

**A NEW SUBCLASS OF UNIVALENT FUNCTIONS CONNECTED
WITH CONVOLUTION DEFINED VIA EMPLOYING A LINEAR
COMBINATION OF TWO GENERALIZED DIFFERENTIAL
OPERATORS INVOLVING SIGMOID FUNCTION**

OYEKAN EZEKIEL ABIODUN^{1*}, AWOLERE IBRAHIM TUNJI²

^{1*,2}DEPARTMENT OF MATHEMATICAL SCIENCES, FACULTY OF SCIENCE,
OLUSEGUN AGAGU UNIVERSITY OF SCIENCE AND TECHNOLOGY (OAUSTECH),
P.M.B. 353, OKITIPUPA, ONDO STATE, NIGERIA.

¹ORCID: [HTTPS://ORCID.ORG/0000-0001-8660-7804](https://orcid.org/0000-0001-8660-7804),

²ORCID: [HTTPS://ORCID.ORG/0000-0002-0771-8037](https://orcid.org/0000-0002-0771-8037)

ABSTRACT. By introducing an operator $E_{\mu}^n(\beta, \lambda, \omega, \varphi, t)f_{\gamma}(z)$ via a linear combination of two generalized differential operators involving modified Sigmoid function, we defined and studied certain geometric properties of a new subclass $T_{\gamma}D_{\lambda, \omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, t; p : n)$ of analytic functions in the open unit disk U . In particular, we give some properties of functions in this subclass such as; coefficient estimates, growth and distortion theorems, closure theorem and Fekete-Szego inequality for functions belonging to the subclass. Some earlier known results are special cases of results established for the new subclass defined.

1. INTRODUCTION AND PRELIMINARIES

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk. In the usual notation, let A denote the class of functions $f(z)$ which are analytic in the open unit disk and of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1.1)$$

which is analytic in the open unit disk U and let

$$\gamma(s) = \frac{2}{(1 + e^{-s})}; \quad s \geq 0 \quad (1.2)$$

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with $\gamma(s) = 1$ for $s = 0$ be the modified Simoid function. (See details in [1, 2, 3, 4, 5]).

Also, we denote by T the class of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0 \tag{1.3}$$

as a subclass of A .

The class T of functions with negative coefficients from second term was first introduced by Silverman [6] and has since then opened up a prolific line of research in that direction among function theorists.

For $f_\gamma(z) \in T_\gamma$, Oluwayemi and Fadipe-Joseph [5] gave the following definition:

$$f_\gamma(z) = z - \sum_{k=2}^{\infty} \gamma(s) a_k z^k, \quad a_k \geq 0 \tag{1.4}$$

as a consequence of (1.3). We note that $\gamma(s) = 1 + \frac{1}{2}s - \frac{1}{24}s^3 + \frac{1}{240}s^5 - \frac{17}{40320}s^7 + \dots$ defined by (1.2). Furthermore, we define identity function for T_γ as

$$e_\gamma(z) = z. \tag{1.5}$$

2. DIFFERENTIAL OPERATORS

2.1. Salagean Differential Operator.

Definition 2.1. [7] For $f \in A, n \in \mathbb{N}$, the operator D^n is defined by $D^n : A \rightarrow A$.

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1 f(z) &= z f'(z) \\ D^{n+1} f(z) &= z(D^n f(z))', \quad z \in U \end{aligned} \tag{2.1}$$

Remark 1 : If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in A$, then

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad z \in U \tag{2.2}$$

2.2. Al-Oboudi Differential Operator.

Definition 2.2. [8] For $f \in A, n \in \mathbb{N} \cup 0$, the Al-Oboudi differential operator D_δ^n is defined by $D_\delta^n : A \rightarrow A$.

$$\begin{aligned} D^0 f(z) &= f(z) \\ D_\delta^1 f(z) &= (1 - \delta)f(z) + \delta z f'(z) = D_\delta f(z) \\ &\dots \\ D_\delta^n f(z) &= D_\delta(D_\delta^n f(z)), \quad z \in U. \end{aligned} \tag{2.3}$$

Remark 2 : If D_δ^n is a differential operator and for $f \in A$:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

we have

$$D_\delta^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\lambda]^n a_k z^k, z \in U \quad (2.4)$$

and

$$(D_\delta^{n+1} f(z))' = (D_\delta^n f(z))' + \delta z (D_\delta^{n+1} f(z))'', \quad z \in U. \quad (2.5)$$

When $\delta = 1$, we get the Salagean differential operator (2.2).

2.3. Opoola New Differential Operator.

Definition 2.3. [9] For a function $f \in A$ with

$$D_t f(z) = 1 + \sum_{k=2}^{\infty} [1 + (k + \beta - \mu - 1)t]^n a_k z^{k-1}, \quad 0 \leq \mu \leq \beta, t \geq 0.$$

Opoola defined the differential operator $D^n(\mu, \beta, t)f(z)$ such that

$$\begin{aligned} D^0(\mu, \beta, t)f(z) &= f(z) \\ D^1(\mu, \beta, t)f(z) &= zD_t f(z) = zt f'(z) - zt(\mu - \beta) + (1 + (\mu - \beta - 1)t)f(z) \quad (2.6) \\ &\dots \\ D^n(\mu, \beta, t)f(z) &= zD_t [D^{n-1}(\mu, \beta, t)f(z)], \quad n \in \mathbb{N} \cup 0. \end{aligned}$$

Remark 3 : If $D^n(\mu, \beta, t)f(z)$ is a linear operator such that for $f \in A$,

$$D^n(\mu, \beta, t)f(z) = z + \sum_{k=2}^{\infty} [1 + (k + \beta - \mu - 1)t]^n a_k z^k, \quad z \in U, 0 \leq \mu \leq \beta, t \geq 0. \quad (2.7)$$

It turns out that the differential operator $D^n(\mu, \beta, t)f(z)$ reduces to the Salagean and al-Oboudi differential operators (2.2) and (2.4) respectively for suitably varied parameters and by letting $t = \lambda$.

2.4. Differential Operator Involving Modified Sigmoid Function.

Definition 2.4. [4], [5] Fadipe-Joseph et al. introduced Salagean differential operator $D^n f_\gamma(z)$ involving modified sigmoid function which is defined as follows: Let $f_\gamma(z) \in T_\gamma$, the Salagean differential operator denoted by $D^n f_\gamma(z)$ is defined by

$$\begin{aligned} D^0 f_\gamma(z) &= f_\gamma(z) \\ D^1 f_\gamma(z) &= zD_t f_\gamma(z) \\ &\dots \\ D^n f_\gamma(z) &= D[D^{n-1} f_\gamma(z)] \\ &\gamma(s)z(D^{n-1} f_\gamma(z)), \quad z \in U. \end{aligned} \quad (2.8)$$

Hence,

$$D^n f_\gamma(z) = \gamma^n(s)z + \sum_{k=2}^{\infty} \gamma^{n+1}(s)k^n a_k z^k, \quad z \in U. \tag{2.9}$$

2.5. Darus and Ibrahim Generalized Differential Operator Involving Sigmoid Function.

Definition 2.5. [5] *Oluwayemi and fadipe-Joseph introduced the generalized differential operator $D_{\lambda,\omega}^n f_\gamma(z)$ involving sigmoid function as a consequence of [10] by following (2.9):*

$$D_{\lambda,\omega}^n f_\gamma(z) = \gamma^n(s)z - \sum_{k=2}^{\infty} \gamma^{n+1}(s)[(k-1)(\lambda-\omega) + k]^n a_k z^k \tag{2.10}$$

for $\lambda, \omega \geq 0$. For more information on this, interested reader may see [4] and [10]

2.6. Ruscheweyh Operator Involving Modified Sigmoid Function.

Definition 2.6. [5] *Recently, Oluyemi and Fadipe-Joseph gave a Ruscheweyh Differential operator involving the modified Sigmoid function $R^n : T_\gamma \rightarrow T_\gamma$, with $n \in \mathbb{N} \cup 0$ such that*

$$R^n f_\gamma(z) = z + \sum_{k=2}^{\infty} \gamma(s)C_{n+k-1}^n a_k z^k, \quad a_k \geq 0, \quad z \in U; \tag{2.11}$$

where $\gamma(s) = \frac{2}{1+e^{-s}}$, $s \geq 0$ with $\gamma(s) = 1$ for $s = 0$.

Moreover,

$$\begin{aligned} C_{n+k-1}^n &= B_k(n) = B(n, k) = \binom{n+k-1}{n} \\ &= \frac{(n+1)(n+2) \cdots (n+k-1)}{(k-1)!} \\ &= \frac{(n+1)_{k-1}}{(1)_{k-1}}. \end{aligned} \tag{2.12}$$

Hence, $B(0, k) = \binom{k-1}{0} = \frac{(1)_{k-1}}{(1)_{k-1}} = 1$.

2.7. Linear Combination of a Generalized Salagean Differential Operator and Ruscheweyh Operator involving modified sigmoid function.

Definition 2.7. [5] *By combining the generalized Salagean differential operator involving modified sigmoid and the Ruscheweyh operator involving modified sigmoid function, the following operator was defined in [5] by Oluwayemi and Fadipe-Joseph as:*

$$\begin{aligned} \Phi_{\lambda,\omega}^n f_\gamma(z) &= \mu D_{\lambda,\omega}^n f_\gamma(z) + (1-\mu)R^n f_\gamma(z) \\ &= [\mu\gamma^n(s) - \mu + 1]z - \sum_{k=2}^{\infty} \gamma(s)\{\mu\gamma^n(s)[(k-1)(\lambda-\omega)]^n + (1-\mu)B_k(n)\}a_k z^k, \end{aligned} \tag{2.13}$$

for $\lambda \in [0, 1]$, $\mu \in [0, 1]$, $z \in U$.

Where $D_{\lambda, \omega}^n f_\gamma(z)$ and $R^n f_\gamma(z)$ are defined respectively in (2.10) and (2.11) respectively.

We note the following in respect of given by (2.13):

- (i) Equation (2.13) corrects the one defined for $\Phi_{\lambda, \omega}^n f_\gamma(z)$ in [5].
- (ii) That the operator defined in (2.13) is consequent upon a generalized different operator defined by Darus and Ibrahim [11].

2.8. New Differential Operator Involving Modified Sigmoid Function.

Definition 2.8. Let $f_\gamma(z) \in T_\gamma$, then from (2.7) and (2.11) we obtain a generalized differential operator involving modified sigmoid function as follows:

$$D^n(\varphi, \beta, t)f_\gamma(z) = \gamma^n(s)z + \sum_{k=2}^{\infty} \gamma^{n+1}(s)[1 + (k + \beta - \varphi - 1)t]^n a_k z^k, \quad z \in U, \quad (2.14)$$

for $0 \leq \varphi \leq \beta$, $n \in \mathbb{N} \cup 0, t \geq 0$.

We note here that μ has been replaced by φ for convenience.

2.9. New Differential Operator Involving Sigmoid Defined by Convolution. For the purpose of defining our new differential operator of interest, the following definition is required:

Definition 2.9. (Hadamard product or convolution): The Hadamard (or convolution) of two analytic functions $f(z)$ given by (1.1) and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ is given by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad z \in U. \quad (2.15)$$

Following (2.15) for (2.10) and (2.15), a certain new differential operator involving sigmoid function defined by convolution is defined as follows:

$$\begin{aligned} D_{\lambda, \omega}^n(\varphi, \beta, t)f_\gamma(z) &= (D_{\lambda, \omega}^n f_\gamma(z)) * (D^n(\varphi, \beta, t)f_\gamma(z)) \\ &= \gamma^n(s)z + \sum_{k=2}^{\infty} \gamma^{n+1}(s)[1 + (k + \beta - \varphi - 1)t]^n [(k - 1)(\lambda - \omega) + k]^n a_k z^k \end{aligned} \quad (2.16)$$

2.10. Linear Combination of the New Differential Operator Involving Sigmoid defined by Convolution and Ruscheweyh Operator involving modified sigmoid function. Following (2.15), we combined equations (2.11) and (2.16)

above to obtain a certain operator define as follows:

$$\begin{aligned}
 & E_\mu^n(\beta, \lambda, \omega, \varphi, t) f_\gamma(z) \\
 &= \mu D_{\lambda, \omega}^n(\varphi, \beta, t) f_\gamma(z) + (1 - \mu) R^n f_\gamma(z) \\
 &= [\mu \gamma^n(s) - \mu + 1] z - \sum_{k=2}^{\infty} \gamma(s) \{ \mu \gamma^n(s) [1 + (k + \beta - \varphi - 1)t]^n [(k - 1)(\lambda - \omega) + k]^n \\
 &+ (1 - \mu) B_k(n) \} a_k z^k
 \end{aligned} \tag{2.17}$$

Remark 4 : (i) For $n = 0, \mu = 1$ in (2.17) we have,

$$f_\gamma(z) = z - \sum_{k=2}^{\infty} \gamma(s) a_k z^k, \quad a_k \geq 0,$$

defined by (1.4).

(ii) For $t = 0, \mu = 1$ in (2.17) we have,

$$D^n(\varphi, \beta, t) f_\gamma(z) = \gamma^n(s) z + \sum_{k=2}^{\infty} \gamma^{n+1}(s) [1 + (k + \beta - \varphi - 1)t]^n a_k z^k$$

defined by (2.16).

(iii) For $\mu = 0$ in (2.17) we have,

$$R^n f_\gamma(z) = z - \sum_{k=2}^{\infty} \gamma(s) C_{n+k-1}^n a_k z^k, \quad a_k \leq 0, \quad z \in U$$

defined by (2.11).

(1v) For $t = 0$ in (2.17) we have,

$$\Phi_{\lambda, \omega}^n f_\gamma(z) = [\mu \gamma^n(s) - \mu + 1] z - \sum_{k=2}^{\infty} \gamma(s) \{ \mu \gamma^n(s) [(k - 1)(\lambda - \omega)]^n + (1 - \mu) B_k(n) \} a_k z^k,$$

as defined in (2.13) and corrects the one defined in [5].

(v) For $s = 0, \mu = 1, t = \delta, \beta = \varphi = 0, \lambda = 1$ and $\omega = 2$ in (2.17) we have,

$$D_\delta^n f(z) = z - \sum_{k=2}^{\infty} [1 + (k - 1)\lambda]^n a_k z^k, \quad z \in U$$

which is Al-Oboudi differential operator for function $f \in T$ of the form (1.3).

(vi) For $s = 0, \mu = 1, t = 0, \lambda = \omega = 0$ in (2.17) we have,

$$D^n f(z) = z - \sum_{k=2}^{\infty} k^n a_k z^k, \quad z \in U,$$

which is Salagean differential operator for functions $f \in T$.

In the field of geometric function theory, various subclasses of the normalized analytic functions which are univalent have been studied from different viewpoints. Many authors such as [[3],[4],[5],[10],[11],[12],[13],[14],[15],[16][17],[18],[19],[20],[21],[22]] have successfully defined and studied various subclasses of univalent functions. Particularly, Joshi and Sangle[13] introduced and investigated subclass $T_\gamma D_{\lambda, \omega}(\alpha, \beta, \xi, \mu; n)$

of univalent functions by using Al-Oboudi operator as a generalized Salagean differential operator in the unit disk U . This was followed by the work of Oluwayemi and Fadipe-Joseph[5] wherein they introduced and investigated subclass $T_\gamma D_{\lambda,\omega} f_\gamma(\alpha, \beta, \xi, \mu; p : n)$ by using the generalized differential operator $\Phi_{\lambda,\omega}^n f_\gamma(z)$. The motivation for this present work are the works of both Joshi and Sangle and Oluwayemi and Fadipe-Joseph. In particular, we introduce and investigate the class $T_\gamma D_{\lambda,\omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, t; p : n)$ as a subclass of univalent functions by using the generalized differential operator $E_\mu^n(\beta, \lambda, \omega, \varphi, t) f_\gamma(z)$.

2.11. The class $T_\gamma D_{\lambda,\omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, t; p : n)$.

Definition 2.10. $\{T_\gamma D_{\lambda,\omega}(\alpha, \beta, \omega, \varphi, \lambda, \eta, \xi, t; p : n) : \}$ A function $f_\gamma(z) \in T_\gamma$ defined by (1.4) is in the class $T_\gamma D_{\lambda,\omega}(\alpha, \beta, \omega, \varphi, \lambda, \eta, \xi, t; p : n)$ if

$$\left| \frac{[E_\mu^n(\beta, \lambda, \omega, \varphi, t) f_\gamma(z)]' - [\mu\gamma^n(s) - \mu + 1]}{p\xi[(E_\mu^n(\beta, \lambda, \omega, \varphi, t) f_\gamma(z))]' - [(E_\mu^n(\beta, \lambda, \omega, \varphi, t) f_\gamma(z))]' - [\mu\gamma^n(s) - \mu + 1]}} \right| < \eta,$$

where $\alpha \in [0, \frac{1}{2})$, $\eta \in (0, 1)$, $\frac{1}{2} \leq \xi \leq 1$, $\mu \in [0, 1]$, $0 \geq \varphi \leq \beta$, $n \in \mathbb{N} \cup 0$, $n, t \geq 0$, $p \geq 2$ and $z \in U$.

3. MAIN RESULTS

In this section we find the coefficient estimates for the functions in the class $T_\gamma D_{\lambda,\omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, t; p : n)$. Our main characterization theorem for this function class is stated as Theorem 3.1 below.

3.1. Coefficient Estimates for class $T_\gamma D_{\lambda,\omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, t; p : n)$.

Theorem 3.1. *If a function $f_\gamma(z)$ belongs to the class $T_\gamma D_{\lambda,\omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, t; p : n)$, then*

$$\begin{aligned} & \sum_{k=2}^{\infty} k\gamma(s)[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n[(k - 1)(\lambda - \omega) + k]^n \\ & + (1 - \mu)B_k(n)\}a_k z^k \\ & \leq p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha] \end{aligned}$$

Proof. Suppose $f_\gamma(z) \in T_\gamma D_{\lambda,\omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, t; p : n)$, by equation (2.17) and definition 2.10, we have that

$$\begin{aligned} & \left| - \sum_{k=2}^{\infty} k\gamma(s)\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n[(k - 1)(\lambda - \omega) + k]^n \right. \\ & \left. + (1 - \mu)B_k(n)\}a_k z^{k-1} \right| \\ & \leq \eta \left| - \sum_{k=2}^{\infty} p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]k\gamma(s)[1 + \eta(1 - p\xi)]\{\mu\gamma^n(s) \right. \\ & \left. [1 + (k + \beta - \varphi - 1)t]^n[(k - 1)(\lambda - \omega) + k]^n + (1 - \mu)B_k(n)\}a_k z^{k-1} \right| \end{aligned}$$

$|z| \leq r$ and as $r \rightarrow 1^+$, then

$$\begin{aligned} & \sum_{k=2}^{\infty} k\gamma(s)\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n[(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\}a_k \\ & \leq \eta p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha]+\sum_{k=2}^{\infty}\eta k\gamma(s)(1-p\xi)\{\mu\gamma^n(s) \\ & [1+(k+\beta-\varphi-1)t]^n[(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\}a_k \\ & \Rightarrow \sum_{k=2}^{\infty}k\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n[(k-1)(\lambda-\omega)+k]^n \\ & +(1-\mu)B_k(n)\}a_k \\ & \leq p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha] \end{aligned} \tag{3.1}$$

Hence,

$$\sum_{k=2}^{\infty} a_k \leq \frac{p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha]}{k\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n[(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\}} \tag{3.2}$$

The result is sharp for

$$f(z) = z - \frac{p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha]}{k\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n[(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\}}z^k$$

□

Corollary 3.2. *Let a function $f_\gamma(z) \in T_\gamma D_{\lambda,\omega}(\alpha, \beta, \mu, \omega, 0, \lambda, \eta, \xi, t; p : n)$, then*

$$\begin{aligned} & \sum_{k=2}^{\infty} k\gamma(s)[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\}a_k z^k \\ & \leq p\xi\eta[\mu\gamma^n(s)-\mu+1-\alpha], \end{aligned}$$

which is the correct form of Theorem 3.1 in [5] when $\eta = \beta$.

Corollary 3.3. *Let $s = 0$, then we have that a function $f_\gamma(z)$ belongs to the class $T_1 D_{\lambda,\omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, t; p : n)$, then*

$$\begin{aligned} & \sum_{k=2}^{\infty} k[1+\eta(p\xi-1)]\{\mu\gamma^n(s)[1+(k+\beta-\varphi-1)t]^n[(k-1)(\lambda-\omega)+k]^n \\ & +(1-\mu)B_k(n)\}a_k z^k \\ & \leq p\xi\eta(1-\alpha), \end{aligned}$$

Corollary 3.4. *If $t = 0$, in corollary 3.3, then we have the following:*

Let a function $f_\gamma(z)$ belongs to the class $T_1 D_{\lambda,\omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, 0; p : n)$, then

$$\begin{aligned} & \sum_{k=2}^{\infty} k[1+\eta(p\xi-1)]\{\mu[(k-1)(\lambda-\omega)+k]^n+(1-\mu)B_k(n)\}a_k z^k \\ & \leq p\xi\eta(1-\alpha), \end{aligned}$$

which is Corollary 3.2 in [5] when $\eta = \beta$.

Corollary 3.5. *If $\mu = 1$, in corollary 3.4, then we have the following:*

Let a function $f_\gamma(z)$ belongs to the class $T_1 D_{\lambda, \omega}(\alpha, \beta, 1, \omega, \varphi, \lambda, \eta, \xi, 0; p : n)$, then

$$\begin{aligned} & \sum_{k=2}^{\infty} k[1 + \eta(p\xi - 1)]\{[(k-1)(\lambda - \omega) + k]^n + (1 - \mu)B_k(n)\}a_k z^k \\ & \leq p\xi\eta(1 - \alpha), \end{aligned}$$

which is Corollary 3.3 in [5] when $\eta = \beta$.

4. GROWTH AND DISTORTION THEOREMS FOR THE CLASS

$$T_\gamma D_{\lambda, \omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, t; p : n)$$

Theorem 4.1. *If a function $f_\gamma(z) \in T_\gamma D_{\lambda, \omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, t; p : n)$, then for $|z| \leq r < 1$, we have*

$$r - \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{2[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n[(\lambda - \omega) + 2]^n + (1 - \mu)B_2(n)\}} r^2 \leq |f_\gamma(z)|$$

$$r + \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{2[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n[(\lambda - \omega) + 2]^n + (1 - \mu)B_2(n)\}} r^2$$

and

$$1 - \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n[(\lambda - \omega) + 2]^n + (1 - \mu)B_2(n)\}} r \leq |f_\gamma(z)|$$

$$1 + \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n[(\lambda - \omega) + 2]^n + (1 - \mu)B_2(n)\}} r$$

Proof. Since $f_\gamma(z) \in T_\gamma D_{\lambda, \omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, t; p : n)$, Theorem 3.1 readily yields the inequality

$$\sum_{k=2}^{\infty} a_k \leq \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{2\gamma(s)[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n[(\lambda - \omega) + 2]^n + (1 - \mu)B_2(n)\}} \quad (4.1)$$

Thus, for $|z| = r < 1$, and by making use of (4.2) we have

$$\begin{aligned} |f_\gamma(z)| & \leq |z| + \sum_{k=2}^{\infty} \gamma(s)a_k |z^k| \leq r + \gamma(s)r^2 \sum_{k=2}^{\infty} a_k \\ & \leq r + \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{2\gamma(s)[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n[(\lambda - \omega) + 2]^n + (1 - \mu)B_2(n)\}} r^2 \end{aligned}$$

and

$$\begin{aligned}
 |f_\gamma(z)| &\geq |z| - \sum_{k=2}^{\infty} \gamma(s)a_k|z^k| \geq r - \gamma(s)r^2 \sum_{k=2}^{\infty} a_k \\
 &\geq r - \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{2[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n[(\lambda - \omega) + 2]^n + (1 - \mu)B_2(n)\}} r^2
 \end{aligned}$$

Also from Theorem 3.1, it follows that

$$\begin{aligned}
 &\gamma(s)[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n[(\lambda - \omega) + 2]^n \\
 &+ (1 - \mu)B_2(n)\} \sum_{k=2}^{\infty} ka_k \\
 &\sum_{k=2}^{\infty} k\gamma(s)[1 + (k + \beta - \varphi - 1)t]^n[(\lambda - \omega) + 2]^n + (1 - \mu)B_2(n)\} a_k \\
 &p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha].
 \end{aligned}$$

Hence,

$$\begin{aligned}
 |f'(z)| &\leq 1 + \sum_{k=2}^{\infty} \gamma(s)ka_k|z^k| \leq 1 + \gamma(s)r \sum_{k=2}^{\infty} a_k \\
 &\leq 1 + \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n[(\lambda - \omega) + 2]^n + (1 - \mu)B_2(n)\}} r
 \end{aligned}$$

and

$$\begin{aligned}
 |f'(z)| &\geq 1 - \sum_{k=2}^{\infty} \gamma(s)ka_k|z^k| \geq 1 - \gamma(s)r \sum_{k=2}^{\infty} a_k \\
 &\geq 1 - \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n[(\lambda - \omega) + 2]^n + (1 - \mu)B_2(n)\}} r.
 \end{aligned}$$

This completes the proof of Theorem 4.1. □

4.1. Closure Theorem.

Theorem 4.2. *If a function $f_\gamma(z) \in T_\gamma D_{\lambda,\omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, t; p : n)$. Let $f_1(z) = z$ and*

$$f_\gamma(z) = z - \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{k\gamma(s)[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n[(k - 1)(\lambda - \omega) + k]^n + (1 - \mu)B_k(n)\}} z^k,$$

$k \geq 2$.

Then the function $f_\gamma(z) \in T_\gamma D_{\lambda,\omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, t; p : n)$ if and only if it can be expressed in the form

$$f_\gamma(z) = \sum_{k=2}^{\infty} \mu_k f_k(z), \tag{4.2}$$

where $\mu_k \geq 0$ and $\sum_{k=1}^{\infty} \mu_k = 1$.

Proof. Let $f_\gamma(z) = \sum_{k=1}^{\infty} \mu_k f_k(z)$, $\mu_k \geq 0$, $k = 1, 2, \dots$, and $\sum_{k=1}^{\infty} \mu_k = 1$.

Thus

$$f_\gamma(z) = \sum_{k=1}^{\infty} \mu_k f_k(z) = \mu_1 f_1(z) + \sum_{k=2}^{\infty} \mu_k f_k(z).$$

Therefore,

$$\begin{aligned} f_\gamma(z) &= \mu_1 f_1(z) \\ &+ \sum_{k=2}^{\infty} \mu_k \left\{ z - \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{k\gamma(s)[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n\}} \right\} z^k \\ &= (\mu_1 + \mu_2 + \mu_3 + \dots)z \\ &- \sum_{k=2}^{\infty} \mu_k \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{k\gamma(s)[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n\}} z^k \\ &\mu_1(z) + \mu_2(z) + \mu_3(z) + \dots = \mu_1(z) + \sum_{k=2}^{\infty} \mu_k f_k(z) \end{aligned}$$

where $\mu_1 + \mu_1 + \mu_1 + \dots = \sum_{k=1}^{\infty} \mu_k = 1$. Then

$$f_\gamma(z) = z - \sum_{k=2}^{\infty} \mu_k \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{k\gamma(s)[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n[(k - 1)(\lambda - \omega) + k]^n + (1 - \mu)B_k(n)\}}.$$

It follows that

$$\begin{aligned} &\sum_{k=2}^{\infty} \mu_k \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{k\gamma(s)[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n[(k - 1)(\lambda - \omega) + k]^n + (1 - \mu)B_k(n)\}} \\ &\times \frac{k\gamma(s)[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n[(k - 1)(\lambda - \omega) + k]^n + (1 - \mu)B_k(n)\}}{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]} \\ &\sum_{k=2}^{\infty} \mu_k = 1 - \mu_1 \leq 1. \end{aligned}$$

In other words,

$$f_\gamma(z) = \mu_1 + \sum_{k=2}^{\infty} \mu_k = 1 \Rightarrow 1 - \mu_1 \leq 1.$$

By Theorem 3.1 therefore,

$$f_\gamma(z) \in T_\gamma D_{\lambda, \omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, t; p : n).$$

Conversely, if $f_\gamma(z) \in T_\gamma D_{\lambda, \omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, t; p : n)$, then by Theorem 3.1,

$$a_k \leq \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{k\gamma(s)[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n[(k - 1)(\lambda - \omega) + k]^n + (1 - \mu)B_k(n)\}}$$

By setting

$$\mu_k \leq \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{k\gamma(s)[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n[(k - 1)(\lambda - \omega) + k]^n + (1 - \mu)B_k(n)\}^{a_k}}$$

and

$$\mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k.$$

So that

$$\mu_k = \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]k\gamma(s)[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n + (1 - \mu)B_k(n)\}}{[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n[(k - 1)(\lambda - \omega) + k]^n + (1 - \mu)B_k(n)\}^{a_k} p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]k\gamma(s)}.$$

Consequently, f_k can be expressed in the form (5.1). Hence, $f_\gamma(z) = \sum_{k=1}^{\infty} \mu_k f_k(z)$. Thus the proof is complete. \square

5. FEKETE-SZEGO INEQUALITY FOR THE CLASS

$$T_\gamma D_{\lambda,\omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, t; p : n)$$

In this section, Fekete-Szego inequality for functions $f_\gamma(z)$ belonging to the class $T_\gamma D_{\lambda,\omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, t; p : n)$ was established.

Theorem 5.1. *If a function $f_\gamma(z)$ belongs to the class $T_\gamma D_{\lambda,\omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, t; p : n)$, and $\Delta \in \mathbb{N}$. Then*

$$\left| a_3 - \Delta a_2^2 \right| \leq \left| \frac{AB^2 - \Delta A^2 C}{CB^2} \right|.$$

Proof. From (??),

$$a_k \leq \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{k\gamma(s)[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (k + \beta - \varphi - 1)t]^n + (1 - \mu)B_k(n)\}}, \quad a_k \geq 2. \quad (5.1)$$

From (5.1),

$$a_2 = \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{2\gamma(s)[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (1 + \beta - \varphi)t]^n + (1 - \mu)B_2(n)\}}, \quad (k = 2),$$

and

$$a_3 = \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{3\gamma(s)[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (2 + \beta - \varphi)t]^n + (1 - \mu)B_3(n)\}}, \quad (k = 2).$$

So that

$$a_3 - \Delta a_2^2 = \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{3\gamma(s)[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (2 + \beta - \varphi)t]^n + (1 - \mu)B_3(n)\}} - \Delta \left\{ \frac{p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]}{2\gamma(s)[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (1 + \beta - \varphi)t]^n + (1 - \mu)B_2(n)\}} \right\}^2$$

Such that

$$A = p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha]$$

$$B = \frac{3\gamma(s)[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (2 + \beta - \varphi)t]^n}{[2(\lambda - \omega) + 3]^n + (1 - \mu)B_3(n)}$$

$$C = \frac{2\gamma(s)[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (1 + \beta - \varphi)t]^n}{[(\lambda - \omega) + 2]^n + (1 - \mu)B_2(n)}.$$

Therefore,

$$\left| a_3 - \Delta a_2^2 \right| \leq \left| \frac{AB^2 - \Delta A^2 C}{CB^2} \right|.$$

□

Let $t = 0$ in Theorem 5.1 we have the following:

Corollary 5.2. *If a function $f_\gamma(z)$ belongs to the class $T_\gamma D_{\lambda, \omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, 0; p : n)$, and $\varphi \in \mathbb{N}$. Then*

$$\left| a_3 - \Delta a_2^2 \right| \leq \left| \frac{AB_1^2 - \Delta A^2 C_1}{C_1 B_1^2} \right| = \left| \frac{R\Omega_2^2 - \varphi R^2 \Omega_1}{\Omega_1 \Omega_2^2} \right|.$$

For

$$A = p\xi\eta[\mu\gamma^n(s) - \mu + 1 - \alpha] = R$$

$$B_1 = \frac{3\gamma(s)[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)}{[2(\lambda - \omega) + 3]^n + (1 - \mu)B_3(n)} = \Omega_2$$

$$C_1 = \frac{2\gamma(s)[1 + \eta(p\xi - 1)]\{\mu\gamma^n(s)[1 + (1 + \beta - \varphi)t]^n}{[(\lambda - \omega) + 2]^n + (1 - \mu)B_2(n)} = \Omega_1.$$

Remark 5:

$$\left| a_3 - \Delta a_2^2 \right| \leq \left| \frac{R\Omega_2^2 - \varphi R^2 \Omega_1}{\Omega_1 \Omega_2^2} \right|$$

where $\Delta = \varphi$ is the result in [5] that is due to Oluwayemi and Fadipe-Joseph.

6. CONCLUSION

This work is a generalization of some earlier well-known (defined) differential operators, some of which were illustrated in this work. Particularly in this work, we studied some geometrical properties of functions in the class

$$T_\gamma D_{\lambda, \omega}(\alpha, \beta, \mu, \omega, \varphi, \lambda, \eta, \xi, t; p : n),$$

and when $t = 0$ we obtained the class $T_\gamma D_{\lambda, \omega}(\alpha, \beta, \xi, \mu; p : n)$ studied in [5]. Furthermore, by suitably specializing the parameters involved, we obtained some of the results in [5] as special cases of our own results. Finally, by suitably varying the parameters involved in the results obtained in this new work, one is guaranteed of some other existing results and presumably new ones.

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OYEKAN EZEKIEL ABIODUN,
DEPARTMENT OF MATHEMATICAL SCIENCES, FACULTY OF SCIENCE, OLUSEGUN AGAGU UNIVERSITY
OF SCIENCE AND TECHNOLOGY (OAUSTECH), P.M.B. 353, OKITIPUPA, ONDO STATE, NIGERIA,
PHONE: +2348034772630

E-mail address: shalomfa@yahoo.com, ea.oyekan@oaustech.ed.ng

AWOLERE IBRAHIM TUNJI,
DEPARTMENT OF MATHEMATICAL SCIENCES, FACULTY OF SCIENCE, OLUSEGUN AGAGU UNIVERSITY
OF SCIENCE AND TECHNOLOGY (OAUSTECH), P.M.B. 353, OKITIPUPA, ONDO STATE, NIGERIA,
PHONE: +2348035316832

E-mail address: awolereibrahim01@gmail.com