IKONION JOURNAL OF MATHEMATICS

YEAR: 2020 VOLUME:2 ISSUE: 2

ISSN: 2687-6531

MULTIPLICATIVE VOLTERRA INTEGRAL EQUATIONS AND THE RELATIONSHIP BETWEEN MULTIPLICATIVE DIFFERENTIAL EQUATIONS

Nihan GÜNGÖR *1 and Hatice DURMAZ[1]

[1] Gümüşhane University, Faculty of Engineering and Naturel Sciences, Department of Mathematical Engineering, Gümüşhane, Turkey.

E-mail: nihangungor@gumushane.edu.tr (corresponding author)

(Received:15.09.2020, Accepted: 01.11.2020, Published Online: 05.11.2020)

Abstract

In this study, the multiplicative Volterra integral equation is defined by using the concept of multiplicative integral. The solution of multiplicative Volterra integral equation of the second kind is researched by using the successive approximations method with respect to the multiplicative calculus and the necessary conditions for the continuity and uniqueness of the solution are given. The main purpose of this study is to investigate the relationship of the multiplicative integral equations with the multiplicative differential equations.

Keywords: Multiplicative calculus; Multiplicative differential equations; Multiplicative Volterra integral equations; Successive approximations method.

1. Introduction

Grossman and Katz [10] have built non-Newtonian calculus between years 1967-1970 as an alternative to classic calculus. They have set an infinite family of calculus, including classic, geometric, harmonic, quadratic, bigeometric, biharmonic and biquadratic calculus. Also, they defined a new kind of derivative and integral by using multiplication and division operations instead of addition and subtraction operations. Later, the new calculus that establish in this way is named multiplicative calculus by Stanley [16]. Multiplicative calculus provide different point of view for applications in science and engineering. It is discussed and developed by many researchers. Stanley [16] developed multiplicative calculus, gave some basic theorems about derivatives, integrals and proved infinite products in this calculus. Aniszewska [1] used the multiplicative version of Runge-Kutta method for solving multiplicative differential equations. Bashirov, Mısırlı and Özyapıcı [2] demonstrated some applications and usefulness of multiplicative calculus for the attention of researchers in the branch of analysis. Rıza, Özyapıcı and Mısırlı [14] studied the finite difference methods for the numerical solutions of multiplicative differential equations and Volterra integral equations. Mısırlı and Gurefe [13] developed multiplicative Adams Bashforth-Moulton methods to obtain the numerical solution of multiplicative differential equations. Bashirov, Rıza [4] discussed multiplicative differentiation for complex valued functions and Bashirov, Norozpour [6] extended the multiplicative integral to complex valued functions. Bashirov [5] studied double integrals in the sense of multiplicative calculus. Bhat et al. [7] defined multiplicative Fourier transform and found the solution of multiplicative differential equations by applying multiplicative Fourier transform. Bhat et al. [8] defined multiplicative Sumudu transform and solved some multiplicative differential equations by using multiplicative Sumudu transform. For more details see in [1-10, 13, 14, 16-20].

Integral equations have used for the solution of many problems in applied mathematics, mathematical physics and engineering since the 18th century. The integral equations have begun to enter the problems of engineering and other fields because of the relationship with differential equations and so their importance has increased in recent years. The reader may refer for relevant terminology on the integral equations to [11, 12, 15, 21, 22].

Now, we will give some necessary definitions and theorems in multiplicative calculus as follows:

Definition 1. Let f be a function whose domain is $\mathbb{R}$ the set of real numbers and whose range is a subset of $\mathbb{R}$. The multiplicative derivative of the f at x is defined as the limit

$$\frac{d^*f(x)}{dx} = f^*(x) = \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}}.$$

The limit is also called $*$-derivative of f at x, briefly.

If f is a positive function on an open set $A \subseteq \mathbb{R}$ and its classical derivative $f'(x)$ exists, then its multiplicative derivative also exists and

$$f^*(x) = e^{\left[\frac{f'(x)}{f(x)}\right]} = e^{(ln\circ f)'(x)}$$

where $ln \circ f(x) = lnf(x)$. Moreover, if f is multiplicative differentiable and $f^*(x) \neq 0$, then its classical derivative exists and

$f'(x) = f(x) \cdot lnf^*(x)$ [16].

The multiplicative derivative of f^* is called the second multiplicative derivative and it is denoted by f^{**}. Likewise, the n-th multiplicative derivative can be defined of f and denoted by $f^{*(n)}$ for $n = 0,1,2,\dots$. If n-th derivative $f^{(n)}(x)$ exists, then its n-th multiplicative derivative $f^{*(n)}(x)$ also exists and

$f^{*(n)}(x) = e^{(ln\circ f)^{(n)}(x)},\ n = 0,1,2,\dots$ [2].

Definition 2. The multiplicative absolute value of $x \in \mathbb{R}$ denoted with the symbol $|x|^*$ and defined by

$$|x|_* = \begin{cases} x, & x \geq 1 \\ \frac{1}{x}, & x < 1. \end{cases}$$

Theorem 1. Let f and g be multiplicative differentiable functions. Then the functions $c.f, f.g, f+g, {}^{f}/_{g}, f^g$ are multiplicative differentiable where c is an arbitrary constant and their multiplicative derivative can be shown as

(1) $(cf)^*(x) = f^*(x)$

(2) $(fg)^*(x) = f^*(x)g^*(x)$

(3) $(f+g)^*(x) = f^*(x)^{\frac{f(x)}{f(x)+g(x)}}g^*(x)^{\frac{g(x)}{f(x)+g(x)}}$

(4) $\left(\frac{f}{g}\right)^*(x) = \frac{f^*(x)}{g^*(x)}$

(5) $(f^g)^*(x) = f^*(x)^{g(x)}f(x)^{g'(x)}$

(6) $[f^*(x)]^n = [f^n(x)]^*$ for $n \in \mathbb{R}$ [16].

Theorem 2. *(Multiplicative Mean Value Theorem)* If the function f is continuous on $[a,b]$ and is *-differentiable on (a,b), then there exits $a < c < b$ such that

$$f^*(c) = \left(\frac{f(b)}{f(a)}\right)^{\frac{1}{b-a}}$$ [3].

Definition 3. Let f be a function with two variables, then its multiplicative partial derivatives are defined as

$$\frac{\partial^* f(x,y)}{\partial x} = f_x^*(x,y) = e^{\frac{\partial}{\partial x} ln(f(x,y))} \quad \text{and} \quad \frac{\partial^* f(x,y)}{\partial y} = f_y^*(x,y) = e^{\frac{\partial}{\partial y} ln(f(x,y))}$$ [5].

Theorem 3. (*Multiplicative Chain Rule*) Suppose that f be a function of two variables y and z with continuous multiplicative partial derivatives. If y and z are differentiable functions on (a,b) such that $f\big(y(x),z(x)\big)$ is defined for every $x \in (a,b)$, then

$$\frac{d^* f\big(y(x),z(x)\big)}{dx} = f_y^*\big(y(x),z(x)\big)^{y'(x)} f_z^*\big(y(x),z(x)\big)^{z'(x)}$$ [2].

Definition 4. Let f be a positive function and continuous on the interval $[a,b]$, then it is multiplicative integrable or briefly $*$-integrable on $[a,b]$ and

$$*\int_a^b f(x)^{dx} = e^{\int_a^b \ln(f(x))dx}$$ [16].

Theorem 4. If f and g are integrable functions on $[a,b]$ in the sense of multiplicative, then

(1) $*\int_a^b \big(f(x)^k\big)^{dx} = \left(*\int_a^b \big(f(x)\big)^{dx}\right)^k$

(2) $*\int_a^b \big(f(x)g(x)\big)^{dx} =* \int_a^b \big(f(x)\big)^{dx} * \int_a^b \big(g(x)\big)^{dx}$

(3) $*\int_a^b \left(\frac{f(x)}{g(x)}\right)^{dx} = \frac{*\int_a^b (f(x))^{dx}}{*\int_a^b (g(x))^{dx}}$

(4) $*\int_a^b \big(f(x)\big)^{dx} =* \int_a^c \big(f(x)\big)^{dx} * \int_c^b \big(f(x)\big)^{dx}$

where $k \in \mathbb{R}$ and $a \le c \le b$ [2,3].

Theorem 5. *(Fundamental Theorem of Multiplicative Calculus)* If the function f has multiplicative derivative on $[a,b]$ and f^* is multiplicative integrable on $[a,b]$, then

$$*\int_a^b f^*(x)^{dx} = \frac{f(b)}{f(a)}$$ [2,3].

Definition 5. The equation of the form

$$y^*(x) = f\big(x,y(x)\big)$$

including the multiplicative derivative of y is called first order multiplicative differential equation. It is equivalent to the ordinary differential equation $y'(x) = y(x)\ln f\big(x,y(x)\big)$. Similarly, n-th order multiplicative differential equation is defined by $F\left(x,y,y^*,\dots,y^{*(n-1)},y^{*(n)}(x)\right) = 1,\ (x,y) \in \mathbb{R} \times \mathbb{R}^+$ [2,3]. The equation of the form

$\left(y^{*(n)}\right)^{a_n(x)}\left(y^{*(n-1)}\right)^{a_{n-1}(x)} \dots (y^{**})^{a_2(x)}(y^{*})^{a_1(x)}y^{a_0(x)} = f(x)$

that f is a positive function, is called multiplicative linear differential equation. If the exponentials $a_n(x)$ are constants, then the equation called as multiplicative linear differential equation with constant exponentials; if not it is called as multiplicative linear differential equation with variable exponentials [17].

2. Multiplicative Volterra Integral Equations

An equation in which an unknown function appears under one or more signs of multiplicative integration is called a multiplicative integral equation (MIE), if the multiplicative integral exists. The equation

$$u(x) = f(x) \ * \int_a^x [u(t)]^{K(x,t)^{dt}}$$

where $f(x)$ and $K(x,t)$ are known functions, $u(x)$ is unknown function, is called linear multiplicative Volterra integral equation (LMVIE) of the second kind. The function $K(x,t)$ is the kernel of multiplicative Volterra integral equation. If $f(x) = 1$, then the equation takes the form

$$u(x) = * \int_a^x [u(t)]^{K(x,t)^{dt}}$$

and it is called LMVIE of the first kind.

Example 1. Show that the function $u(x) = e^{2x}$ is a solution of the MVIE $u(x) = e^x * \int_0^x \left[\left(u(t)\right)^{\frac{1}{x}}\right]^{dt}$.

Solution. Substituting the function e^{2x} in place of $u(x)$ into the right side of the equation, we obtain

$$e^x \ * \int_0^x \left[\left(u(t)\right)^{\frac{1}{x}}\right]^{dt} = e^x \ * \int_0^x \left[(e^{2t})^{\frac{1}{x}}\right]^{dt} = e^x \, e^{\int_0^x \ln e^{\frac{2t}{x}} dt} = e^x \, e^{\int_0^x \frac{2t}{x} dt} = e^x \, e^{\frac{2}{x}\cdot\frac{t^2}{2}\Big|_0^x} = e^{2x} = u(x)$$

So, this means that the function $u(x) = e^{2x}$ is a solution of the MVIE.

2.1. The Successive Approximation Method For Solving Multiplicative Volterra Integral Equations

Theorem 6. Consider LMVIE of the second kind as

$$u(x) = f(x) \ * \int_0^x [u(t)]^{K(x,t)^{dt}}. \tag{1}$$

If $f(x)$ is positive and continuous on $[0, a]$ and $K(x,t)$ is continuous on the rectangle $0 \le t \le x$ and $0 \le x \le a$, then there exists an unique continuous solution of (1) as

$$u(x) = \prod_{n=0}^{\infty} \varphi_n(x) = e^{\sum_{n=0}^{\infty} ln\varphi_n(x)}$$

such that the series $\sum_{n=0}^{\infty} ln\varphi_n(x)$ is absolute and uniform convergent where

$$\varphi_0(x) = f(x)\,,\ \varphi_n(x) = * \int_0^x [\,\varphi_{n-1}(t)]^{K(x,t)^{dt}},\ n = 1,2,\dots .$$

Proof: Take the initial approximation as

$$u_0(x) = f(x) = \ \varphi_0(x)\,. \tag{2}$$

If we write $u_0(x)$ instead of $u(x)$ in equation (1), then we get the new function showed with $u_1(x)$ as

$$u_1(x) = f(x) \ast\int_0^x [u_0(t)]^{K(x,t)^{dt}}. \qquad (3)$$

Since the multiplicative integral which is in equation (3) depends on variable x, we can show it with

$$\varphi_1(x) = \ast\int_0^x [u_0(t)]^{K(x,t)^{dt}} = \ast\int_0^x [\varphi_0(t)]^{K(x,t)^{dt}}$$

and write the equation (3) as follow

$$u_1(x) = f(x)\varphi_1(x) = \varphi_0(x)\varphi_1(x) \qquad (4)$$

by using (2). Therefore the third approximation is obtained as

$$u_2(x) = f(x) \ast\int_0^x [u_1(t)]^{K(x,t)^{dt}}.$$

By the equation (4), we find

$$u_2(x) = f(x) \ast\int_0^x [\varphi_0(t).\varphi_1(t)]^{K(x,t)^{dt}}$$

$$= f(x) \ast\int_0^x \left([\varphi_0(t)]^{K(x,t)}[\varphi_1(t)]^{K(x,t)}\right)^{dt}$$

$$= f(x) \ast\int_0^x [\varphi_0(t)]^{K(x,t)^{dt}} \ast\int_0^x [\varphi_1(t)]^{K(x,t)^{dt}}.$$

If we set $\varphi_2(x) = \ast\int_0^x [\varphi_1(t)]^{K(x,t)^{dt}}$, then $u_2(x) = \varphi_0(x)\,\varphi_1(x)\,\varphi_2(x)$. In a similar way, we get

$$u_n(x) = \varphi_0(x)\,\varphi_1(x)\,\varphi_2(x)\ \dots\ \varphi_n(x) \qquad (5)$$

where

$\varphi_0(x) = f(x)$, $\varphi_n(x) = \ast\int_0^x [\,\varphi_{n-1}(t)]^{K(x,t)^{dt}}$, $n = 1,2,\dots$.

Continuing this process, we get the series

$$u(x) = \varphi_0(x)\,\varphi_1(x)\,\varphi_2(x)\ \dots\ \varphi_n(x).\dots = \prod_{n=0}^{\infty}\varphi_n(x) = e^{\sum_{n=0}^{\infty} ln\varphi_n(x)}. \qquad (6)$$

From (5) and (6), it is clear that $\lim_{n\to\infty} u_n(x) = u(x)$.

Assume that $F = \max_{x\in[0,a]} f(x)$ and $K = \max_{0\le t\le x\le a} |K(x,t)|$. Then we find

$$\varphi_0(x) = f(x) \le F$$

$$\varphi_1(x) = \ast\int_0^x [\varphi_0(t)]^{K(x,t)^{dt}} = e^{\int_0^x K(x,t)ln\varphi_0(t)dt} \le e^{\int_0^x |K(x,t)|\,|ln\varphi_0(t)|dt} = e^{\int_0^x |K(x,t)|\,ln|\varphi_0(t)|_\ast dt}$$

$$\le e^{K.lnF.x}$$

$$\varphi_2(x) = \ast\int_0^x [\varphi_1(t)]^{K(x,t)^{dt}} = e^{\int_0^x K(x,t)ln\varphi_1(t)dt} \le e^{K.lnF\int_0^x |K(x,t)|.tdt} \le e^{K^2.lnF\int_0^x tdt} = e^{lnF.K^2.\frac{x^2}{2!}}$$

$\vdots$

$$\varphi_n(x) \le e^{lnF.\frac{K^n x^n}{n!}}.$$

Also, we can see that $|ln\varphi_n(x)| = ln|\varphi_n(x)|_* \leq lnF\ \frac{K^n a^n}{n!}$ for $= 1,2, \dots$. Because of $|ln\varphi_n(x)| \leq lnF\ \frac{K^n a^n}{n!}$, the series $\sum_{n=0}^{\infty} ln\varphi_n(x)$ is absolute and uniform convergence from the Weierstrass M-test. Since each terms of this series are continuous, the function which this series convergences uniformly is continuous. Hence $u(x)$ is continuous function. Now, we will show $u(x)$ is a solution of the equation (1). Since $\varphi_n(x) = *\int_0^x \varphi_{n-1}(t)^{K(x,t)^{dt}}$ and $\varphi_0(x) = f(x)$, we find

$$\varphi_0(x)\prod_{n=1}^{N}\varphi_n(x) = f(x)\prod_{n=1}^{N}\left(*\int_0^x \varphi_{n-1}(t)^{K(x,t)^{dt}}\right)$$

$$\prod_{n=0}^{N}\varphi_n(x) = f(x) * \int_0^x \left(\prod_{n=1}^{N-1}\varphi_{n-1}(t)\right)^{K(x,t)^{dt}}$$

$$\prod_{n=0}^{N}\varphi_n(x) = f(x) * \int_0^x \left(\prod_{n=0}^{N}\varphi_n(t)\right)^{K(x,t)^{dt}} \tag{7}$$

From (6) and (7), we obtain

$$u(x) = \lim_{N\to\infty}\prod_{n=0}^{N}\varphi_n(x) = \lim_{N\to\infty} f(x)\ e^{\int_0^x K(x,t)\sum_{n=0}^{N} ln\varphi_n(t)dt} = f(x)\ e^{\int_0^x K(x,t)\lim_{N\to\infty}\left(\sum_{n=0}^{N} ln\varphi_n(t)\right)dt}$$

$$= f(x)\ e^{\int_0^x K(x,t) ln e^{\sum_{n=0}^{\infty} ln\varphi_n(t)} dt} = f(x)\ e^{\int_0^x K(x,t) lnu(t)dt} = f(x)\ * \int_0^x u(t)^{K(x,t)^{dt}}.$$

by using the uniform convergence of the series $\sum_{n=0}^{\infty} ln\varphi_n(x)$. This indicates that $u(x)$ is the solution of the equation (1). Now, we will show the uniqueness of the solution. Assume that $u(x)$ and $v(x)$ are different solutions of the equation (1). Since

$$u(x) = f(x)\ * \int_0^x [u(t)]^{K(x,t)^{dt}}$$

$$v(x) = f(x) * \int_0^x [v(t)]^{K(x,t)^{dt}}$$

we find

$$\frac{u(x)}{v(x)} = * \int_0^x \left[\frac{u(t)}{v(t)}\right]^{K(x,t)^{dt}} = e^{\int_0^x K(x,t)\left(lnu(t) - lnv(t)\right)dt}.$$

If we set $\frac{u(x)}{v(x)} = \phi(x)$, we can write $\phi(x) = * \int_0^x [\phi(t)]^{K(x,t)^{dt}} = e^{\int_0^x K(x,t) ln\phi(t)dt}$. Because of $ln\phi(x) = \int_0^x K(x,t) ln\phi(t)dt$, we find

$$|ln\phi(x)| = \left|\int_0^x K(x,t) ln\phi(t)dt\right| \leq \int_0^x |K(x,t)||ln\phi(t)|dt \leq K\int_0^x |ln\phi(t)|dt.$$

It is taken as $h(x) = \int_0^x |ln\phi(t)|dt$, then we write

$$|ln\phi(x)| \leq Kh(x)$$

$$|ln\phi(x)| - Kh(x) \leq 0.$$

By multiplication with e^{-Kx} both sides of the inequality, then

$e^{-Kx}|ln\phi(x)| - e^{-Kx}Kh(x) \leq 0$

$\frac{d}{dx}\left(e^{-Kx}h(x)\right) \leq 0$

and by integration both sides of this inequality from 0 to x we find

$e^{-Kx}h(x) - e^{-K0}h(0) \leq 0$

$e^{-Kx}h(x) \leq 0.$

Since $h(x) \leq 0$ and $h(x) \geq 0$, we find $h(x) = 0$. Therefore $|ln\phi(x)| = 0$ for every $x \in [0, a]$, i.e., $\phi(x) = 1$ for every $x \in [0, a]$. Thus $\phi(x) = \frac{u(x)}{v(x)} = 1$ and we obtain $u(x) = v(x)$. This completes the proof.

Remark 1. If the following iterations of method of successive approximations are set by

$u_0(x) = f(x)$

$$u_n(x) = f(x) * \int_0^x [u_{n-1}(t)]^{K(x,t)^{dt}}, \quad n = 1,2,3, \ldots$$

for the multiplicative integral equation

$$u(x) = f(x) * \int_0^x [u(t)]^{K(x,t)^{dt}}$$

where $f(x)$ is positive and continuous on $[0, a]$ and $K(x, t)$ is continuous for $0 \leq x \leq a$, $0 \leq t \leq x$, then the sequence of successive approximations $u_n(x)$ converges to the solution $u(x)$.

Example 2. Solve the multiplicative Volterra integral equation

$$u(x) = e^x * \int_0^x \left[\left(u(t)\right)^{(t-x)}\right]^{dt}$$

with using the successive approximations method.

Solution. Let taken $u_0(x) = e^x$, then the first approximation is obtained as

$$u_1(x) = e^x * \int_0^x \left[(e^t)^{(t-x)}\right]^{dt} = e^x e^{\int_0^x ln\left(e^{t.(t-x)}\right)dt} = e^x e^{\int_0^x t.(t-x)dt} = e^{\left(x - \frac{x^3}{3!}\right)}$$

and by using this approximation it can be obtained as

$$u_2(x) = e^x * \int_0^x \left[\left(e^{\left(t - \frac{t^3}{3!}\right)}\right)^{(t-x)}\right]^{dt} = e^{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)}.$$

By proceeding similarly, the n^{th} approximation is

$$u_n(x) = e^{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}\right)}.$$

Since the expression $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$ is the Maclaurin series of $sinx$, $\lim_{n \to \infty} u_n(x) = e^{sinx}$. Therefore the solution of the equation $u(x) = e^{sinx}$.

3. The Relationship Between Multiplicative Differential Equations

We will investigate the relationship of the multiplicative Volterra integral equations with the multiplicative differential equations.

3.1. The Conversion of the Multiplicative Volterra Integral Equations to Multiplicative Differential Equations

In this section, we demonstrate the method of converting a multiplicative Volterra integral equation into a multiplicative differential equation. For this, we need the Leibniz Formula in the sense of multiplicative calculus.

Firstly, we will give necessary lemma with using proof of multiplicative Leibniz formula.

Lemma 1. Let Ω be an open set in $\mathbb{R}^2$. Suppose that $f:\Omega\to\mathbb{R}$ be a function such that the multiplicative partial derivatives $f_{xy}^{**}(x,y)$, $f_{yx}^{**}(x,y)$ exists in Ω and are continuous, then we have

$$\frac{\partial^*}{\partial x}\left(\frac{\partial^*}{\partial y}f(x,y)\right)=\frac{\partial^*}{\partial y}\left(\frac{\partial^*}{\partial x}f(x,y)\right).$$

Proof. Fix x and y. $F(h,k)$ is taken as

$$F(h,k)=\left(\frac{f(x+h,y+k)\,f(x,y)}{f(x,y+k)\,f(x+h,y)}\right)^{\frac{1}{hk}}$$

By using the multiplicative mean value theorem, we find

$$F(h,k)=\left(\frac{f(x+h,y+k)\,f(x,y)}{f(x,y+k)\,f(x+h,y)}\right)^{\frac{1}{hk}}=\left(\left(\frac{\frac{f(x+h,y+k)}{f(x,y+k)}}{\frac{f(x+h,y)}{f(x,y)}}\right)^{\frac{1}{k}}\right)^{\frac{1}{h}}=\left(\frac{\partial^*}{\partial y}\left(\frac{f(x+h,y+\lambda_1 k)}{f(x,y+\lambda_1 k)}\right)\right)^{\frac{1}{h}}$$

$$=\frac{\partial^*}{\partial y}\left(\left(\frac{f(x+h,y+\lambda_1 k)}{f(x,y+\lambda_1 k)}\right)^{\frac{1}{h}}\right)=\frac{\partial^*}{\partial y}\left(\frac{\partial^*}{\partial x}f(x+\lambda_2 h,y+\lambda_1 k)\right)$$

and

$$F(h,k)=\left(\frac{f(x+h,y+k)\,f(x,y)}{f(x,y+k)\,f(x+h,y)}\right)^{\frac{1}{hk}}=\left(\left(\frac{\frac{f(x+h,y+k)}{f(x+h,y)}}{\frac{f(x,y+k)}{f(x,y)}}\right)^{\frac{1}{h}}\right)^{\frac{1}{k}}=\left(\frac{\partial^*}{\partial x}\left(\frac{f(x+\lambda_3 h,y+k)}{f(x+\lambda_3 h,y)}\right)\right)^{\frac{1}{k}}$$

$$=\frac{\partial^*}{\partial x}\left(\left(\frac{f(x+\lambda_3 h,y+k)}{f(x+\lambda_3 h,y)}\right)^{\frac{1}{k}}\right)=\frac{\partial^*}{\partial y}\left(\frac{\partial^*}{\partial y}f(x+\lambda_3 h,y+\lambda_4 k)\right)$$

for some $0<\lambda_1,\lambda_2,\lambda_3,\lambda_4<1$ which all of them depend on x,y,h,k. Therefore,

$$\frac{\partial^*}{\partial y}\left(\frac{\partial^*}{\partial x}f(x+\lambda_2 h,y+\lambda_1 k)\right)=\frac{\partial^*}{\partial y}\left(\frac{\partial^*}{\partial y}f(x+\lambda_3 h,y+\lambda_4 k)\right)$$

for all h and k. Taking the limit $h,k\to 0$ and using the assumed continuity of both partial derivatives, it gives

$$\frac{\partial^*}{\partial y}\left(\frac{\partial^*}{\partial x}f(x,y)\right)=\frac{\partial^*}{\partial y}\left(\frac{\partial^*}{\partial y}f(x,y)\right).$$

Theorem 7. *(Multiplicative Leibniz Formula)* Let $A, I \subseteq \mathbb{R}$ be open set and f be a continuous function on $A\times I$ into $\mathbb{R}$. If f_x^* exists and is continuous on $A\times I$, $h(x), v(x)$ are continuously differentiable functions of A into I, then we have

$$\frac{d^*}{dx}\left(*\int_{h(x)}^{v(x)} f(x,t)^{dt}\right)=*\int_{h(x)}^{v(x)} f_x^*(x,t)^{dt}\ \frac{f\big(x,v(x)\big)^{v'(x)}}{f\big(x,h(x)\big)^{h'(x)}}$$

Proof. Let $f(x,t)=\frac{\partial^*}{\partial t}F(x,t)=F_t^*(x,t)$. Hence we can write $*\int_{h(x)}^{v(x)} f(x,t)^{dt}=*\int_{h(x)}^{v(x)} F_t^*(x,t)^{dt}$. Since $*\int_{h(x)}^{v(x)} f(x,t)^{dt}=\frac{F(x,v(x))}{F(x,h(x))}$, we find

$$\frac{d^*}{dx}\left(*\int_{h(x)}^{v(x)} f(x,t)^{dt}\right)=\frac{d^*}{dx}\left(\frac{F\big(x,v(x)\big)}{F\big(x,h(x)\big)}\right)=\frac{\frac{d^*}{dx}F\big(x,v(x)\big)}{\frac{d^*}{dx}F\big(x,h(x)\big)}$$

by using properties of multiplicative derivative. Therefore we get

$$\frac{d^*}{dx}\left(*\int_{h(x)}^{v(x)} f(x,t)^{dt}\right)=\frac{F_x^*\big(x,v(x)\big)^1\left[F_{v(x)}^*\big(x,v(x)\big)\right]^{v'(x)}}{F_x^*\big(x,h(x)\big)^1\left[F_{h(x)}^*\big(x,h(x)\big)\right]^{h'(x)}} \tag{8}$$

with multiplicative chain rule. By using Lemma 1, we obtain

$$\begin{aligned}\frac{d^*}{dx^*}\left(*\int_{h(x)}^{v(x)} f(x,t)^{dt}\right)&=*\int_{h(x)}^{v(x)}\left(\frac{\partial^*}{\partial t}F_x^*(x,t)\right)^{dt}\frac{f\big(x,v(x)\big)^{v'(x)}}{f\big(x,h(x)\big)^{h'(x)}}\\ &=*\int_{h(x)}^{v(x)}\left(\frac{\partial^*}{\partial t}\left(\frac{\partial^*}{\partial x}F(x,t)\right)\right)^{dt}\frac{f\big(x,v(x)\big)^{v'(x)}}{f\big(x,h(x)\big)^{h'(x)}}\\ &=*\int_{h(x)}^{v(x)}\frac{\partial^*}{\partial x}\left(\frac{\partial^*}{\partial t}F(x,t)\right)^{dt}\frac{f\big(x,v(x)\big)^{v'(x)}}{f\big(x,h(x)\big)^{h'(x)}}\\ &=*\int_{h(x)}^{v(x)}\frac{\partial^*}{\partial x}\big(F_t^*(x,t)\big)^{dt}\frac{f\big(x,v(x)\big)^{v'(x)}}{f\big(x,h(x)\big)^{h'(x)}}\\ &=*\int_{v(x)}^{h(x)} f_x^*(x,t)^{dt}\ \frac{f\big(x,v(x)\big)^{v'(x)}}{f\big(x,h(x)\big)^{h'(x)}}.\end{aligned}$$

from the equality (8). This completes the proof.

Example 3. Show that the multiplicative integral equation $u(x) = \sin x * \int_0^x ([u(t)]^{x\tan t})^{dt}$ can be transformed to a multiplicative differential equation.

Solution. If we consider the equation $u(x) = \sin x \; * \int_0^x ([u(t)]^{x\tan t})^{dt}$ and differentiate it by using multiplicative Leibniz formula, we write

$$u^*(x) = \frac{d^*}{dx}(\sin x)\ \frac{d^*}{dx}\left(* \int_0^x ([u(t)]^{x\tan t})^{dt} \right)$$

$$= e^{\frac{\cos x}{\sin x}} * \int_0^x \left[\frac{\partial^*}{\partial x}([u(t)]^{x\tan t})\right]^{dt} \frac{(u(x)^{x\tan x})^{x'}}{(u(0)^{x\tan 0})^{0'}}$$

$$= e^{\cot x} * \int_0^x [u(t)^{\tan t}]^{dt}\, u(x)^{x\tan x}$$

To take derivative is continued until the expression gets rid of the integral sign. Hence, we obtain

$$u^{**}(x) = \frac{d^*}{dx}(e^{\cot x})\frac{d^*}{dx}\left(* \int_0^x [u(t)^{\tan t}]^{dt} \right)\frac{d^*}{dx}(u(x)^{x\tan x})$$

$$= e^{-cosec^2 x} * \int_0^x [1]^{dt} \frac{(u(x)^{\tan x})^{x'}}{(u(0)^{\tan 0})^{0'}} e^{\left((x.tanx)'\, lnu(x) + \frac{u'(x)}{u(x)}\, x\, tanx\right)}$$

$$= e^{-cosec^2 x}\ [u(x)]^{\tan x}\ e^{\left(lnu(x)^{(x\,tanx)'} + \frac{u'(x)}{u(x)}\, x\, tanx\right)}$$

$$= e^{-cosec^2 x}\ [u(x)]^{(2\tan x + x\, sec^2 x)}\ \left(e^{\frac{u'(x)}{u(x)}}\right)^{xtanx}$$

$$= e^{-cosec^2 x}\ [u(x)]^{(2\tan x + x\, sec^2 x)}\ [u^*(x)]^{x\ \tan x}.$$

Thus the multiplicative integral equation is equivalent to the multiplicative differential equation $u^{**}(x) = e^{-cosec^2 x}\, u(x)^{(2\tan x + x\, sec^2 x)}.[u^*(x)]^{x\tan x}$.

3.2. The Conversion of the Multiplicative Linear Differential Equations to Multiplicative Integral Equations

In this section, we prove that the multiplicative linear differential equation with constant or variable exponentials is converted to MVIE. We need to following theorem for converting n^{th} order multiplicative differential equation to MVIE.

Theorem 8. If n is a positive integer and a is a constant with $x \geq a$, then we have

$$* \int_a^x \ldots (n) \ldots * \int_a^x u(t)^{dt \ldots dt} = * \int_a^x \left[\left(u(t)\right)^{\frac{(x-t)^{(n-1)}}{(n-1)!}} \right]^{dt}$$

Proof. Let

$$I_n =* \int_a^x [u(t)]^{(x-t)^{(n-1)}dt} . \tag{9}$$

If it is taken $F(x,t) = [u(t)]^{(x-t)^{(n-1)}}$, we can write that

$$\begin{aligned}\frac{d^* I_n}{dx} &=* \int_a^x F_x^*(x,t)^{dt} \frac{[F(x,x)]^1}{[F(x,a)]^0} \\ &=* \int_a^x F_x^*(x,t)^{dt}\end{aligned}$$

by using the multiplicative Leibniz formula to equation (9). Then we find

$$\begin{aligned}\frac{d^* I_n}{dx} &=* \int_a^x \left(e^{\frac{\partial}{\partial x} ln\, F(x,t)}\right)^{dt} \\ &=* \int_a^x \left(e^{(n-1)(x-t)^{(n-2)} ln\, u(t)}\right)^{dt} \\ &=* \int_a^x \left(e^{ln\left([u(t)]^{(n-1)(x-t)^{(n-2)}}\right)}\right)^{dt} \\ &=* \int_a^x \left([u(t)]^{(n-1)(x-t)^{(n-2)}}\right)^{dt}.\end{aligned}$$

Hence we get

$$\frac{d^* I_n}{dx} = \left(* \int_a^x \left([u(t)]^{(x-t)^{(n-2)}}\right)^{dt} \right)^{(n-1)} = (I_{n-1})^{(n-1)} \tag{10}$$

where $n > 1$. Since $I_1(x) =* \int_a^x u(t)^{dt}$ for $n = 1$, then we can write

$$\frac{d^* I_1}{dx} = \frac{d^*}{dx}\left(* \int_a^x \left(u(t)\right)^{dt} \right) = u(x). \tag{11}$$

If it is taken multiplicative derivative of the equation (10) by using multiplicative Leibniz formula, then

$$\begin{aligned}\frac{d^{**} I_n}{dx^{(2)}} &= \frac{d^*}{dx}\left(* \int_a^x \left([u(t)]^{(x-t)^{(n-2)}}\right)^{dt} \right)^{(n-1)} \\ &= \left(\frac{d^*}{dx}\left(* \int_a^x \left([u(t)]^{(x-t)^{(n-2)}}\right)^{dt} \right) \right)^{(n-1)}\end{aligned}$$

$$= \left(* \int_a^x \left[\frac{\partial^*}{\partial x} \left([u(t)]^{(x-t)^{(n-2)}} \right) \right]^{dt} \frac{\left[[u(x)]^{(x-x)^{(n-2)}} \right]^1}{\left[[u(a)]^{(x-a)^{n-2}} \right]^0} \right)^{(n-1)}$$

$$= \left(* \int_a^x \left(\frac{\partial^*}{\partial x} \left([u(t)]^{(x-t)^{(n-2)}} \right) \right)^{dt} \right)^{(n-1)}$$

$$= \left(* \int_a^x \left(e^{\frac{\partial}{\partial x}\left(ln\left([u(t)]^{(x-t)^{(n-2)}} \right) \right)} \right)^{dt} \right)^{(n-1)}$$

$$= \left(* \int_a^x \left[e^{\left(ln\left((u(t))^{(n-2)(x-t)^{(n-3)}} \right) \right)} \right]^{dt} \right)^{(n-1)}$$

$$= \left(* \int_a^x \left[(u(t))^{(n-2)(x-t)^{(n-3)}} \right]^{dt} \right)^{(n-1)}$$

$$= \left(* \int_a^x \left[(u(t))^{(x-t)^{(n-3)}} \right]^{dt} \right)^{(n-1).(n-2)}$$

$$= (I_{n-2})^{(n-1)(n-2)} .$$

By proceeding similarly, we obtain

$$\frac{d^{*(n-1)} I_n}{dx^{(n-1)}} = (I_1)^{(n-1)!}$$

Hence, we write

$$\frac{d^{*(n)} I_n}{dx^{(n)}} = \left(\frac{d^* I_1}{dx} \right)^{(n-1)!} = [u(x)]^{(n-1)!}$$

from the equation (11). Now, we will take multiplicative integral by considering the above relations.

From the equation (11), $I_1(x) =* \int_a^x u(t)^{dt}$. Also, we have

$$I_2(x) =* \int_a^x I_1(x_2)^{dx_2} =* \int_a^x * \int_a^{x_2} u(x_1)^{dx_1 dx_2}$$

where x_1 and x_2 are parameters. By proceeding similarly, we obtain

$$I_n(x) = \left(* \int_a^x * \int_a^{x_n} \dots * \int_a^{x_3} * \int_a^{x_2} u(x_1)^{dx_1 dx_2 \dots dx_n} \right)^{(n-1)!}$$

where $x_1, x_2, \dots, x_n$ are parameters. If we write the equation (9) instead of the statement I_n, then it is find

$$*\int_a^x \left[\left(u(t)\right)^{(x-t)^{n-1}}\right]^{dt} = \left(*\int_a^x *\int_a^{x_n} \dots *\int_a^{x_3} *\int_a^{x_2} u(x_1)^{dx_1 dx_2 \dots dx_n} \right)^{(n-1)!}$$

Hence we can write

$$\left(*\int_a^x \left[\left(u(t)\right)^{(x-t)^{n-1}}\right]^{dt} \right)^{\frac{1}{(n-1)!}} =*\int_a^x *\int_a^{x_n} \dots *\int_a^{x_3} *\int_a^{x_2} u(x_1)^{dx_1 dx_2 \dots dx_n}$$

If it is taken $= x_1 = x_2 = \cdots = x_n$, therefore we obtain

$$*\int_a^x \dots (n) \dots *\int_a^x u(t)^{dt \dots dt} =*\int_a^x \left[\left(u(t)\right)^{\frac{(x-t)^{n-1}}{(n-1)!}}\right]^{dt}.$$

This completes the proof.

Let the n^{th}- order multiplicative linear differential equation

$$\frac{d^{*(n)}y}{dx^{(n)}} \left(\frac{d^{*(n-1)}y}{dx^{(n-1)}}\right)^{a_1(x)} \left(\frac{d^{*(n-2)}y}{dx^{(n-2)}}\right)^{a_2(x)} \dots \left(\frac{d^{*}y}{dx}\right)^{a_{n-1}(x)} (y)^{a_n(x)} = f(x) \tag{12}$$

that given the initial conditions

$$y(0) = c_0 \, , y^*(0) = c_1 \, , y^{*(n-1)}(0) = c_{n-1} \tag{13}$$

It can be transformed the multiplicative Volterra integral equation. Hence the solution of (12)-(13) may be reduced to a solution of some multiplicative Volterra integral equation.

Take $\frac{d^{*(n)}y}{dx^{(n)}} = u(x)$. By integrating both sides of the equality $\frac{d^*}{dx}\left(\frac{d^{*(n-1)}y}{dx^{(n-1)}}\right) = u(x)$, we write

$$*\int_0^x d^*\left(\frac{d^{*(n-1)}y}{dx^{(n-1)}}\right) = *\int_0^x u(t)^{dt}$$

$$\frac{y^{*(n-1)}(x)}{y^{*(n-1)}(0)} = *\int_0^x u(t)^{dt}$$

$$y^{*(n-1)}(x) = c_{n-1} *\int_0^x u(t)^{dt}$$

By proceeding similarly, we find

$$*\int_0^x d^*\left(\frac{d^{*(n-2)}y}{dx^{(n-2)}}\right) =*\int_0^x \left(c_{n-1} *\int_0^x u(t)^{dt}\right)^{dt}$$

$$\frac{y^{*(n-2)}(x)}{y^{*(n-2)}(0)} =*\int_0^x c_{n-1}{}^{dt} *\int_0^x *\int_0^x u(t)^{dtdt}$$

$$\frac{y^{*(n-2)}(x)}{c_{n-2}} = e^{\int_0^x \ln c_{n-1} dt} *\int_0^x *\int_0^x u(t)^{dtdt}$$

$$\frac{y^{*(n-2)}(x)}{c_{n-2}}=(c_{n-1})^{x}*\int_0^x *\int_0^x u(t)^{dtdt}$$

$$y^{*(n-2)}(x)=c_{n-2}\,(c_{n-1})^{x}*\int_0^x *\int_0^x u(t)^{dtdt}$$

$$*\int_0^x d^*\left(\frac{d^{*(n-3)}y}{dx^{(n-3)}}\right)=*\int_0^x [(c_{n-1})^{x}.c_{n-2}]^{dt}*\int_0^x *\int_0^x *\int_0^x u(t)^{dtdtdt}$$

$$y^{*(n-3)}(x)=c_{n-3}\,(c_{n-2})^{x}\,(c_{n-1})^{x^2}*\int_0^x *\int_0^x *\int_0^x u(t)^{dtdtdt}$$

$$\vdots$$

$$y^{*}=c_1\,(c_2)^{x}\,(c_3)^{x^2}\dots(c_{n-2})^{x^{(n-3)}}\,(c_{n-1})^{x^{(n-2)}}*\int_0^x \dots(n-1)\dots *\int_0^x u(t)^{dt\dots dt}$$

Hence, we get

$$y=c_0\,(c_1)^{x}\,(c_2)^{x^2}\dots(c_{n-1})^{x^{(n-1)}}*\int_0^x \dots(n)\dots *\int_0^x u(t)^{dt\dots dt}$$

If we take into account the above expressions, the multiplicative linear differential equation (12) is written as follows

$$u(x)\left[(c_{n-1})^{a_1(x)}\left(*\int_0^x u(t)^{dt}\right)^{a_1(x)}\right]\left[(c_{n-2})^{a_2(x)}\,(c_{n-1})^{x\,a_2(x)}\left(*\int_0^x *\int_0^x u(t)^{dtdt}\right)^{a_2(x)}\right]\cdots$$

$$\left[(c_0)^{a_n(x)}\,(c_1)^{xa_n(x)}\,(c_2)^{x^2a_n(x)}.\dots.(c_{n-1})^{x^{n-1}a_n(x)}\quad\left(*\int_0^x \dots(n)\dots *\int_0^x u(t)^{dt\dots dt}\right)^{a_n(x)}\right]=f(x)$$

$$u(x)\,(c_0)^{a_n(x)}\,(c_1)^{x\,a_n(x)+a_{n-1}(x)}\dots(c_{n-1})^{x^{n-1}a_n(x)+\dots+a_1(x)}\left(*\int_0^x u(t)^{dt}\right)^{a_1(x)}\left(*\int_0^x *\int_0^x u(t)^{dtdt}\right)^{a_2(x)}\dots$$

$$\left(*\int_0^x \dots(n)\dots *\int_0^x u(t)^{dt\dots dt}\right)^{a_n(x)}=f(x) \qquad (14)$$

If we set

$$a_1(x)+a_2(x)\,x+\dots+a_n(x)\,x^{n-1}=f_{n-1}(x)$$

$$a_2(x)+a_3(x)\,x+\dots+a_n(x)\,x^{n-2}=f_{n-2}(x)$$

$$\vdots$$

$$a_{n-1}(x)+a_n(x)\,x=f_1(x)$$

$$a_n(x)=f_0(x)$$

and

$$F(x)=\frac{f(x)}{(c_0)^{f_0(x)}\,(c_1)^{f_1(x)}\ldots(c_{n-1})^{f_{n-1}(x)}}$$

then we can edit the equation (14) in the form as follows

$$u(x)\left(*\int_0^x u(t)^{dt}\right)^{a_1(x)}\left(*\int_0^x*\int_0^x u(t)^{dtdt}\right)^{a_2(x)}\ldots\left(*\int_0^x\ldots(n)\ldots*\int_0^x u(t)^{dt\ldots dt}\right)^{a_n(x)}=F(x)\,.$$

By using Theorem 8, we get

$$u(x)\left(*\int_0^x u(t)^{dt}\right)^{a_1(x)}\left(*\int_0^x[u(t)^{x-t}]^{dt}\right)^{a_2(x)}\ldots\left(*\int_0^x u(t)^{\frac{(x-t)^{n-1}}{(n-1)!}dt}\right)^{a_n(x)}=F(x).$$

Then we find the equation

$$u(x)*\int_0^x\left(u(t)^{\left[a_1(x)+(x-t)a_2(x)+\cdots+a_n(x)\frac{(x-t)^{n-1}}{(n-1)!}\right]}\right)^{dt}=F(x).$$

If we put $K(x,t)=a_1(x)+(x-t)a_2(x)+\cdots+a_n(x)\frac{(x-t)^{n-1}}{(n-1)!}$ as the kernel function, then the equation (12) is turned into

$$u(x)*\int_0^x u(t)^{K(x,t)dt}=F(x)$$

which is a MVIE of the second kind.

Example 4. Form a multiplicative Volterra integral equation corresponding to the multiplicative differential equation $\frac{d^{*2}y(x)}{dx^{(2)}}=y(x)^{cosx}$ with the initial conditions $y(0)=1, y^*(0)=1$.

Solution. Let $\frac{d^{*2}y(x)}{dx^{(2)}}=u(x)$. Then we write

$$*\int_0^x d^*y^*=*\int_0^x u(t)^{dt}$$

$$\frac{y^*(x)}{y^*(0)}=*\int_0^x u(t)^{dt}$$

$$y^*(x)=*\int_0^x u(t)^{dt}.$$

Therefore we find

$$*\int_0^x y^*(t)^{dt}=*\int_0^x*\int_0^x u(t)^{dt}$$

$$\frac{y(x)}{y(0)} =* \int_0^x * \int_0^x u(t)^{dt}$$

$$y(x) =* \int_0^x \left[u(t)^{(x-t)}\right]^{dt}$$

If we replace the equation $y(x) =* \int_0^x \left[u(t)^{(x-t)}\right]^{dt}$ into the given multiplicative differential equation, we obtain $u(x) =* \int_0^x \left[u(t)^{cos\,x\,(x-t)}\right]^{dt}$.

4. Conclusion

In this paper, the multiplicative Voltterra integral equation is defined by using the concept of multiplicative integral. The solution of multiplicative Volterra integral equation is obtained with the successive approximations method. The multiplicative Leibniz formula is proved and the multiplicative Volterra integral equation is converted to a multiplicative differential equation by aid of multiplicative Leibniz formula. The multiplicative linear differential equation with constant or variable exponentials is converted to a multiplicative Volterra integral equation is proved.

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