IKONION JOURNAL OF MATHEMATICS

YEAR: 2020 VOLUME :2 ISSUE: 1

ISSN: 2687-6531

A NEW CONCEPT FOR FRACTIONAL QUANTUM CALCULUS: $(\beta, q)-$CALCULUS AND ITS PROPERTIES

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(Received: 04.08.2020, Accepted: 06.10.2020, Published Online: 16.10.2020)

Abstract

In this article, authors expressed a new definition called $(\beta, q)-$derivative and integral which is a new type of fractional quantum derivative and integral. Also theorems and proofs on some basic properties of $(\beta, q)-$derivative and integral such as product rule, quotient rule, linearity and etc. are stated. These new definitions can be used in many different models arising in physical applications.

Keywords: $q-$calculus; $(\beta, q)-$derivative; $(\beta, q)-$integral.

1 Introduction

Fractional calculus which has drawn attention of scientists, has gained considerable interest from both theoretical and the applied points of view in recent years. There are numerous applications in many fields such as electrical networks, chemical physics, fluid flow, economics, signal and image processing, viscoelasticity, porous media, aerodynamics, modeling for physical phenomena exhibiting anomalous diffusion, and so on. In contrast to integer-order differential and integral operators, fractional-order differential operators give us chance to model nonlinear and complex phenomenons in nature and make understand the hereditary properties of several processes. The monographs [1, 2, 3] are commonly cited for the theory of fractional derivatives and integrals and applications to differential equations of fractional order. Indeed, studies on the theory of non-integer derivatives and integrals are not new. There are many different definitions of fractional derivatives and integrals. Among these Riemann-Liouville, Caputo, Grünwald-Letnikov, Riesz-Fischer derivatives and integrals are known commonly. But recently a new fractional derivative and its anti-derivative definition is presented by Atangana [4] called "Beta derivative" and "Atangana Beta integral" respectively.

Definition 1 *Let $f : [0, \infty) \to \mathbb{R}$ be a function. "Beta derivative" of f is defined by*

$$ {}_0^A D_t^{\beta}(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon\,(t + 1/\Gamma(\beta))^{1-\beta}) - f(t)}{\varepsilon} \tag{1} $$

for all $t > 0, \beta \in (0, 1)$. The anti-derivative of a function $f : [a, \infty) \to$ is defined as

$$ {}_0^A I_t^{\beta}(f)(t) = \int_a^t f(x) \left(x + \frac{1}{\Gamma(\beta)} \right)^{\beta - 1} dx $$

where $\beta \in (0, 1]$.

Quantum calculus or q-calculus which was firstly declared by FH Jackson in the early twentieth century is known as calculus without limits, but this kind of calculus had already been worked out by Euler and Jacobi. Absence of limit lets us to study on sets of non-differentiable functions. Recently it arose great interest in the mathematical modeling in quantum calculus. Hence $q-$calculus is seen as a connection between mathematics and physics, scientists have been paying great attention to it because of its different and huge application areas such as number theory, combinatorics, orthogonal polynomials, basic hyper-geometric functions, quantum theory, mechanics and the theory of relativity. Kac and Cheung [5] gave many of the basic features of quantum calculus in their book.

Let us give some basic definitions of q-calculus.

Definition 2 *[6] Let f be a function defined on a q-geometric set I, i.e., $qt \in I$ for all $t \in I$. For $0 < q < 1$, we define the q-derivative as*

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \in I \setminus \{0\}, \tag{2}$$

$$D_q(f)(0) = \lim_{t \to 0} D_q f(t). \tag{3}$$

Note that

$$\lim_{q \to 1} D_q f(t) = \lim_{q \to 1} \frac{f(t) - f(qt)}{(1-q)t} = \frac{df}{dt}.$$

Definition 3 *[6] For $t \geq 0$, we set $J_t = \{tq^n : n \in N \cup \{0\}\} \cup \{0\}$ and define the definite q-integral of a function $f : J_t \to \mathbb{R}$ by*

$$I_q f(t) = \int_0^t f(s) d_q s = (1-q)t \sum_{n=0}^{\infty} q^n f(q^n t).$$

2 Properties of $(\beta, q)-$Derivative and Integral

Definition 4 ($(\beta, q)-$ Derivative) *$(\beta, q)-$ Derivative of a function f can be expressed as*

$$T^{(\beta,q)} f(t) = \left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} \frac{f(qt) - f(t)}{(q-1)t} \tag{4}$$

where

$$q = 1 + \frac{\varepsilon (t + 1/\Gamma(\beta))^{1-\beta}}{t}.$$

Example 5 *Assume that $f(t) = t^n$. The $(\beta, q)-$derivative of function f can be evaluated as follows.*

$$\begin{aligned} T^{(\beta,q)}(f)(t) &= \left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} \frac{t^n - (qt)^n}{(1-q)t} \\ &= \left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} \frac{t^n - q^n t^n}{(1-q)t} \\ &= \left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} t^{n-1} \; [n]_q \end{aligned}$$

where $[n]_q = \frac{1-q^n}{1-q}$.

Theorem 6 *Let $f(t), g(t)$ are $(\beta, q)-$differentiable. $T^{(\beta,q)}$ operator satisfies following properties.*

1. $T^{(\beta,q)}(af(t) + bg(t)) = aT^{(\beta,q)} f(t) + bT^{(\beta,q)} g(t), \; a, b \in \mathbb{R}.$
2. $T^{(\beta,q)}(fg)(t) = g(t) \; T^{(\beta,q)} f(t) + f(qt) T^{(\beta,q)} g(t).$

3. $T^{(\beta,q)}\left(\frac{f}{g}\right)(t) = \frac{g(t)T^{(\beta,q)}f(t) - f(t)T^{(\beta,q)}g(t)}{g(t)g(qt)}$.

Proof.

1. *The proof can be easily seen from the definition of* $(\beta, q)-$ *derivative.*

2. *From the definition of* (β, q)*-derivative,*

$$\begin{aligned} T^{(\beta,q)}(fg)(t) &= (t + 1/\Gamma(\beta))^{1-\beta} \frac{f(t)g(t) - f(qt)g(qt)}{(1-q)t} \\ &= \frac{(t + 1/\Gamma(\beta))^{1-\beta}}{(1-q)t} \left[f(t)g(t) - f(qt)g(qt) + f(qt)g(t) - f(qt)g(t)\right] \\ &= g(t)(t + 1/\Gamma(\beta))^{1-\beta} \frac{f(t) - f(qt)}{(1-q)t} + f(qt)(t + 1/\Gamma(\beta))^{1-\beta} \frac{g(t) - g(qt)}{(1-q)t} \\ &= g(t)T^{(\beta,q)}f(t) + f(qt)T^{(\beta,q)}g(t) \end{aligned}$$

can be obtained.

3. *By using the definition of* $(\beta, q)-$*derivative we acquire*

$$\begin{aligned} T^{(\beta,q)}\left(\frac{f}{g}\right)(t) &= (t + 1/\Gamma(\beta))^{1-\beta} \frac{\frac{f(t)}{g(t)} - \frac{f(qt)}{g(qt)}}{(1-q)t} \\ &= \frac{(t + 1/\Gamma(\beta))^{1-\beta}}{g(t)g(qt)} \left(g(t)\frac{f(t) - f(qt)}{(1-q)t} - f(t)\frac{g(t) - g(qt)}{(1-q)t}\right) \\ &= \frac{g(t)T^{(\beta,q)}f(t) - f(t)T^{(\beta,q)}g(t)}{g(t)g(qt)}. \end{aligned}$$

■

Definition 7 *Let* $f : [0, \infty) \to \mathbb{R}$ *be a continuous function. Thus the* $(\beta, q)-$*integral of function* f *can be declared as follows.*

$$I^{(\beta,q)}f(t) = \int_0^t f(s)d_q^\beta s = \int_0^t (s + 1/\Gamma(\beta))^{\beta-1} f(s)d_q s = (1-q)t \sum_{n=0}^{\infty} q^n f(q^n t)(q^n t + 1/\Gamma(\beta))^{\beta-1}.$$

Theorem 8 *Let function* f *is* $(\beta, q)-$ *differentiable and integrable function. The following properties are satisfied for* $t > 0$.

1. $T^{(\beta,q)}(I^{(\beta,q)}f(t)) = f(t)$.

2. $I^{(\beta,q)}(T^{(\beta,q)}f(t)) = f(t)$.

3. $\int_a^t T^{(\beta,q)}f(s)d_q^\beta s = f(t) - f(a)$ *for* $a \in (a, t)$.

Proof.

1. From definition of $(\beta,q)-$derivative and integral we obtain

$$\begin{aligned}
T^{(\beta,q)}(I^{(\beta,q)}f(t)) &= T^{(\beta,q)}\int_0^t f(s)d_q^{\beta}s \\
&= T^{(\beta,q)}\int_0^t (s+1/\Gamma(\beta))^{\beta-1}f(s)d_q s \\
&= T^{(\beta,q)}\left[(1-q)t\sum_{n=0}^{\infty} q^n f(q^n t)\left(q^n t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}\right] \\
&= \frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}}{t}\left[t\sum_{n=0}^{\infty} q^n f(q^n t)\left(q^n t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1} - qt\sum_{n=0}^{\infty} q^n f(q^n(qt))\left(q^n(qt)+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}\right] \\
&= \left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\left[\sum_{n=0}^{\infty} q^n f(q^n t)\left(q^n t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1} - q\sum_{n=0}^{\infty} q^n f(q^{n+1} t)\left(q^{n+1}t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}\right] \\
&= \left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\left[\sum_{n=0}^{\infty} q^n f(q^n t)\left(q^n t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1} - \sum_{n=0}^{\infty} q^{n+1} f(q^{n+1} t)\left(q^{n+1}t+\frac{1}{\Gamma(\beta)}\right)^{\beta-1}\right] \\
&= f(t).
\end{aligned}$$

2. By using the definition of $(\beta,q)-$derivative and integral we get

$$\begin{aligned}
I^{(\beta,q)}(T^{(\beta,q)}f(t)) &= \int_0^t T^{(\beta,q)}f(s)d_q^{\beta}s \\
&= \int_0^t \left(s+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\frac{f(s)-f(qs)}{(1-q)s}d_q^{\beta}s \\
&= \int_0^t \frac{f(s)-f(qs)}{(1-q)s}d_q s \\
&= (1-q)t\sum_{n=0}^{\infty} q^n\left(\frac{f(q^n t)}{(1-q)q^n t}-\frac{f(q(q^n t))}{(1-q)q^n t}\right) \\
&= \sum_{n=0}^{\infty}\left[f(q^n t)-f(q^{n+1}t)\right] \\
&= f(t).
\end{aligned}$$

3. It can easily seen by the help of (2).

■

Theorem 9 *Let $f,g:[0,\infty)\to\mathbb{R}$ are continuous and $(\beta,q)-$differentiable and integrable functions. For $t\in[0,\infty)$,*

1. $\int_0^t [af(s)+bg(s)]\,d_q^{\beta}s = a\int_0^t f(s)d_q^{\beta}s + b\int_0^t g(s)d_q^{\beta}s,\ a,b\in\mathbb{R}.$

2. $\int_a^t f(s)T^{(\beta,q)}g(s)d_q^{\beta}s = (fg)(t) - \int_a^t g(qs)T^{(\beta,q)}f(s)d_q^{\beta}s.$

Proof.

1. One can easily see the proof from the definition.

2. From Theorem 6 (2), it can be deducted

$$T^{(\beta,q)}(fg)(t) = f(t)T^{(\beta,q)}g(t) + g(qt)Df(t).$$

By taking $(\beta, q)-$integral of two sides; we have

$$\int_0^t T^{(\beta,q)}(fg)(s)d_q^\beta s = \int_0^t f(s)T^{(\beta,q)}g(s)d_q^\beta s + \int_0^t g(qs)T^{(\beta,q)}f(s)d_q^\beta s.$$

Then using Theorem 8 (2), we get

$$\int_0^t f(s)T^{(\beta,q)}g(s)d_q^\beta s = f(t)g(t) - \int_0^t g(qs)T^{(\beta,q)}f(s)d_q^\beta s.$$

■

3 Conclusion

In this paper the authors give a new definition entitled "$(\beta, q)-$derivative and integral" by establishing relation between with quantum calculus and beta fractional calculus. By using these new definitions, many applications on different branches of science can be made. Possible applications by using "$(\beta, q)-$derivative and integral" are left as open problem.

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