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BIVARIATE BERNSTEIN POLYNOMIALS THAT REPRODUCE EXPONENTIAL FUNCTIONS

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Abstract. In this paper, we construct Bernstein type operators that reproduce exponential functions on simplex with one moved curved side. The operator interpolates the function at the corner points of the simplex. Used function sequence with parameters α and β not only are gained more modeling flexibility to operator but also satisfied to preserve some exponential functions. We examine the convergence properties of the new approximation processes. Later, we also state its shape preserving properties by considering classical convexity. Finally, a Voronovskaya-type theorem is given and our results are supported by graphics.

1. Introduction

Over the last 60 years, the study of linear approximation has been revealed powerful and important tools in approximation theory, mainly due to their possible applications not only in mathematics but also in other fields such as statistics, engineering and computer science. One of the most vital aspects of the linear approximation is the construction of sequences of linear positive operators to obtain a new approximation process. One of them is Bernstein-type operators on a triangle with a curved side. The operators have been studied extensively and have important applications in many areas such as computer-aided geometric design (see e.g [6] and [7]). In particular, several bivariate extensions of Bernstein operators have been proposed in literature (see for instance Refs., [9], [1], [11] and references therein). Remember that given $n \in \mathbb{N}$, the bivariate Bernstein polynomial of order n on the

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simplex $\mathcal{S} \equiv \left\{(x,y) \in \mathbb{R}^2; x,y \geq 0, x+y \leq 1\right\}$ is given, for $f \in C(\mathcal{S})$, by

$$B_n f(x,y) = \sum_{k=0}^{n} \sum_{l=0}^{n-k} f\left(\frac{k}{n}, \frac{l}{n}\right) p_{n,k,l}(x,y), \tag{1}$$

where

$$p_{n,k,l}(x,y) = \binom{n}{k}\binom{n-k}{l} x^k y^l (1-x-y)^{n-k-l}.$$

To obtain an improvement of the error of convergence in certain subsets of the simplex, in [9], authors generalized the operators $B_n f(x,y)$ as

$$B_{n,\alpha,\beta} f(x,y) := B_n f(t_{n,\alpha}(x), t_{n,\beta}(y)), \tag{2}$$

where

$$t_{n,\gamma}(z) = \frac{-1-\gamma n + \sqrt{(\gamma n+1)^2 + 4n(n-1)(z^2+\gamma z)}}{2(n-1)},$$

with $\gamma \in [0,\infty)$ and $n \in \mathbb{N}, n > 1$. The operators $B_{n,\alpha,\beta} f(x,y)$ fix the polynomials of the form $p_1^2 + \alpha p_1$ and $p_2^2 + \beta p_2$ for $\alpha, \beta \in [0,\infty)$, where $p_0(x,y) = 1$, $p_1(x,y) = x$ and $p_2(x,y) = y$. These operators can be considered as generalization of the operators defined in [8] for the two variable function.

On the other hand, in recent years, there is an increasing interest in modifying linear operators so that the new versions reproduce certain exponential functions. Corresponding modifications of the different operators have been extensively studied nowadays, among the others, we refer the readers to [4], [14], [15], [2], [5]. In [3], the authors proposed the modification of Bernstein operators to reproduce some exponential functions and perform better compared to the classical Bernstein operators, under sufficient conditions. These operators are defined by

$$G_n f(x) = G_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) e^{-\mu k/n} e^{\mu x} p_{n,k}(a_{n,\mu}(x)), x \in [0,1], n \in \mathbb{N}, \tag{3}$$

where

$$a_{n,\mu}(x) = \frac{e^{\mu x/n} - 1}{e^{\mu/n} - 1}.$$

In this paper, motivated by the operators (2) and (3), we modify the operators (1) so that they preserve some exponential functions.

The present work is organized as follows. In the second section, we give definition of a new family of generalized Bernstein operators and their certain elementary properties. In the third section, certain shape preserving properties including generalized convexity for bivariate functions are obtained. Uniform and quantitative type convergence of the mentioned operators and a Voronovskaya type theorem are given in fourth section. In the last two section, we have an inequality showing that the new operator is closer to function f and present examples of graphics supporting the results.

2. The new generalized Bernstein operators

For each integer $n > 1$, let $r_n : (0,\infty) \times [0,1] \to [0,1]$ be the function defined by

$$r_n(\gamma, z) := \frac{e^{\frac{\gamma z}{n}} - 1}{e^{\frac{\gamma}{n}} - 1},$$

for $\gamma \in (0,\infty)$ and $z \in [0,1]$. We introduce a new family of operators as follows.

Definition 1. *Let $\mathcal{S}_{\alpha,\beta} \equiv \left\{(x,y) \in \mathbb{R}^2; x,y \geq 0, r_n(\alpha,x) + r_n(\beta,y) \leq 1\right\} \subset \mathcal{S}$ for each integer n and $\alpha, \beta > 0$. We define the Bernstein operators on $C(\mathcal{S}_{\alpha,\beta})$ as*

$$\mathcal{B}_n^{\alpha,\beta} f(x,y) := \sum_{k=0}^{n} \sum_{l=0}^{n-k} f\left(\frac{k}{n}, \frac{l}{n}\right) p_{n,k,l}^{\alpha,\beta}(x,y),$$

where

$$p_{n,k,l}^{\alpha,\beta}(x,y) = \binom{n}{k}\binom{n-k}{l} r_n(\alpha,x)^k \, r_n(\beta,y)^l \, (1 - r_n(\alpha,x) - r_n(\beta,y))^{n-k-l}.$$

It is obvious that $0 \leq r_n(\alpha,x) \leq x$ and $0 \leq r_n(\beta,y) \leq y$. For each $n > 1$ and $x,y \in [0,1]$, if we accept $r_n(\alpha,x) = x$ and $r_n(\beta,y) = y$ (it can be take as $n \to \infty$), then $\mathcal{B}_n^{\alpha,\beta} f(x,y)$ becomes to $B_n f(x,y)$ on $\mathcal{S}$.

Throughout the paper, $\alpha, \beta > 0$ represent fixed real parameters and $exp_{i,j}^{\alpha,\beta}$ represents the exponential function defined by $exp_{i,j}^{\alpha,\beta}(t_1,t_2) := e^{i\alpha t_1 + j\beta t_2}$ for $i,j = 0,1,2$. The inverse of the exponential function with respect to first variable t_1 is denoted by $\log_\alpha^\beta$ and for second variable t_2, we use the representation $\log_\beta^\alpha$.

Note that, for the $\mathcal{B}_n^{\alpha,\beta} f(x,y)$ to be positive operator, it must be defined on the triangular region $\mathcal{S}_{\alpha,\beta}$ with curved side. This situation is shown in the Figure 1. The equalities

$$r_n(\gamma, 0) = 0 \text{ and } r_n(\gamma, 1) = 1$$

are hold. The bivariate Bernstein operators $\mathcal{B}_n^{\alpha,\beta} f(x,y)$ interpolate $f(x,y)$ at the corner points of the simplex, namely

$$\mathcal{B}_n^{\alpha,\beta} f(0,0) = f(0,0), \quad \mathcal{B}_n^{\alpha,\beta} f(1,0) = f(1,0) \text{ and } \mathcal{B}_n^{\alpha,\beta} f(0,1) = f(0,1).$$

Let $\alpha, \beta \in (0,\infty)$ and $n > 1$. Proceeding as it is usually done for the classical Bernstein polynomials, it is easily attained that

$$\mathcal{B}_n^{\alpha,\beta} \exp_{0,0}^{\alpha,\beta}(x,y) = 1, \quad \mathcal{B}_n^{\alpha,\beta} \exp_{1,0}^{\alpha,\beta}(x,y) = e^{\alpha x}, \quad \mathcal{B}_n^{\alpha,\beta} \exp_{0,1}^{\alpha,\beta}(x,y) = e^{\beta y}, \tag{4}$$

$$\mathcal{B}_n^{\alpha,\beta} \exp_{2,0}^{\alpha,\beta}(x,y) = \left(\left(e^{\frac{\alpha}{n}} + 1\right)\left(e^{\frac{\alpha x}{n}} - 1\right) + 1\right)^n, \tag{5}$$

$$\mathcal{B}_n^{\alpha,\beta} \exp_{0,2}^{\alpha,\beta}(x,y) = \left(\left(e^{\frac{\beta}{n}} + 1\right)\left(e^{\frac{\beta y}{n}} - 1\right) + 1\right)^n. \tag{6}$$

On the other hand, for each $\gamma \in (0,\infty)$, $z \mapsto r_n(\gamma,z)$ is an increasing and convex real function satisfying $r_n(\gamma,0) = 0$, $r_n(\gamma,1) = 1$ and $0 < r_n(\gamma,z) < z < 1$ for $0 < z < 1$. As a direct consequence, for $\alpha,\beta \in (0,\infty)$, $\mathcal{B}_n^{\alpha,\beta}$ is a positive operator which interpolates f at the vertices of $\mathcal{S}_{\alpha,\beta}$.

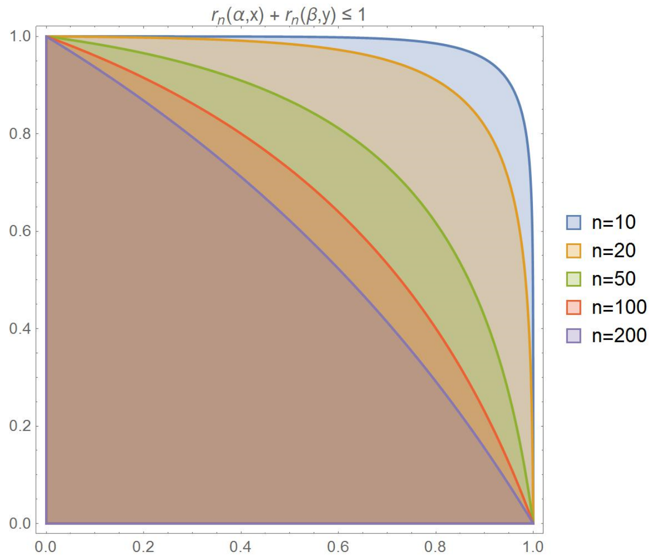

FIGURE 1. The domain areas of $\mathcal{B}_n^{\alpha,\beta}(f;x,y)$ with different values of n.

3. Shape Preserving Properties

In this section, it is convenient to recover these visual shape preserving properties and add some others, in terms of generalized convexities with respect to the functions $\exp_{0,0}^{\alpha,\beta}$, $\exp_{1,0}^{\alpha,\beta}$ and $\exp_{0,1}^{\alpha,\beta}$.

In addition, for use in other shape preserving properties, we first remind a classical definition of convexity for bivariate functions:

For $f \in C(\mathcal{S})$, $(x,y) \in \mathcal{S}_{\alpha,\beta}$ and $h \in \mathbb{R}^+$, we define (whenever it has sense):

$$\begin{aligned}
\Delta_h^{(1,0)} f(x,y) &= f(x+h,y) - f(x,y), \quad \Delta_h^{(0,1)} f(x,y) = f(x,y+h) - f(x,y), \\
\Delta_h^{(1,1)} f(x,y) &= f(x+h,y+h) + f(x,y) - f(x+h,y) - f(x,y+h), \\
\Delta_h^{(2,0)} f(x,y) &= f(x+2h,y) - 2f(x+h,y) + f(x,y), \\
\Delta_h^{(0,2)} f(x,y) &= f(x,y+2h) - 2f(x,y+h) + f(x,y).
\end{aligned}$$

Definition 2. *If for $h \in \mathbb{R}^+$, $\Delta_h^{(i,j)} f \geq 0$, then $f(x,y)$ is convex of order (i,j), $i,j \in \mathbb{N}$, $0 < i+j \leq 2$.*

Now, with the aim of obtaining the shape preserving properties that the operator $\mathcal{B}_n^{\alpha,\beta}$ possesses, we investigate expressions of first two partial derivatives of $\mathcal{B}_n^{\alpha,\beta} f$ according to both x and y. To derive them needs tedious but elementary computation (for details see Ref. [11]).

$$\frac{\partial \mathcal{B}_n^{\alpha,\beta} f}{\partial x}(x,y) = n\frac{\partial r_n(\alpha,x)}{\partial x}\sum_{k=0}^{n-1}\sum_{l=0}^{n-k-1}\left\{f\left(\frac{k+1}{n},\frac{l}{n}\right)-f\left(\frac{k}{n},\frac{l}{n}\right)\right\}p_{n-1,k,l}^{\alpha,\beta}(x,y),$$

$$\begin{aligned}\frac{\partial^2 \mathcal{B}_n^{\alpha,\beta} f}{\partial x^2}(x,y) &= n\frac{\partial^2 r_n(\alpha,x)}{\partial x^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-k-1}\left\{f\left(\frac{k+1}{n},\frac{l}{n}\right)-f\left(\frac{k}{n},\frac{l}{n}\right)\right\}p_{n-1,k,l}^{\alpha,\beta}(x,y)\\ &\quad +n(n-1)\left(\frac{\partial r_n(\alpha,x)}{\partial x}\right)^2\sum_{k=0}^{n-2}\sum_{l=0}^{n-k-2}p_{n-2,k,l}^{\alpha,\beta}(x,y)\\ &\quad \times\left\{f\left(\frac{k+2}{n},\frac{l}{n}\right)-2f\left(\frac{k+1}{n},\frac{l}{n}\right)+f\left(\frac{k}{n},\frac{l}{n}\right)\right\},\end{aligned}$$

$$\frac{\partial \mathcal{B}_n^{\alpha,\beta} f}{\partial y}(x,y) = n\frac{\partial r_n(\beta,y)}{\partial y}\sum_{k=0}^{n-1}\sum_{l=0}^{n-k-1}\left\{f\left(\frac{k}{n},\frac{l+1}{n}\right)-f\left(\frac{k}{n},\frac{l}{n}\right)\right\}p_{n-1,k,l}^{\alpha,\beta}(x,y),$$

$$\begin{aligned}\frac{\partial^2 \mathcal{B}_n^{\alpha,\beta} f}{\partial y^2}(x,y) &= n\frac{\partial^2 r_n(\beta,y)}{\partial y^2}\sum_{k=0}^{n-1}\sum_{l=0}^{n-k-1}p_{n-1,k,l}^{\alpha,\beta}(x,y)\left\{f\left(\frac{k}{n},\frac{l+1}{n}\right)-f\left(\frac{k}{n},\frac{l}{n}\right)\right\}\\ &\quad +n(n-1)\left(\frac{\partial r_n(\beta,y)}{\partial y}\right)^2\sum_{k=0}^{n-2}\sum_{l=0}^{n-k-2}p_{n-2,k,l}^{\alpha,\beta}(x,y)\\ &\quad \times\left\{f\left(\frac{k}{n},\frac{l+2}{n}\right)-2f\left(\frac{k}{n},\frac{l+1}{n}\right)+f\left(\frac{k}{n},\frac{l}{n}\right)\right\}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 \mathcal{B}_n^{\alpha,\beta} f}{\partial x\partial y}(x,y) &= n(n-1)\frac{\partial r_n(\alpha,x)}{\partial x}\frac{\partial r_n(\beta,y)}{\partial y}\sum_{k=0}^{n-2}\sum_{l=0}^{n-k-2}p_{n-2,k,l}^{\alpha,\beta}(x,y)\\ &\quad \times\left\{f\left(\frac{k+1}{n},\frac{l+1}{n}\right)-f\left(\frac{k}{n},\frac{l+1}{n}\right)-f\left(\frac{k+1}{n},\frac{l}{n}\right)+f\left(\frac{k}{n},\frac{l}{n}\right)\right\}.\end{aligned}$$

Also, following equality should not be forgotten:

$$r_n''(\gamma,z)=\frac{\gamma}{n}r_n'(\gamma,z).$$

From these expressions, taking into account the aforementioned properties of the function $z\mapsto r_n(\gamma,z)$, the following results follows:

Proposition 3. *Let $\alpha,\beta\in(0,\infty)$ and $f\in C(\mathcal{S})$.*

1. If $f(x,y)$ is convex of order $(1,0)$ (resp. $(0,1)$), then so is $\mathcal{B}_n^{\alpha,\beta}f$.
2. The convexity of order $(2,0)$ (resp. $(0,2)$) of the function $f(x,y)$ does not imply the one of $\mathcal{B}_n^{\alpha,\beta}f$.
3. If $f(x,y)$ is synchronically positive, decreasing and convex of order $(2,0)$ (resp. $(0,2)$), then $\mathcal{B}_n^{\alpha,\beta}f$ is convexity of order $(2,0)$ (resp. $(0,2)$).

(4) If $f(x,y)$ is simultaneously classical convex of order$(1,0)$ and $(2,0)$ (resp. $(0,1)$ and $(0,2)$), then $\mathcal{B}_n^{\alpha,\beta} f$ is classical convex of order $(2,0)$ (resp. $(0,2)$).
(5) If $f(x,y)$ is convex of order $(1,1)$, then so is $\mathcal{B}_n^{\alpha,\beta} f$.

4. Convergence Properties

We dedicate this section to go through some usual topics related to the convergence of linear approximation processes.

Theorem 4. *Let $\alpha, \beta \in (0,\infty)$ and $f \in C(\mathcal{S})$. Then, we get*

$$\lim_{n\to\infty} \mathcal{B}_n^{\alpha,\beta}(f;x,y) = f(x,y).$$

Proof. It is enough to verify the conditions

$$\lim_{n\to\infty} \mathcal{B}_n^{\alpha,\beta}\left(exp_{i,j}^{\alpha,\beta};x,y\right) = exp_{i,j}^{\alpha,\beta}(x,y)$$

for the pairs of $(i,j) \in \{(0,0),(1,0),(0,1),(2,0),(0,2)\}$. We have previously shown that $\mathcal{B}_n^{\alpha,\beta}\left(exp_{0,0}^{\alpha,\beta};x,y\right) = 1$, $\mathcal{B}_n^{\alpha,\beta}\left(exp_{1,0}^{\alpha,\beta};x,y\right) = e^{\alpha x}$, $\mathcal{B}_n^{\alpha,\beta}\left(exp_{0,1}^{\alpha,\beta};x,y\right) = e^{\beta y}$. For $(0,0),(1,0),(0,1)$,the conditions follows from above equalities. For $(i,j) = (0,2)$ and $(i,j) = (2,0)$, we get

$$\begin{aligned}\lim_{n\to\infty} \mathcal{B}_n^{\alpha,\beta}\left(exp_{2,0}^{\alpha,\beta} + exp_{0,2}^{\alpha,\beta};x,y\right) &= \lim_{n\to\infty}\left[\left(e^{\frac{\alpha}{n}(x+1)} + e^{\frac{\alpha x}{n}} - e^{\frac{\alpha}{n}}\right)^n \right.\\ &\quad \left. + \left(e^{\frac{\beta}{n}(y+1)} + e^{\frac{\beta y}{n}} - e^{\frac{\beta}{n}}\right)^n\right] \\ &= e^{2\alpha x} + e^{2\beta y}\end{aligned}$$

and

$$\sup_{x\in\mathcal{S}_{\alpha,\beta}} \left|\mathcal{B}_n^{\alpha,\beta}\left(\exp_{2,0}^{\alpha,\beta}\right) - e^{2\alpha x}\right| = \sup_{0\le x\le 1} e^{2\alpha x} \sup_{0\le x\le 1}\left|\left(e^{\frac{\alpha}{n}-\frac{\alpha x}{n}} + e^{-\frac{\alpha x}{n}} - e^{\frac{\alpha}{n}-\frac{2\alpha x}{n}}\right)^n - 1\right|.$$

Since the critical point of the function $\left(e^{\frac{\alpha}{n}-\frac{\alpha x}{n}} + e^{-\frac{\alpha x}{n}} - e^{\frac{\alpha}{n}-\frac{2\alpha x}{n}}\right)^n$ is $x_0 = \frac{n}{\alpha}\ln\left(\frac{2e^{\frac{\alpha}{n}}}{1+e^{\frac{\alpha}{n}}}\right)$, we obtain that

$$\sup_{0\le x\le 1}\left|\left(e^{\frac{\alpha}{n}-\frac{\alpha x}{n}} + e^{-\frac{\alpha x}{n}} - e^{\frac{\alpha}{n}-\frac{2\alpha x}{n}}\right)^n - 1\right| = e^{-\alpha}\left(\frac{e^{\frac{\alpha}{n}}+1}{2}\right)^{2n} - 1,$$

$$\begin{aligned}\sup_{x\in\mathcal{S}_{\alpha,\beta}}\left|\mathcal{B}_n^{\alpha,\beta}\left(\exp_{2,0}^{\alpha,\beta}\right) - e^{2\alpha x}\right| &\le e^{2\alpha}\left(e^{-\alpha}\left(\frac{e^{\frac{\alpha}{n}}+1}{2}\right)^{2n} - 1\right) \\ &\to 0 \text{ as } n\to\infty.\end{aligned}$$

Consider the sequence of operators

$$\mathcal{B}_n(f;x,y) = \begin{cases} \mathcal{B}_n^{\alpha,\beta}(f;x,y), & (x,y)\in\mathcal{S}_{\alpha,\beta} \\ f(x,y), & (x,y)\in[0,1]\times[0,1]\setminus\mathcal{S}_{\alpha,\beta}.\end{cases}$$

Then, we obtain

$$\left\|\mathcal{B}_n\left(f;x,y\right)-f\left(x,y\right)\right\|_{C([0,1]\times[0,1])}=\left\|\mathcal{B}_n^{\alpha,\beta}\left(f;x,y\right)-f\left(x,y\right)\right\|_{C(\mathcal{S}_{\alpha,\beta})}. \quad (7)$$

Therefore, we get

$$\lim_{n\to\infty}\left\|\mathcal{B}_n^{\alpha,\beta}\left(exp_{i,j}^{\alpha,\beta};x,y\right)-exp_{i,j}^{\alpha,\beta}\left(x,y\right)\right\|_{C(\mathcal{S}_{\alpha,\beta})}=0$$

for the pairs of $(i,j)\in\{(0,0),(1,0),(0,1),(2,0),(0,2)\}$. Applying Korovkin theorem to sequence $\mathcal{B}_n(f)$, we have

$$\lim_{n\to\infty}\left\|\mathcal{B}_n\left(f;x,y\right)-f\left(x,y\right)\right\|_{C([0,1]\times[0,1])}=0.$$

Therefore, from (7), we have desired result. □

When it comes to quantitative sight, Censor's result yields the estimate

$$\left|B_nf\left(x,y\right)-f\left(x,y\right)\right|\leq\left(1+\frac{x\left(1-x\right)+y\left(1-y\right)}{n\delta^2}\right)\omega\left(f,\delta\right).$$

Herein, $\omega(f,\delta)$ is the bivariate Euclidean modulus of continuity which is defined by

$$\omega\left(f,\delta\right)=\sup\left\{\left|f\left(x_1,y_1\right)-f\left(x_2,y_2\right)\right|:\left(x_i,y_i\right)\in\mathcal{S},\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2\leq\delta\right\}.$$

Theorem 5. *Let $f\in C(\mathcal{S})$. Then, following inequality holds*

$$\begin{aligned}&\left|\mathcal{B}_n^{\alpha,\beta}\left(f;x,y\right)-f\left(x,y\right)\right|\leq\left(1+\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)\\&\times\omega\left(f;\sqrt{\left(\left(e^{\frac{\alpha}{n}}+1\right)\left(e^{\frac{\alpha x}{n}}-1\right)+1\right)^n-e^{2\alpha x}+((e^{\frac{\beta}{n}}+1)(e^{\frac{\beta y}{n}}-1)+1)^n-e^{2\beta y}}\right).\end{aligned}$$

Proof. We have

$$\left|\mathcal{B}_n^{\alpha,\beta}\left(f;x,y\right)-f\left(x,y\right)\right|\leq\left(1+\frac{\mathcal{B}_n^{\alpha,\beta}\left(\left(t_1-x\right)^2+\left(t_2-y\right)^2\right)\left(x,y\right)}{\delta^2}\right)\omega\left(f,\delta\right).$$

Using mean value theorem, we get

$$\begin{aligned}\left|\mathcal{B}_n^{\alpha,\beta}\left(f;x,y\right)-f\left(x,y\right)\right|\leq&\left(1+\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)\right.\\&\left.\times\frac{B_n^{\alpha,\beta}\left(\left(exp_{1,0}^{\alpha,\beta}-e^{\alpha x}\right)^2+\left(exp_{0,1}^{\alpha,\beta}-e^{\beta y}\right)^2\right)\left(x,y\right)}{\delta^2}\right)\\&\times\omega\left(f,\delta\right).\end{aligned}$$

Letting

$$\begin{aligned}\delta^2 &= \mathcal{B}_n^{\alpha,\beta}\left(\left(exp_{1,0}^{\alpha,\beta}-e^{\alpha x}\right)^2+\left(exp_{0,1}^{\alpha,\beta}-e^{\beta y}\right)^2\right)(x,y)\\ &= \mathcal{B}_n^{\alpha,\beta}\left(exp_{2,0}^{\alpha,\beta}\right)(x,y)-e^{2\alpha x}+\mathcal{B}_n^{\alpha,\beta}\left(exp_{0,2}^{\alpha,\beta}\right)(x,y)-e^{2\beta y}\\ &= \left(\left(e^{\frac{\alpha}{n}}+1\right)\left(e^{\frac{\alpha x}{n}}-1\right)+1\right)^n-e^{2\alpha x}+((e^{\frac{\beta}{n}}+1)(e^{\frac{\beta y}{n}}-1)+1)^n-e^{2\beta y},\end{aligned}$$

we have

$$\begin{aligned}&\left|\mathcal{B}_n^{\alpha,\beta}(f;x,y)-f(x,y)\right|\leq\left(1+\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)\\ &\times\omega\left(f;\sqrt{\left(\left(e^{\frac{\alpha}{n}}+1\right)\left(e^{\frac{\alpha x}{n}}-1\right)+1\right)^n-e^{2\alpha x}+((e^{\frac{\beta}{n}}+1)(e^{\frac{\beta y}{n}}-1)+1)^n-e^{2\beta y}}\right).\end{aligned}$$

□

We are going to prove a Voronovskaya-type theorem for $\mathcal{B}_n^{\alpha,\beta}$.

Theorem 6. *Let $f\in C(\mathcal{S})$. Then, we have*

$$\begin{aligned}\lim_{n\to\infty}2n\left(\mathcal{B}_n^{\alpha,\beta}(f;x,y)-f(x,y)\right) &= \left(\frac{\partial^2 f(x,y)}{\partial x^2}-\alpha\frac{\partial f(x,y)}{\partial x}\right)x(1-x)\\ &\quad -2xy\frac{\partial f(x,y)}{\partial y\partial x}\\ &\quad +\left(\frac{\partial^2 f(x,y)}{\partial y^2}-\beta\frac{\partial f(x,y)}{\partial y}\right)y(1-y)\end{aligned}$$

uniformly in $(x,y)\in\mathcal{S}_{\alpha,\beta}$.

Proof. Let $(x,y)\in\mathcal{S}_{\alpha,\beta}$. By the Taylor's theorem, we get

$$\begin{aligned}f(t_1,t_2) &= f(x,y)+\frac{\partial f\left(\log_\alpha^\beta,\cdot\right)\left(e^{\alpha x},e^{\beta y}\right)}{\partial x}\left(e^{\alpha t_1}-e^{\alpha x}\right)\\ &\quad+\frac{\partial f\left(\cdot,\log_\beta^\alpha\right)\left(e^{\alpha x},e^{\beta y}\right)}{\partial y}\left(e^{\beta t_2}-e^{\beta y}\right) \qquad (8)\\ &\quad+\frac{1}{2}\left\{\frac{\partial^2 f\left(\log_\alpha^\beta,\cdot\right)\left(e^{\alpha x},e^{\beta y}\right)}{\partial x^2}\left(e^{\alpha t_1}-e^{\alpha x}\right)^2\right.\\ &\quad+2\frac{\partial^2 f\left(\log_\alpha^\beta,\log_\beta^\alpha\right)\left(e^{\alpha x},e^{\beta y}\right)}{\partial y\partial x}\left(e^{\alpha t_1}-e^{\alpha x}\right)\left(e^{\beta t_2}-e^{\beta y}\right)\\ &\quad\left.+\frac{\partial^2 f\left(\cdot,\log_\beta^\alpha\right)\left(e^{\alpha x},e^{\beta y}\right)}{\partial y^2}\left(e^{\beta t_2}-e^{\beta y}\right)^2\right\}\\ &\quad+\eta(t_1,t_2;x,y)\left\{\left(e^{\alpha t_1}-e^{\alpha x}\right)^2+\left(e^{\beta t_2}-e^{\beta y}\right)^2\right\}, \qquad (9)\end{aligned}$$

where $\eta(t_1, t_2; x, y) \to 0$, as $(t_1, t_2) \to (x, y)$.

Operating $\mathcal{B}_n^{\alpha,\beta}(.; x, y)$ on both sides of (9), we obtain

$$\begin{aligned}
\mathcal{B}_n^{\alpha,\beta}(f; x, y) &= f(x, y) + \frac{\partial f\left(\log_\alpha^\beta, \cdot\right)(e^{\alpha x}, e^{\beta y})}{\partial x}\mathcal{B}_n^{\alpha,\beta}\left(\left(e^{\alpha t_1} - e^{\alpha x}\right); x, y\right) \\
&+ \frac{\partial f\left(\cdot, \log_\beta^\alpha\right)(e^{\alpha x}, e^{\beta y})}{\partial y}\mathcal{B}_n^{\alpha,\beta}\left(\left(e^{\beta t_2} - e^{\beta y}\right); x, y\right) \\
&+ \frac{1}{2}\left\{\frac{\partial^2 f\left(\log_\alpha^\beta, \cdot\right)(e^{\alpha x}, e^{\beta y})}{\partial x^2}\mathcal{B}_n^{\alpha,\beta}\left(\left(e^{\alpha t_1} - e^{\alpha x}\right)^2; x, y\right)\right. \\
&+ 2\frac{\partial^2 f\left(\log_\alpha^\beta, \log_\beta^\alpha\right)(e^{\alpha x}, e^{\beta y})}{\partial y \partial x}\mathcal{B}_n^{\alpha,\beta}\left(\left(e^{\alpha t_1} - e^{\alpha x}\right)\left(e^{\beta t_2} - e^{\beta y}\right); x, y\right) \\
&\left. + \frac{\partial^2 f\left(\cdot, \log_\beta^\alpha\right)(e^{\alpha x}, e^{\beta y})}{\partial y^2}\mathcal{B}_n^{\alpha,\beta}\left(\left(e^{\beta t_2} - e^{\beta y}\right)^2; x, y\right)\right\} \\
&+ \mathcal{B}_n^{\alpha,\beta}\left(\eta(t_1, t_2; x, y)\left\{\left(e^{\alpha t_1} - e^{\alpha x}\right)^2 + \left(e^{\beta t_2} - e^{\beta y}\right)^2\right\}; x, y\right). \quad (10)
\end{aligned}$$

Since

$$\begin{aligned}
\frac{\partial f\left(\log_\alpha^\beta, \cdot\right)(e^{\alpha x}, e^{\beta y})}{\partial x} &= \alpha^{-1}e^{-\alpha x}\frac{\partial f(x, y)}{\partial x}, \\
\frac{\partial^2 f\left(\log_\alpha^\beta, \cdot\right)(e^{\alpha x}, e^{\beta y})}{\partial x^2} &= e^{-2\alpha x}\left(\alpha^{-2}\frac{\partial^2 f(x, y)}{\partial x^2} - \alpha^{-1}\frac{\partial f(x, y)}{\partial x}\right), \\
\frac{\partial^2 f\left(\log_\alpha^\beta, \log_\beta^\alpha\right)(e^{\alpha x}, e^{\beta y})}{\partial y \partial x} &= \alpha^{-1}\beta^{-1}e^{-(\alpha x + \beta y)}\frac{\partial f(x, y)}{\partial y \partial x}
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} n\mathcal{B}_n^{\alpha,\beta}\left(\left(e^{\alpha t_1} - e^{\alpha x}\right)^2; x, y\right) &= \lim_{n\to\infty} n\left(\left(e^{\frac{\alpha}{n}(x+1)} + e^{\frac{\alpha x}{n}} - e^{\frac{\alpha}{n}}\right)^n - e^{2\alpha x}\right) \\
&= -x(x-1)\alpha^2 e^{2\alpha x},
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to\infty} n\mathcal{B}_n^{\alpha,\beta}\left(\left(e^{\alpha t_1} - e^{\alpha x}\right)\left(e^{\beta t_2} - e^{\beta y}\right); x, y\right) &= \lim_{n\to\infty} n\left(\left(e^{\frac{\alpha}{n}x} + e^{\frac{\beta y}{n}} - 1\right)^n - e^{\alpha x + \beta y}\right) \\
&= -xy\alpha\beta e^{\alpha x + \beta y},
\end{aligned}$$

directly from (9), (4), (5), (6), we get

$$\begin{aligned}
\lim_{n\to\infty} 2n\left(\mathcal{B}_n^{\alpha,\beta}(f; x, y) - f(x, y)\right) &= \left(\frac{\partial^2 f(x, y)}{\partial x^2} - \alpha\frac{\partial f(x, y)}{\partial x}\right)x(1-x) \\
&\quad -2xy\frac{\partial f(x, y)}{\partial y \partial x}
\end{aligned}$$

$$+\left(\frac{\partial^2 f(x,y)}{\partial y^2}-\beta\frac{\partial f(x,y)}{\partial y}\right)y(1-y)$$
$$+\lim_{n\to\infty}2n\mathcal{B}_n^{\alpha,\beta}\left(\eta(t_1,t_2;x,y)\right.$$
$$\left.\times\left\{\left(e^{\alpha t_1}-e^{\alpha x}\right)^2+\left(e^{\beta t_2}-e^{\beta y}\right)^2\right\};x,y\right).$$

Now, by applying Cauchy-Schwarz inequality to the last term of (10), we attain

$$\begin{aligned}&\mathcal{B}_n^{\alpha,\beta}\left(\eta(t_1,t_2;x,y)\left\{\left(e^{\alpha t_1}-e^{\alpha x}\right)^2+\left(e^{\beta t_2}-e^{\beta y}\right)^2\right\};x,y\right)\\ \leq\ &\left\{\mathcal{B}_n^{\alpha,\beta}\left(\eta^2(t_1,t_2;x,y);x,y\right)\right\}^{1/2}\\ &\times\left\{\sqrt{\mathcal{B}_n^{\alpha,\beta}\left(\left(e^{\alpha t_1}-e^{\alpha x}\right)^4;x,y\right)}+\sqrt{\mathcal{B}_n^{\alpha,\beta}\left(\left(e^{\beta t_2}-e^{\beta y}\right)^4;x,y\right)}\right\}.\end{aligned}$$

Since $\eta(t_1,t_2;x,y)\to 0$, as $(t_1,t_2)\to(x,y)$, applying Korovkin Theorem, we have

$$\lim_{n\to\infty}\mathcal{B}_n^{\alpha,\beta}\left(\eta^2(t_1,t_2;x,y);x,y\right)=0$$

uniformly in $(x,y)\in\mathcal{S}_{\alpha,\beta}$.

From calculations with Mathematica, we get

$$\mathcal{B}_n^{\alpha,\beta}\left(\left(e^{\alpha t_1}-e^{\alpha x}\right)^4;x,y\right)=O\left(\frac{1}{n^2}\right)\text{ and }\mathcal{B}_n^{\alpha,\beta}\left(\left(e^{\beta t_2}-e^{\beta y}\right)^4;x,y\right)=O\left(\frac{1}{n^2}\right)$$

uniformly in $(x,y)\in\mathcal{S}_{\alpha,\beta}$.

Therefore,

$$2n\mathcal{B}_n^{\alpha,\beta}\left(\eta(t_1,t_2;x,y)\left\{\left(e^{\alpha t_1}-e^{\alpha x}\right)^2+\left(e^{\beta t_2}-e^{\beta y}\right)^2\right\};x,y\right)\to 0,$$

as $n\to\infty$, uniformly in $\mathcal{S}_{\alpha,\beta}$.

Thus, the desired result is obtained. □

5. Comparison with Bernstein Operators

In this section, we compare the operators $\mathcal{B}_n^{\alpha,\beta}(f;x,y)$ with Bernstein operators.

Definition 7. *Let $f\in C^2(\mathcal{S})$.*

i) $f(x,y)$ is $\alpha-$convex of order $(1,0)$ if $\frac{\partial^2 f(x,y)}{\partial x^2}-\alpha\frac{\partial f(x,y)}{\partial x}\geq 0$,
ii) $f(x,y)$ is $\beta-$convex of order $(0,1)$ if $\frac{\partial^2 f(x,y)}{\partial y^2}-\beta\frac{\partial f(x,y)}{\partial y}\geq 0$.

Theorem 8. *Let $f\in C^1(S_{\alpha,\beta})$. Suppose that $f(x,y)$ is $\alpha-$convex of order $(1,0)$, $\beta-$convex of order $(0,1)$ and $(1,1)$ concave. Then, there exists $n_0\in\mathbb{N}$ such that*

$$\mathcal{B}_n^{\alpha,\beta}(f;x,y)\geq f(x,y)$$

for all $n\geq n_0$ and $(x,y)\in S_{\alpha,\beta}$.

Theorem 9. *Let $f \in C^1(S_{\alpha,\beta})$. Suppose that there exists $n_0 \in \mathbb{N}$ such that*

$$f(x,y) \le \mathcal{B}_n^{\alpha,\beta}(f;x,y) \le B_n(f;x,y)$$

for all $n \ge n_0$ and $(x,y) \in S_{\alpha,\beta}$. Then,

$$\frac{\partial^2 f(x,y)}{\partial x^2} \ge \alpha \frac{\partial f(x,y)}{\partial x} \ge 0, \tag{11}$$

$$\frac{\partial^2 f(x,y)}{\partial y^2} \ge \beta \frac{\partial f(x,y)}{\partial y} \ge 0 \tag{12}$$

and

$$\frac{\partial f(x,y)}{\partial y \partial x} \le 0. \tag{13}$$

Conversely, if (11), (12) and (13) hold with strict inequalities at a given point $(x,y) \in S_{\alpha,\beta}$, then there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$

$$f(x,y) < \mathcal{B}_n^{\alpha,\beta}(f;x,y) < B_n(f;x,y).$$

Remark 10. *We can easily see that following equality, if f is $(1,0)$-convex and $(0,1)$-convex, then the functions $\alpha \to \mathcal{B}_n^{\alpha,\beta}(f;x,y)$ and $\beta \to \mathcal{B}_n^{\alpha,\beta}(f;x,y)$ are decreasing:*

$$\begin{aligned}\frac{\partial r_n(\alpha,x)}{\partial \alpha} &= \frac{xe^{\frac{x\alpha}{n}}\left(e^{\frac{\alpha}{n}}-1\right) - e^{\frac{\alpha}{n}}\left(e^{\frac{\alpha x}{n}}-1\right)}{n\left(e^{\frac{\alpha}{n}}-1\right)^2} \le \frac{xe^{\frac{x\alpha}{n}}\left(e^{\frac{\alpha}{n}}-1\right) - e^{\frac{\alpha x}{n}}\left(e^{\frac{\alpha}{n}}-1\right)}{n\left(e^{\frac{\alpha}{n}}-1\right)^2} \\ &= \frac{(x-1)e^{\frac{x\alpha}{n}}\left(e^{\frac{\alpha}{n}}-1\right)}{n\left(e^{\frac{\alpha}{n}}-1\right)^2} \le 0,\end{aligned}$$

$$\frac{\partial \mathcal{B}_n^{\alpha,\beta} f}{\partial \alpha}(x,y) = n\frac{\partial r_n(\alpha,x)}{\partial \alpha}\sum_{k=0}^{n-1}\sum_{l=0}^{n-k-1}\left\{f\left(\frac{k+1}{n},\frac{l}{n}\right) - f\left(\frac{k}{n},\frac{l}{n}\right)\right\} p_{n-1,k,l}^{\alpha,\beta}(x,y).$$

Since $\mathcal{B}_n^{\alpha,\beta}(f;x,y)$ converges uniformly in $S_{\alpha,\beta}$ towards $B_n(f;x,y)$ as $\alpha,\beta \to 0$ and the convergence is decreasing, then $\mathcal{B}_n^{\alpha,\beta}(f;x,y) \le B_n(f;x,y)$. That is the operators $\mathcal{B}_n^{\alpha,\beta}(f;x,y)$ provide a better approximation in a certain sense than the classical Bernstein operator for mentioned class of functions.

6. Graphical and Numerical Analysis

In this section, we give some graphs and numerical examples to show the convergence of $\mathcal{B}_n^{\alpha,\beta}(f;x,y)$ to $f(x,y)$ with the different values of α, β and n.

Let $f(x,y) = e^{x^2+y^2}$. The graphs of $\mathcal{B}_n^{\alpha,\beta}(f;x,y)$ with the different values of n are shown in Figure 2 and different values of a and β are demonsrated in Figure 3.

In Figure 2, we intend to show how the operators approximate to $f(x,y) = e^{x^2+y^2}$ for increasing n. Figure 3 shows that $\mathcal{B}_n^{\alpha,\beta}(f;x,y)$ approximates to $f(x,y) = e^{x^2+y^2}$ for decreasing α and β.

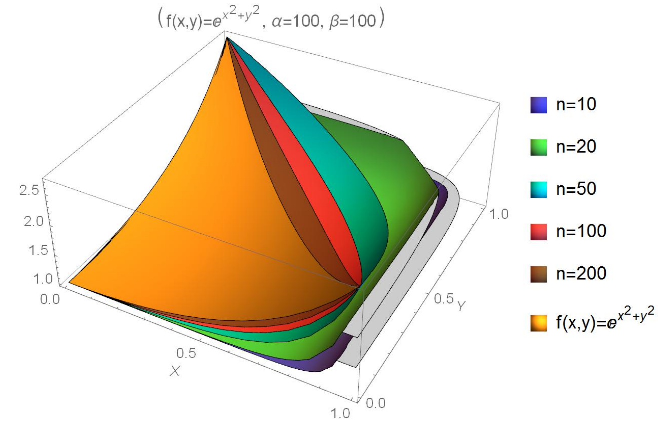

FIGURE 2. The graphic is about how the $\mathcal{B}_n^{\alpha,\beta}(f;x,y)$ approximates to $f(x,y)$ with different values of n for $\alpha = 100$ and $\beta = 100$.

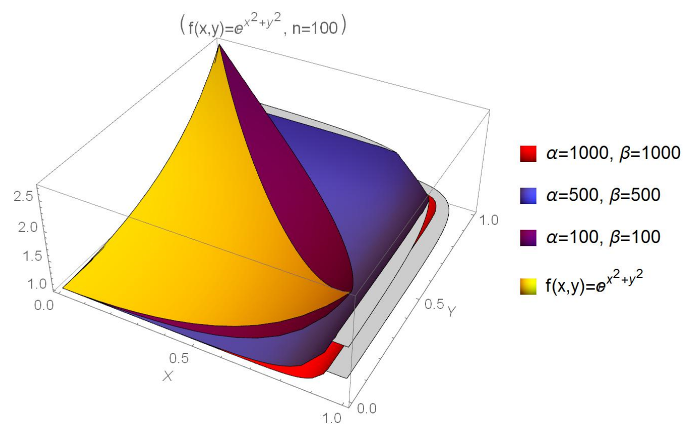

FIGURE 3. The graphs of $\mathcal{B}_n^{\alpha,\beta}(f;x,y)$ with different values of α and β for $n = 100$.

We can see from Table 1 the errors of the operator $\mathcal{B}_n^{\alpha,\beta}(f;x,y)$. In the Table 1, the errors of $\left\|\mathcal{B}_n^{\alpha,\beta}(f) - f\right\|$ for some values of n, α and β are demonstrated.

TABLE 1. The errors of approximation

$\left\|\mathcal{B}_n^{\alpha,\beta}(f) - f\right\|$	$n = 10$	$n = 20$	$n = 50$	$n = 100$	$n = 200$
$\alpha = 1$	0.1869760	0.0929331	0.0370372	0.0184957	0.0092420
$\alpha = 0.5$	0.0170407	0.0085077	0.0033999	0.0016994	0.0008495
$\alpha = 0.1$	0.0003053	0.0001526	0.0000610	0.0000305	0.0000152
$\alpha = 0.05$	0.0000690	0.0000345	0.0000138	0.0000069	0.0000034
$\alpha = 0.01$	0.0000025	0.0000012	0.0000005	0.0000002	0.0000001

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