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Boole approximation method with residual error function to solve linear Volterra integro-differential equations

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Abstract

In this study, a numerical method is developed for the approximate solution of the linear Volterra integro-differential equations. This method is based Boole polynomial, its derivatives and the collocation points. The aim is to reduce the given problem, as the linear algebraic equation, to the matrix equation. This matrix equation is solved using Boole collocation points. As a result, the approximate solution is obtained in the truncated Boole series in the interval $[a, b]$. The exact solution and the approximate solution are included in the study. Also, the error analysis and residual correction calculations are performed in the study. The results have been obtained by using computer program MATLAB.

Keywords: Boole polynomials, linear Volterra integro-differential equation, collocation points, approximate solutions, Residual error analysis

1. Introduction

Integro-differential equations are a tool used to representing problems in fields such as physics, biology, chemistry, mechanics, astronomy, electrostatic, natural science, potential theory, economics. Since integro-differential equation classes are difficult to solve by analytical methods, numerical methods are preferred. For the numerical solution of integro-differential equation classes, the method based on the Bernoulli polynomial by Erdem Biçer et al. [1, 13], the method based on the Bessel polynomial by Yüzbaşı et al. [11, 28], the method based on the Laguerre polynomial by Baykuş Savaşaneril and Sezer [27] and the method based on the Dickson polynomial by Kürkçü [15] have been developed [14, 16-24]. In addition, the numerical methods such as Taylor collocation method [2], a multiscale Galerkin method [3], Bernstein polynomials method[4], Legendre collocation method [5], Euler wavelet method [6], Newton-Product method [7], homotopy-perturbation method [8], improved Bessel collocation method [9], Spectral collocation method [10], Hybrid Euler-Taylor matrix method [29] and Tau method [12] are included in the literature.

In this study, the numerical method is developed using Boole polynomial, its derivatives and collocation points for the approximate solution of the mth order linear Volterra integro-differential equation which is a type of the integro-differential equations.

Charles Jordan has stated general form of Boole polynomial as follows [26,30-31]

$$R_n(x) = \sum_{m=0}^{n+1} \frac{(-1)^m}{2^m} \binom{x}{n-m}. \qquad \mathbf{(1.1)}$$

The defined form of the Boole polynomial is written as

$$\sum_{n=0}^{\infty} \frac{R_n(x)}{n!} t^n = \frac{2(1+t)^x}{2+t}. \qquad \mathbf{(1.2)}$$

The aim is to get Boole solution of the mth order linear Volterra integro-differential equation

$$\sum_{k=0}^{m} P_k(x) y^{(k)}(x) = g(x) + \lambda \int_a^x K(x,t) y(t) dt, \quad a \le x, t \le b \qquad \mathbf{(1.3)}$$

with the initial boundary conditions

$$\sum_{k=0}^{m-1}\left(a_{jk}y^{k}(a)+b_{jk}y^{k}(b)\right)=\lambda_j,$$
$$j=0,1,2,3,\dots m-1. \qquad (\mathbf{1.4})$$

The Boole solution of Eq. (1.3) is obtained in the following Boole series form

$$y(x)\cong y_N(x)=\sum_{n=0}^{N}a_nR_n(x) \qquad (\mathbf{1.5})$$

where $R_n(x)$ is Boole polynomial and a_n, $n=0,1,2,\dots,N$ are the unknown Boole coefficients.

2. Matrix relations of the linear Volterra integro-differential equation

In this section, the matrix relations to reduce the Eq. (1.3) to the matrix equation system and the matrix relation of conditions (1.4) are given. Firstly, the matrix form of the Boole polynomial, $\mathbf{R}(x)$, is written as

$$\mathbf{R}(x)=\mathbf{X}(x)\mathbf{H}^{T} \qquad (\mathbf{2.1})$$

where

$$\mathbf{R}(x)=[R_0(x)\quad R_1(x)\quad R_2(x)\quad \dots\quad R_N(t)],$$
$$\mathbf{X}(\mathrm{x})=[1\quad x\quad x^2\quad \dots\quad x^N]$$

and

$$\mathbf{H}=\begin{bmatrix}1 & 0 & 0 & \cdots\\ -\frac{1}{2} & 1 & 0 & \cdots\\ \frac{1}{2} & -2 & 1 & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{bmatrix}.$$

The matrix relation of the Boole series form (1.5) is written as

$$y(x)=\mathbf{R}(x)\mathbf{A} \qquad (\mathbf{2.2})$$

and its kth derivative is

$$y^{(k)}(x)=\mathbf{R}^{(k)}(x)\mathbf{A}. \qquad (\mathbf{2.3})$$

The matrix form (2.1) is placed in the matrix relation (2.3) and the matrix relation is obtained as

$$y^{(k)}(x)=\mathbf{X}^{(k)}(x)\mathbf{H}^{\mathrm{T}}\mathbf{A} \qquad (\mathbf{2.4})$$

where

$$\mathbf{X}^{(k)}(x)=\mathbf{X}(x)\mathbf{E}^{k}. \qquad (\mathbf{2.5})$$

First, the matrix form (2.5) is placed in the matrix relation (2.4), the new matrix relation is obtained as

$$y^{(k)}(x)=\mathbf{X}(x)\mathbf{E}^{k}\mathbf{H}^{\mathrm{T}}\mathbf{A} \qquad (\mathbf{2.6})$$

where the matrix $\mathbf{E}$ is derivative transition matrix of $\mathbf{X}(x)$,

$$\mathbf{E}=\begin{bmatrix}0 & 1 & 0 & 0 & \dots & 0\\ 0 & 0 & 2 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \dots & N\\ 0 & 0 & 0 & 0 & \dots & 0\end{bmatrix},\ \mathbf{A}=\begin{bmatrix}a_0\\ a_1\\ a_2\\ \vdots\\ a_N\end{bmatrix}.$$

In Eq. (1.3), the kernel function $K(x,t)$ is defined as follows for the Taylor polynomial and Boole polynomial, respectively

$$\left.\begin{aligned}K(x,t)&=\mathbf{X}(x)^{t}K\mathbf{X}^{\mathrm{T}}(t)\\ K(x,t)&=\mathbf{R}(x)^{\mathrm{R}}K\mathbf{R}^{\mathrm{T}}(t)\end{aligned}\right\} \qquad (\mathbf{2.7})$$

From these forms, the matrix relation is obtained as follows

$$^{R}\mathbf{K}=\ (\mathbf{H}^{\mathrm{T}})^{-1}\,{}^{t}K\mathbf{H}^{-1} \qquad (\mathbf{2.8})$$

where

$$^{t}K(x,t)=\sum_{m=0}^{N}\sum_{n=0}^{N}{}^{t}k_{mn}\,x^{m}t^{n},$$

$$^{R}K(x,t)=\sum_{m=0}^{N}\sum_{n=0}^{N}{}^{R}k_{mn}\,\mathrm{R_m(x)R_n(t)},$$

and

$$^{t}k_{mn}=\frac{1}{m!\,n!}\frac{\partial^{m+n}K(0,0)}{\partial x^{m}\partial t^{n}}\quad m,n=0,1,2,\dots,N.$$

According to the relation (2.4), the corresponding matrix form of the conditions (1.4) is written as

$$\sum_{k=0}^{m-1}(a_{jk}\mathbf{X}(a)+b_{jk}\,\mathbf{X}(b))\mathbf{E}^{k}\boldsymbol{H}^{T}\mathbf{A}=\lambda_k,$$
$$j=0,1,2,\dots,m-1. \qquad (\mathbf{2.9})$$

3. Collocation method

The matrix relation (2.6), the kernel function for the Taylor polynomial and the matrix form (2.1) are placed in the Eq. (1.3). Then the matrix equation is obtained as

$$\sum_{k=0}^{m}P_k(x)\,\mathbf{X}(x)\mathbf{E}^{\mathbf{k}}\mathbf{H}^{T}\mathbf{A}$$
$$=g(x)+\lambda\,\mathbf{X}(x)^{t}K\boldsymbol{C}(x)\mathbf{H}^{T}\boldsymbol{A} \qquad (\mathbf{3.1})$$

where

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$$\mathbf{C}(\mathrm{x}) = \int_a^x \mathbf{X}^T(t)\mathbf{X}(t)dt = [c_{i,j}(x)],$$

$$c_{i,j}(x) = \frac{x^{i+j+1} - a^{i+j+1}}{i+j+1}, i,j = 0,1,2,\dots,N.$$

By using in Eq. (15) the collocation points x_i defined by

$$x_i = a + \frac{b-a}{N} i, \quad i = 0,1,\dots,N \qquad (\mathbf{3.2})$$

the system of the matrix equations is gained as

$$\sum_{k=0}^{m} P_k(x_i)\,\mathbf{X}(x_i)\mathbf{E}^k\mathbf{H}^{\mathbf{T}}\mathbf{A}$$
$$= g(x_i) + \lambda\,\mathbf{X}(x_i)^t\mathbf{K}\boldsymbol{C}(x_i)\mathbf{H}^T\boldsymbol{A} \qquad (\mathbf{3.3})$$

or briefly the fundamental matrix equation is shown as

$$\left\{\sum_{k=0}^{m} \mathbf{P}_k\,\mathbf{X}\mathbf{E}^{\mathrm{k}}\mathbf{H}^{\mathbf{T}} - \lambda\,\overline{\mathbf{X}}\overline{\mathbf{K}}\overline{\mathbf{C}}\overline{\mathbf{H}^T}\right\}\mathbf{A} = \mathbf{G} \qquad (\mathbf{3.4})$$

Where

$$\mathbf{P}_k = \begin{bmatrix} \mathbf{P}_k(x_0) & 0 & \dots & 0 \\ 0 & \mathbf{P}_k(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{P}_k(x_N) \end{bmatrix}_{(\mathrm{N+1})\mathrm{x}(\mathrm{N+1})},$$

$$\mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}_{(\mathrm{N+1})\mathrm{x}1}, \qquad \mathbf{R} = \begin{bmatrix} \mathbf{R}(x_0) \\ \mathbf{R}(x_1) \\ \vdots \\ \mathbf{R}(x_N) \end{bmatrix}_{(\mathrm{N+1})\mathrm{x}(\mathrm{N+1})},$$

$$\overline{\mathbf{X}} = \begin{bmatrix} \mathbf{X}(x_0) & 0 & \dots & 0 \\ 0 & \mathbf{X}(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{X}(x_N) \end{bmatrix}_{(\mathrm{N+1})\mathrm{x}(\mathrm{N+1})^2},$$

$$\overline{\mathbf{K}} = \begin{bmatrix} \mathbf{K} & 0 & \dots & 0 \\ 0 & \mathbf{K} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{K} \end{bmatrix}_{(\mathrm{N+1})^2\mathrm{x}(\mathrm{N+1})^2},$$

$$\overline{\mathbf{H}^T} = \begin{bmatrix} \mathbf{H}^T \\ \mathbf{H}^T \\ \vdots \\ \mathbf{H}^T \end{bmatrix}_{(\mathrm{N+1})^2\mathrm{x}(\mathrm{N+1})} \quad \text{and}$$

$$\overline{\mathbf{C}} = \begin{bmatrix} \mathbf{C}(x_0) & 0 & \dots & 0 \\ 0 & \mathbf{C}(x_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{C}(x_N) \end{bmatrix}_{(\mathrm{N+1})^2\mathrm{x}(\mathrm{N+1})^2}.$$

The fundamental matrix relation (3.4) is written as

$$\mathbf{WA} = \mathbf{G} \text{ or } [\mathbf{W};\mathbf{G}] \qquad (\mathbf{3.5})$$

where

$$\mathbf{W} = \sum_{k=0}^{m} \mathbf{P}_k\,\mathbf{X}\mathbf{E}^{\mathrm{k}}\mathbf{H}^{\mathbf{T}} - \lambda\,\overline{\mathbf{X}}\overline{\mathbf{K}}\overline{\mathbf{C}}\overline{\mathbf{H}^T}.$$

As a result of the operations, a system of $(N+1)$ of the linear algebraic equation is obtained with the unknown Boole coefficients $a_0, a_1, \dots, a_N$. The matrix form for the conditions (2.9) are written as

$$\mathbf{U}_j\mathbf{A} = [\lambda_j] \text{ or } [\mathbf{U}_j;\ \lambda_j]; \quad \mathrm{j} = 0,1,2,\dots,\mathrm{m}-1 (\mathbf{3.6})$$

where

$$\mathbf{U}_j = \sum_{k=0}^{m-1} (a_{jk}\mathbf{X}(a) + b_{jk}\,\mathbf{X}(b))\mathbf{E}^{\mathrm{k}}\mathbf{H}^{\mathbf{T}}$$
$$= [u_{j0} \quad u_{j1} \quad u_{j2} \quad \dots \quad u_{jN}],$$
$$j = 0,1,2,\dots,m-1.$$

To reach the solution of the Eq. (1.3) under conditions (1.4), the rows matrices (3.5) are replaced with m rows of the matrix (3.6). So, the new augmented matrix form is obtained as

$$[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}] = \begin{bmatrix} w_{00} & \dots & w_{0N} & ; & g(x_0) \\ w_{10} & \dots & w_{1N} & ; & g(x_1) \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ w_{(N-m)0} & \dots & w_{(N-m)N} & ; & g(x_{N-m}) \\ u_{00} & \dots & u_{0N} & ; & \lambda_0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ u_{(m-1)0} & \dots & u_{(m-1)N} & ; & \lambda_{m-1} \end{bmatrix}.$$

If $rank\widetilde{\mathbf{W}} = rank[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}] = N+1$ is, the augmented matrix is written as

$$\mathbf{A} = (\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{G}}. \qquad (\mathbf{3.7})$$

Finally, the unknown Boole coefficients from the matrix (3.7) solution are found and placed in the series (1.5), the Boole polynomial solution is obtained as

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n R_n(x)$$

4. Residual correction and error estimation

The error estimation of the Boole collocation method for the Eq. (1.3) and the residual correction of the Boole approximate solution are given in this section. The residual function of the Boole collocation method is written as

$$\Re_N(x) = L[y_N(x)] - g(x),$$

where $y_N(x)$, which is the Boole polynomial solution defined by (1.5), is the approximate solution of the problem (1.3). Therefore $y_N(x)$ is improved the equation

$$L[y_N(x)] = \sum_{k=0}^{m} P_k(x) y_N^{(k)}(x) - \lambda \int_a^x K(x,t) y_N(t) dt = g(x) + \Re_N(x), \qquad (\mathbf{4.1})$$

The error function $e_N(x)$ is defined as

$$e_N(x) = y(x) - y_N(x), \qquad (\mathbf{4.2})$$

where $y(x)$ is the exact solution of the problem (1.3). Substituting (4.2) into (4.1), and by simplifying the error differential equation is found

$$\sum_{k=0}^{m} P_k(x) e_N^{(k)}(x) - \lambda \int_a^x K(x,t) e_N(t) dt = -\Re_N(x), \qquad (\mathbf{4.3})$$

with homogeneous conditions

$$\sum_{k=0}^{m-1} \left(a_{jk} e_N^{(k)}(a) + b_{jk} e_N^{(k)}(b) \right) = 0, \quad j = 0, 1, 2, \dots, m-1$$

Boole collocation method is applied to Eq. (4.3), the approximation $e_{N,M}(x)$ to $e_N(x)$ is obtained, $(M \geq N)$ which is the error function based on the residual function $\Re_N(x)$ [1,13]. The corrected Boole polynomial solution $y_{N,M}(x) = y_N(x) + e_{N,M}(x)$. The corrected Boole error function is obtained with the Boole error function $e_N(x)$ and the estimated error function $e_{N,M}(x)$ as follows

$$E_{N,M}(x) = e_N(x) - e_{N,M}(x) = y_N(x) - y_{N,M}(x).$$

5. Numerical examples

In order to demonstrate the applicability and validity of the numerical method developed in this section, first exact solution example and then approximate solution examples are given.

Example 1. In the first example, the exact solution of the linear Volterra integro differential equation given by

$$2xy''(x) - xy'(x) - y(x) = \frac{x^5}{3} - x^4 + x^3 - 3x^2 + 4x - 1 - \int_0^x x^2 y(t) dt, \qquad (\mathbf{5.1})$$

with the initial conditions $y(0) = 1$ and $y'(0) = -2$ is obtained in the interval $0 \leq x, t \leq 1$. The approximate solution $y(x)$ by the truncated Boole series

$$y(x) = \sum_{n=0}^{3} a_n R_n(x),$$

where $m = 2, N = 3, a = 0, b = 1, P_0(x) = -1$, $P_1(x) = -x, P_2(x) = 2x$, $g(x) = \frac{x^5}{3} - x^4 + x^3 - 3x^2 + 4x - 1$, $\lambda = -1$ and $K(x,t) = x^2$. The collocation points (3.2) for $N = 3$, $a = 0$ and $b = 1$ are calculated as

$$\left\{ x_0 = 0, x_1 = \frac{1}{3}, x_2 = 1, x_3 = \frac{2}{3}, x_4 = 1 \right\}.$$

The fundamental matrix equation of the Eq. (5.1) is written as

$$\left[\mathbf{P_2 X E^2 H^T} + \mathbf{P_1 X E^1 H^T} + \mathbf{P_0 X E^0 H^T} - \lambda\, \mathbf{\overline{X} K \overline{C} \overline{H^T}}\right]\mathbf{A} = \mathbf{G},$$

where

$$\mathbf{P}_0 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}_{4x4},$$

$$\mathbf{P}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}_{4x4},$$

$$\mathbf{P}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}_{4x4}, \mathbf{E} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{4x4},$$

$$\mathbf{H}^T = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{3}{4} \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -\frac{9}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4x4}, \overline{\mathbf{H}^T} = \begin{bmatrix} \mathbf{H}^T \\ \mathbf{H}^T \\ \mathbf{H}^T \\ \mathbf{H}^T \end{bmatrix}_{16x4},$$

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$$\mathbf{K}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{bmatrix}_{4x4},\ \bar{\mathbf{K}}=\begin{bmatrix}\mathbf{K}&0&0&0\\0&\mathbf{K}&0&0\\0&0&\mathbf{K}&0\\0&0&0&\mathbf{K}\end{bmatrix}_{16x16},$$

$$\bar{\mathbf{C}}=\begin{bmatrix}\mathbf{C}(0)&0&0&0\\0&\mathbf{C}\left(\frac{1}{3}\right)&0&0\\0&0&\mathbf{C}\left(\frac{2}{3}\right)&0\\0&0&0&\mathbf{C}(1)\end{bmatrix}_{16x16},$$

$$\mathbf{R}=\begin{bmatrix}\mathbf{R}(0)\\\mathbf{R}\left(\frac{1}{3}\right)\\\mathbf{R}\left(\frac{2}{3}\right)\\\mathbf{R}(1)\end{bmatrix}_{4x1},\mathbf{G}=\begin{bmatrix}g(0)\\g\left(\frac{1}{3}\right)\\g\left(\frac{2}{3}\right)\\g(1)\end{bmatrix}_{4x1}\text{ and}$$

$$\bar{\mathbf{X}}=\begin{bmatrix}\mathbf{X}(0)&0&0&0\\0&\mathbf{X}\left(\frac{1}{3}\right)&0&0\\0&0&\mathbf{X}\left(\frac{2}{3}\right)&0\\0&0&0&\mathbf{X}(1)\end{bmatrix}_{4x16}.$$

The fundamental matrix relation is calculated, the augmented matrix is obtained as

$$[\mathbf{W};\mathbf{G}]=\begin{bmatrix}-1&\frac{1}{2}&-\frac{1}{2}&\frac{3}{4}&;&-1\\-\frac{26}{27}&-\frac{29}{162}&\frac{1342}{729}&-\frac{17207}{2916}&;&\frac{991}{729}\\-\frac{19}{27}&-\frac{143}{162}&\frac{5095}{1458}&-\frac{22373}{2916}&;&\frac{2291}{729}\\0&-\frac{3}{2}&\frac{13}{3}&-\frac{21}{4}&;&\frac{13}{3}\end{bmatrix}.$$

According to Eq. (3.6), the matrix form of the initial conditions is found as

$[\mathbf{U_0};\ \lambda_0]=\begin{bmatrix}1&-\frac{1}{2}&\frac{1}{2}&-\frac{3}{4}&;&1\end{bmatrix}$ and $[\mathbf{U_1};\ \lambda_1]=[0\quad 1\quad -2\quad 5\quad ;\quad -2]$.

In that case, the new augmented matrix based on conditions is become as

$$[\mathbf{W};\mathbf{G}]=\begin{bmatrix}-1&\frac{1}{2}&-\frac{1}{2}&\frac{3}{4}&;&-1\\-\frac{26}{27}&-\frac{29}{162}&\frac{1342}{729}&-\frac{17207}{2916}&;&\frac{991}{729}\\1&-\frac{1}{2}&\frac{1}{2}&-\frac{3}{4}&;&1\\0&1&-2&5&;&-2\end{bmatrix}.$$

with the solution of this augmented matrix, unknown Boole coefficients are found as

$$\mathbf{A}=\begin{bmatrix}\frac{1}{2}&0&1&0\end{bmatrix}.$$

Finally, found Boole coefficients is placed in the solution (1.5). The Boole solution of the Eq. (5.1) for $N=3$ is gained as $y(x)=x^2-2x+1$ and this is exact solution.

Example 2. In the second example, the approximate solution of the linear Volterra integro differential equation given by

$$y^{(iv)}(x)=\mathrm{x}(1+e^x)+3e^x+y(x)-\int_0^x y(t)dt,\quad 0<x<1 \qquad \mathbf{(5.2)}$$

with initial conditions $y(0)=1, y(1)=1+e, y'(0)=1$ and $y'(1)=2e$ [16]. The exact solution of the problem is $y(x)=1+xe^x$. The results of the exact solution $y(x)$, Boole solution $y_N(x)$ and the corrected Boole solution $y_{N,M}(x)$ for the various values N,M have been calculated in the computer program. The calculated results are given in the Table 1 and compared in the Figure 1 and 2. In addition, the absolute error function $|e_N|$, the estimated error function $|e_{N,M}|$ and the corrected Boole error function $|E_{N,M}|$ have been calculated for the values N,M. The results are given Table 2.

Table 1. The comparison of the exact solution $y(x)=1+xe^x$, Boole solutions $y_N(x)$ and corrected Boole solutions $y_{N,M}(x)$ for the Example 2.

x_i	Exact Solution $y(x)=1+xe^x$	Boole Solution $y_N(x)$		Corrected Boole Solution $y_{N,M}(x)$	
		N=5	N=12	N=5, M=6	N=12, M=13
0	1.0	1.0	1.0	1.0	1.0
0.2	1.244280552	1.243648041	1.244280552	1.244203327	1.244280552
0.4	1.596729879	1.595029888	1.596729879	1.596514071	1.596729879
0.6	2.09327128	2.091228319	2.09327128	2.092994076	2.09327128
0.8	2.780432743	2.779339894	2.780432743	2.78027061	2.780432743
1.0	3.718281828	3.718281828	3.718281828	3.718281828	3.718281828

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Table 2. The comparison of the error function $|e_N|$, the estimated error function $|e_{N,M}|$ and the corrected Boole error function $|E_{N,M}|$ for the Example 2.

x_i	Absolute error function $\|e_N\|$		Estimated error function $\|e_{N,M}\|$		Corrected Boole error function $\|E_{N,M}\|$	
	N=5	N=12	N=5, M=6	N=12, M=13	N=5, M=6	N=12, M=13
0	0	0	0	0	0	0
0.2	6.3251e-04	3.2300e-12	5.5529e-04	3.0890e-12	7.7224e-05	1.4105e-13
0.4	1.7000e-03	9.4901e-12	1.4842e-03	9.0739e-12	2.1581e-04	4.1622e-13
0.6	2.0430e-03	1.3640e-11	1.7658e-03	1.3037e-11	2.7720e-04	6.0285e-13
0.8	1.0928e-03	1.0559e-11	9.3072e-04	1.0082e-11	1.6213e-04	4.7673e-13
1.0	0	0	3.5293e-22	1.9590e-28	0	0

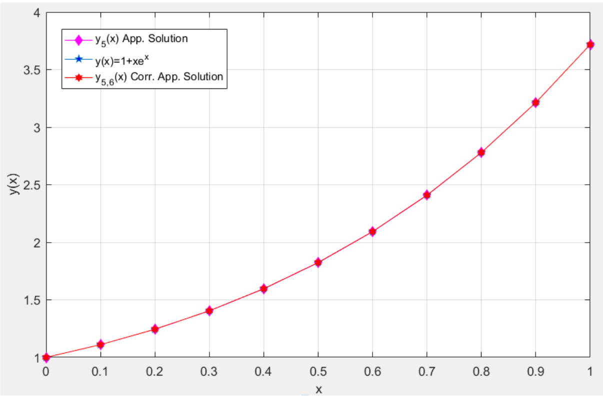

Figure 1. The comparison of the exact solution $y(x) = 1 + xe^x$, Boole solution $y_N(x)$ and corrected Boole solution $y_{N,M}(x)$ for the values $N, M = 5, 6$ for the Example 2.

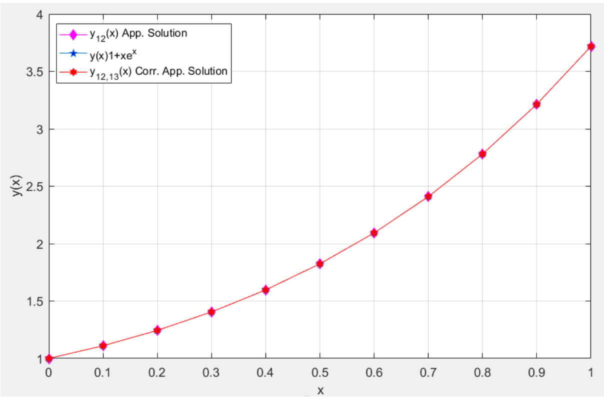

Figure 2. The comparison of the exact solution $y(x) = 1 + xe^x$, Boole solution $y_N(x)$ and corrected Boole solution $y_{N,M}(x)$ for the values $N, M = 12, 13$ for the Example 2.

Example 3. Finally, the linear Volterra integro differential equation given by

$$y^{u}(x) + y(x) + \int_0^x xtan(t)y(t)dt = x(1 - cosx),$$
$$x \in [0,1] \qquad \mathbf{(5.3)}$$

with initial conditions $y(0) = 1$, $y(1) = cos1$ and the analytical solution $y(x) = cosx$ [25]. The absolute error function $|e_N|$, the estimated error function $|e_{N,M}|$ and the corrected Boole error function $|E_{N,M}|$ have been calculated in the computer program for the various values N, M and the results are given in the Table 3. Also, the results of the exact solution $\mathrm{y(x) = cosx}$, Boole solution $\mathrm{y_N(x)}$ and corrected Boole solution $\mathrm{y_{N,M}(x)}$ for the various values N, M have been compared in the Figure 3 and 4.

Table 3. The comparison of the error function $|e_N|$, the estimated error function $|e_{N,M}|$ and the corrected Boole error function $|E_{N,M}|$ for the Example 3.

x_i	Absolute error function $\|e_N\|$		Estimated error function $\|e_{N,M}\|$		Corrected Boole error function $\|E_{N,M}\|$	
	N=4	N=10	N=4, M=5	N=10, M=11	N=4, M=5	N=10, M=11
0	0	0	0	0	0	0
0.2	1.3042e-04	7.5296e-07	2.7030e-04	8.8365e-12	4.0072e-04	7.5297e-07
0.4	2.5308e-04	1.4758e-06	5.3168e-04	8.3035e-13	7.8476e-04	1.4758e-06
0.6	3.6707e-04	2.1365e-06	7.4133e-04	6.3750e-13	1.1084e-03	2.1365e-06
0.8	3.8882e-04	2.5889e-06	7.1303e-04	4.8659e-12	1.1019e-03	2.5889e-06
1.0	0	0	2.6470e-23	1.0500e-24	0	3.6825e-56

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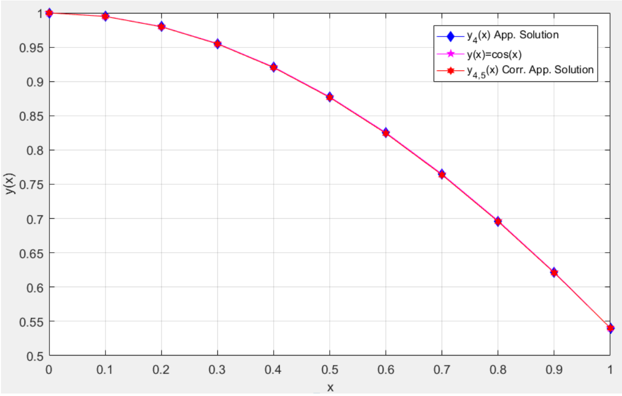

Figure 3. The comparison of the exact solution $y(x) = cosx$, Boole solution $y_N(x)$ and corrected Boole solution $y_{N,M}(x)$ for the values $N, M = 4,5$ for the Example 3.

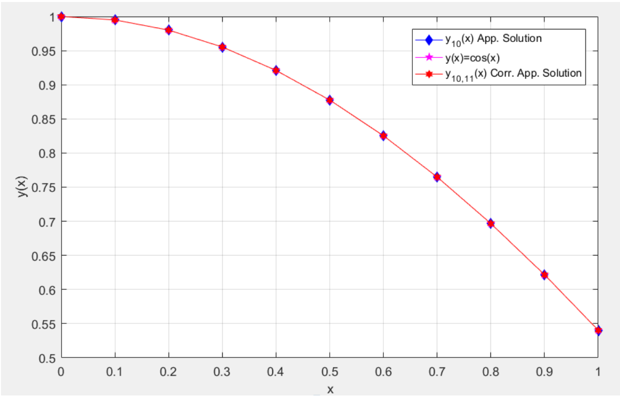

Figure 4. The comparison of the exact solution $\mathrm{y(x)} = \mathrm{cosx}$, Boole solution $\mathrm{y_N(x)}$ and corrected Boole solution $\mathrm{y_{N,M}(x)}$ for the values $\mathrm{N, M} = 10, 11$ for the Example 3.

Author's Contributions

Kübra Erdem Biçer Drafted and wrote the manuscript.

Ethics

There are no ethical issues after the publication of this manuscript.

6. Conclusions

In this study, a new method was developed by using the Boole polynomial to find the solution of Volterra integro differential equation. Numerical results were obtained with the developed this method. The method was used in the exact solution and approximate solution the examples. The exact solution of Example 1 was obtained the using the present method for value $N = 3$. The exact solutions, Boole solutions and corrected Boole solutions of the Example 2 and 3 were gained for various values N, M. In addition, the error estimations based on residual function of the Example 2 and 3 have been calculated. According to the tables and figures, the good results were obtained with this method. The advantage of present method is the solutions and calculations can be obtained easily the using computer code in MATLAB program. The present method can be develop to find solutions of other integro differential equation classes.

References

1. Erdem, K, Yalçınbaş, S, Sezer, M. 2013. A Bernoulli polynomial approach with residual correction for solving mixed linear Fredholm integro-differentialdifference equations. *Journal of Difference Equations and Applications*; 19: 1619-1631.

2. Laib, H, Bellour, A, Bousselsal, A. 2019. Numerical solution of high-order linear Volterra integro-differential equations by using Taylor collocation method. *International Journal of Computer Mathematics*; 96 (5): 1066–1085.

3. Chen, J, He, M, Zeng, T. 2019. A multiscale Galerkin method for second-order boundary value problems of Fredholm integro-differential equation II: Efficient algorithm for the discrete linear system. *J. Vis. Commun. Image R.*; 58: 112–118.

4. Hesameddini, E, Shahbazi, M. 2019. Solving multipoint problems with linear Volterra–Fredholm integro-differential equations of the neutral type using Bernstein polynomials Method. *Applied Numerical Mathematics;* 136: 122–138.

5. Rohaninasab, N, Maleknejad, K, Ezzati, R. 2018. Numerical solution of high-order Volterra–Fredholm integro-differential equations by using Legendre collocation method. *Applied Mathematics and Computation*; 328: 171–188.

6. Wang, Y, Zhu, L. 2017. Solving nonlinear Volterra integro-differential equations of fractional order by using Euler wavelet method. *Advances in Difference Equations*; 2017(1): 27.

7. Babayar-Razlighi, B, Soltanalizadeh, B. 2013. Numerical solution for system of singular nonlinear Volterra integro-differential equations by Newton-Product method. *Applied Mathematics and Computation;* 219: 8375–8383.

8. Roul, P, Meyer, P. 2011. Numerical solutions of systems of nonlinear integro-differential equations by Homotopy-perturbation method. *Applied Mathematical Modelling;* 35: 4234–4242.

9. Yüzbaşı, Ş. 2016. Improved Bessel collocation method for linear Volterra integro-differential equations with piecewise intervals and application of a Volterra population model. *Applied Mathematical Modelling*; 40: 5349–5363.

10. Gu, Z. 2019. Spectral collocation method for weakly singular Volterra integro-differential equations. *Applied Numerical Mathematics*; 143: 263–275.

11. Yuzbasi, S, Sahin, N, Sezer, M. 2011. Bessel matrix method for solving high-order linear Fredholm integro-differential equations, Journal of Advanced Research in Applied Math.; 3(23): 1-25.

12. Hosseini, SM, Shahmorad, S. 2003. Tau numerical solution of Fredholm integro-differential equations with arbitrary polynomial bases. *Applied Mathematical Modelling;* 27: 145–154.

13. Erdem, Biçer, K, Sezer, M. 2017. Bernoulli matrix-collocation method for solving general functional integro- differential equations with Hybrid delays. *Journal of Inequalities and Special Functions*; 8(3): 85-99.

14. Turkyilmazoglu, M. 2014. An effective approach for numerical solutions of high-order Fredholm integro-differential equations. *Appl. Math. Comput*; 227: 384–398.

15. Kürkçü, ÖK, Aslan, E, Sezer, M. 2016. A numerical approach with error estimation to solve general integro-differential–difference equations using Dickson polynomials. *Applied Mathematics and Computation*; 276: 324–339

16. Yıldırım, A. 2008. Solution of BVPs for fourth-order integro-differential equations by using homotopy perturbation method. *Computers and Mathematics with Applications*; 56: 3175-3180.

17. Mirzaee, F, Hoseini, SF. 2017. A new collocation approach for solving systems of high-order linear Volterra integro-differential equations with variable coefficients. *Applied Mathematics and Computation;* 311: 272–282.

18. Yalçınbaş, S, Erdem, K. 2014. A New Approximation Method for the Systems of Nonlinear Fredholm Integral Equations. *Applied Mathematics and Physic;* 2(2): 40-48.

19. Mustafa, MM, Muhammad, AM. 2014. Numerical Solution of Linear Volterra-Fredholm Integro-Differential Equations Using Lagrange Polynomials. *Mathematical Theory and Modelin;* 4(5): 137-146.

20. Dastjerdi, HL, Maalek, Ghaini FM. 2012. Numerical solution of Volterra–Fredholm integral equations by moving least square method and Chebyshev polynomials. *Applied Mathematical Modelling*; 36: 3283–3288.

21. Maleknejad, K, Basirat, B, Hashemizadeh E. 2012. A Bernstein operational matrix approach for solving a system of high order linear Volterra–Fredholm integro-differential equations. *Mathematical and Computer Modelling;* 55: 1363–1372.

22. Yüzbaşı, Ş, Şahin, N, Yildirim, A. 2012. A collocation approach for solving high-order linear Fredholm–Volterra integro-differential equations. *Mathematical and Computer Modelling*; 55: 547–563.

23. Rashidinia, J, Tahmasebi, A. 2013. Approximate solution of linear integro-differential equations by using modified Taylor expansion method. *World Journal of Modelling and Simulation*; 9(4): 289-301.

24. Erdem, Bicer, K, Yalcinbas, S. 2016. A Matrix Approach to Solving Hyperbolic Partial Differential Equations Using Bernoulli Polynomials, *Published by Faculty of Sciences and Mathematics*, 30(4): 993–1000.

25. Cravero, I, Pittaluga, G, Sacripante, L. 2012. An algorithm for solving linear Volterra integro-differential equations. *Numer Algor;* 60: 101–114,

26. Jordan, C. Calculus of Finite Differences; Chelsea Publishing Company: New York, 1950; pp 318.

27. Baykus, Savasaneril, N, Sezer, M. 2016. Laguerre polynomial solution of high- order linear Fredholm integro-differential equations. *NTMSCI;* 4(2), 273-284.

28. Yüzbaşı, Ş, Şahin, N, Sezer, M. 2011. Bessel polynomial solutions of high-order linear Volterra integro-differential equations. *Computers and Mathematics with Applications;* 62: 1940–1956.

29. Balcı, MA, Sezer, M. 2016. Hybrid Euler-Taylor matrix method for solving of generalized linear Fredholm integro-differential difference equations. *Applied Mathematics and Computation*; 273: 33–41.

30. Roman, S., The Umbral Calculus; ACADEMIC PRESS: New York, 1984; pp 12.

31. Kim, DS. 2014. A note on Boole polynomials. *Integral Transforms and Special Functions,* 25(8): 627-633.