies the following axioms:
(UP-1) $(\forall x, y, \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
(UP-2) $(\forall x \in A)(0 \cdot x = x)$,
(UP-3) $(\forall x \in A)(x \cdot 0 = 0)$,
(UP-4) $(\forall x, y \in A)((x \cdot y = 0 \ \wedge \ y \cdot x = 0) \implies x = y)$.

A UP-algebra A is said to be meet-commutative (see [14], Definition 1.15) if it satisfies the condition

$$(\forall x, y \in A)((x \cdot y) \cdot y = (y \cdot x) \cdot x).$$

This term also appears in the paper [5], Definition 3.

In a UP-algebra, the order relation '$\leqslant$' is defined as follows

$$(\forall x, y \in A)(x \leqslant y \iff x \cdot y = 0).$$

A subset F of a UP-algebra A is called a *UP-filter* of A (see [16]) if it satisfies the following conditions:
(F-1) $0 \in F$,
(F-2) $(\forall x, y \in A)((x \in F \ \wedge \ x \cdot y \in F) \implies y \in F)$.

It is clear that every UP-filter F of a UP-algebra A satisfies:
(1) $(\forall x, y \in A)((x \in F \ \wedge \ x \leqslant y) \implies y \in F)$.

We first characterize the meet-commutative UP-algebras.

Theorem 2.1 ([8]). *Let A be a meet-commutative UP-algebra. Then the following holds*
(2) $(\forall x, y \in A)(x \leqslant y \implies y = (y \cdot x) \cdot x)$.

Theorem 2.2 ([8]). *Let A be a meet-commutative UP-algebra. For any $x, y \in A$, the element $x \sqcup y := (x \cdot y) \cdot y = (y \cdot x) \cdot x$ is the least upper bound of x and y.*

Proposition 2.3 ([8]). *Let A be a meet-commutative UP-algebra. Then*
(3) $(\forall x, y \in A)(0 \sqcup x = x,\ x \sqcup 0 = 0,\ x \sqcup x = x,$ *and* $x \sqcup y = y \sqcup x)$.
(4) $(\forall x, y, z \in A)((x \sqcup y) \sqcup z \leqslant (x \sqcup z) \sqcup (y \sqcup z))$.
(5) $(\forall x, y, z \in A)((z \cdot x) \sqcup (z \cdot y) \leqslant z \cdot (x \sqcup y))$.
(6) $(\forall x, y, z \in A)((x \sqcup y) \cdot z \leqslant (x \cdot z) \sqcup (y \cdot z))$.
(7) $(\forall x, y \in A)(x \sqcup y \leqslant (y \cdot x) \sqcup (x \cdot y))$.

We end this section with one important result about meet-commutative UP-algebras.

Proposition 2.4. *If A is a meet-commutative UP-algebra, then $(A, \sqcup)$ is an upper distributive semi-lattice.*

Proof. Let $x, y, z \in A$ be arbitrary elements of a meet-commutative UP-algebras A. From the valid inequalities $x \leqslant x \sqcup y$ and $z \leqslant z$ it follows $x \sqcup z \leqslant (x \sqcup y) \sqcup z$ according to the definition of the notion of last upper bound. Analogous to the previous one, $y \sqcup z \leqslant (x \sqcup y) \sqcup z$ it can be obtained. Thus $(x \sqcup z) \sqcup (y \sqcup z) \leqslant (x \sqcup y) \sqcup z$. The obtained inequality together with the inequality (4) gives the required equation what proves the statement of this theorem. ◻

3. The Main Results: Two Concepts of Prime UP-Filters

The notion of prime UP-filters of a meet-commutative UP-algebra was introduced in article [8]. For the purposes of this paper, we will recognize such a UP filter as a 'prime UP-filter of the first kind'.

Definition 3.1. Let F be a UP-filter of a meet-commutati- ve UP-algebra A. Then F is said to be a *prime UP-filter of the first kind* of A if the following holds
(PF1) $(\forall x, y \in A)(x \sqcup y \in F \implies (x \in F \ \vee \ y \in F))$.

Example 3.2. Let $A = \{0, a, b, c\}$ and operation '$\cdot$' is defined on A as follows:

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	c	0
b	0	c	0	c
c	0	b	b	0

Then $A = (A, \cdot, 0)$ is a meet-commutative UP-algebra. Subsets $\{0\}$, $\{0, b\}$ and $\{0, c\}$ are UP-filters of A. It is not difficult to verify that UP-filters $\{0, b\}$ and $\{0, c\}$ are prime of the first kind. It is clear that $\{0\}$ is not a prime UP-filter of the first kind of A because $b \sqcup c = 0 \in \{0\}$ but $b \notin \{0\}$ and $c \notin \{0\}$.

The following definition gives another type of prime UP-filter in meet-commuta- tive UP-algebras.

Definition 3.3. Let F be a UP-filter of a meet-commutative UP-algebra A. Then F is said to be a *prime UP-filter of the second kind* of A if the following holds
(PF2) $(\forall x, y \in A)(x \cdot y \in F \ \vee \ y \cdot x \in F)$.

Example 3.4. Let A be as in Example 3.2. Then $A = (A, \cdot, 0)$ is a meet-commutative UP-algebra. Subset $\{0, c\}$ is a prime UP-filter of the second kind of A. The subset $F := \{0, b\}$ is a prime UP-filter of the first kind but it is not a prime UP-filter of the second kind because, for example, hold $a \cdot b = c \notin F$ and $b \cdot a = c \notin F$.

In the previous example it was shown that a UP-filter can be a prime UP-filter of the first kind but it does not have to be a prime UP-filter of the second kind. However, the following theorem shows that if F satisfies the condition (PF2), then it satisfies the condition (PF1) also, i.e. any prime UP-filter of the second kind of a meet-commutative UP-algebra A is a prime UP-filter of the first kind of A.

Theorem 3.5. *Any prime UP-filter of the second kind of a meet-commutative UP-algebra A is a prime UP-filter of the first kind of A.*

Proof. Let a UP-filter F satisfy the condition (PF2) and let $x, y \in A$ be such that $x \sqcup y \in F$ are valid. Then $(x \cdot y) \cdot y = (y \cdot x) \cdot x \in F$. On the other hand, according to (PF2), we have $x \cdot y \in F$ or $y \cdot x \in F$. If $x \cdot y \in F$, then from $x \cdot y \in F$ and $(x \cdot y) \cdot y \in F$ it follows $y \in F$ by (F-2). If $y \cdot x \in F$, then from $y \cdot x \in F$ and $(y \cdot x) \cdot x \in F$ it follows $x \in F$ by (F-2). Hence (PF1) holds. □

Theorem 3.6 (Extension property for prime UP-filters of the second kind)**.** *Let A be a meet-commutative UP-algebra and let F and G be UP-filter of A such that $F \subseteq G$. If F is a prime UP-filter of the second kind, then G is a prime UP-filter of the second kind also.*

Proof. Since F is a prime UP-filter of the second kind of A that satisfies the condition (PF2), it follows that the UP-filter G also satisfies the condition (PF2). Therefore G is a prime UP-filter of the second kind of A. □

The following theorem gives one sufficient condition that every UP-filter of a meet-commutative UP-algebra be a prime UP-filter of the second kind.

Theorem 3.7. *If the order relation in a meet-commutative UP-algebra A is a linear relation, then each UP-filter of A is a prime UP-filter of the second kind of A.*

Proof. Suppose that the order relation in a meet-commutative UP-algebra A is a linear order. Then for each elements $x, y \in A$ the following $x \leqslant y$ or $y \leqslant x$ holds. Thus $x \cdot y = 0$ or $y \cdot x = 0$. Hence, for every UP-filter F of A, we have either $x \cdot y \in F$ or $y \cdot x \in F$ by (F-1). This shows that F is a prime UP-filter of the second kind of A. □

The question it is expected to be asked is: When a prime UP-filter of the first kind of a meet-commutative UP-algebra will be a prime UP-filter of the second kind?
The following theorem shows one of the possible answers to this question.

Theorem 3.8. *If a meet-commutative UP-algebra A satisfies the following condition*
(U) $(\forall x, y \in A)((x \cdot y) \sqcup (y \cdot x) = 0)$,
then any prime UP-filter of the first kind of A is a prime UP-filter of the second kind of A.

Proof. Let F be a prime UP-filter of the first kind of a meet-commutative UP-algebra A. Let $x, y \in A$ be such that $(x \cdot y) \sqcup (y \cdot x) = 0$. Then $(x \cdot y) \sqcup (y \cdot x) \in F$ and thus $x \cdot y \in F$ or $y \cdot x \in F$ by (PF1). So, the UP-filter F is a prime UP-filter of the second kind of A. ◻

To the question 'Is there a meet-commutative UP-algebra that satisfies condition (U)?' we have the following answer.

Theorem 3.9. *If the order relation in a meet-commutative UP-algebra A is a linear order, then A satisfies the condition (U).*

Proof. Suppose that the order relation in a meet-commutative UP-algebra A is a linear order. Then for each elements $x, y \in A$ the following $x \leqslant y$ or $y \leqslant x$ holds. Thus $x \cdot y = 0$ or $y \cdot x = 0$. Hence, we have $(x \cdot y) \sqcup (y \cdot x) = 0$. ◻

It is not difficult to prove the following theorem.

Theorem 3.10. *Let A be a meet-commutative UP-algebra. The condition (U) is equivalent the following condition*
(V) $(\forall x, y \in A)(y \cdot x = (x \cdot y) \cdot (y \cdot x))$.

Proof. According to the claim (6) of Proposition 1.7 in [2], we have

$$y \cdot x = (x \cdot y) \cdot (y \cdot x) \iff ((y \cdot x)(x \cdot y)) \cdot (x \cdot y) = 0.$$

Since the equality on the right in the previous equivalence has meaning $(x \cdot y) \sqcup (y \cdot x) = 0$, we have obtained that the condition (U) is equivalent to the condition (V). ◻

4. Conclusion and Further Work

A. Iampan introduced the concept of UP-algebras as one generalization of KU-algebras. This class of logical algebras has been in the interest of the academic public in recent years. The concept of meet-commutative UP-algebras was introduced in the article [14] by Sawika et al. In such UP-algebra, the concept of prime UP-filter (of the first kind) was introduced by Muhiuddin et al. ([8]). In this paper, the concept of prime UP-filter of the second kind (Definition 3.3) is introduced and the relationship of these two types of prime UP-filters in these UP-algebras is considered. Our experience in meet-commutative UP-algebras research in which the order relation is a linear relation initiated our belief that this type of UP-algebra can play a more significant role in our perception of the properties of this class of UP-algebras. This could be the subject of our next research. In addition to the above, the search for new types of UP-filters in meet-commutative UP-algebras should be continued.

The results of this paper are incorporated in the articles [7, 13] which comes chronologically after this report.

Acknowledgement

The author thanks the reviewer for helpful suggestions on the first version of the text whose incorporation has raised the clarity and consistency of the second version of this paper.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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