Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 50 (4) (2021), 1047 – 1062
DOI : 10.15672/hujms.796762

Research Article

$\mathcal{F}-$relative $\mathcal{A}-$summation process for double sequences and abstract Korovkin type theorems

Sevda Yıldız

Department of Mathematics, Sinop University, Sinop, Turkey

Abstract

In this paper, we first introduce the notions of $\mathcal{F}-$relative modular convergence and $\mathcal{F}-$relative strong convergence for double sequences of functions. Then we prove some Korovkin-type approximation theorems via $\mathcal{F}-$relative $\mathcal{A}-$summation process on modular spaces for double sequences of positive linear operators. Also, we present a non-trivial application such that our Korovkin-type approximation results in modular spaces are stronger than the classical ones and we present some estimates of rates of convergence for abstract Korovkin-type theorems. Furthermore, we relax the positivity condition of linear operators in the Korovkin theorems and study an extension to non-positive operators.

Mathematics Subject Classification (2020). 41A35, 46E30, 47B38

Keywords. modular spaces, double sequence, matrix summability, filter convergence, abstract Korovkin theorem

1. Introduction and preliminaries

Since the Korovkin ([18]) theorems' discovery, the simplicity and, at the same time, the power of them impressed several mathematicians. A considerable amount of research extended these theorems to the setting of different function spaces by different convergence methods (see e.g. [4, 11, 13–16, 22, 23]). In some Korovkin type theorems, in the case of the lack of convergence, it is effective to use the matrix summability methods or more generally, summation methods. That's why Nishishiraho introduced and studied the notion of $\mathcal{A}-$summation process on a compact Hausdorff space ([27, 28]). Afterwards Korovkin-type theorems are studied via $\mathcal{A}-$summation process in various spaces like weighted spaces (see [2, 3]), modular spaces ([10, 17, 29, 31, 35, 36]). In the present paper, we introduce the notions of $\mathcal{F}-$relative modular convergence and $\mathcal{F}-$relative strong convergence for double sequences of functions and we prove our main Korovkin-type theorems via $\mathcal{F}-$relative $\mathcal{A}-$summation process on modular spaces. Then, we present a non-trivial application. Besides surveying the rates of convergence, the paper also contains an extension of the Korovkin-type theorem to non-positive operators.

Now, we start by giving some definitions and notations that we will use in the sequal.

A double sequence $x = (x_{mn})$ is said to be convergent in Pringsheim's sense if, for every $\varepsilon > 0$, there exists $M = M(\varepsilon) \in \mathbb{N}$, the set of all natural numbers, such that $|x_{mn} - L| < \varepsilon$ whenever $m, n > M$, where L is called the Pringsheim limit of x and denoted

Email address: sevdaorhan@sinop.edu.tr
Received: 18.09.2020; Accepted: 20.02.2021

by $P-\lim_{m,n} x_{mn}=L$ (see [33]). We shall call such an x, briefly, "$P-$convergent". A double sequence is called bounded if there exists a positive number K such that $|x_{mn}|\leq K$ for all $(m,n)\in\mathbb{N}^2=\mathbb{N}\times\mathbb{N}$. As it is well known, a convergent single sequence is bounded whereas a convergent double sequence need not to be bounded.

A nonempty family $\mathcal{F}$ of subsets of $\mathbb{N}^2$ is a *filter* of $\mathbb{N}^2$ iff $\emptyset\notin\mathcal{F}$, $A\cap B\in\mathcal{F}$ whenever $A,B\in\mathcal{F}$ and for every $A\in\mathcal{F}$ and $B\supset A$ we get $B\in\mathcal{F}$. For any $\mathbf{m}\in\mathbb{N}^2$ set $M_{\mathbf{m}}=\left\{\mathbf{n}\in\mathbb{N}^2:\mathbf{n}\geq\mathbf{m}\right\}$. Here $(\mathbb{N}^2,\geq)$ is meant with respect to the usual componentwise order (see also [5]). A filter $\mathcal{F}$ of $\mathbb{N}^2$ is said to be *free* iff $M_{\mathbf{m}}\in\mathcal{F}$ for every $\mathbf{m}\in\mathbb{N}^2$.

From now on, we always suppose that $\mathcal{F}$ is a free filter of $\mathbb{N}^2$.

A real double sequence $x=(x_{mn})$ is said to be $\mathcal{F}-$convergent to L iff for every $\varepsilon>0$ we get $\left\{(m,n)\in\mathbb{N}^2:|x_{mn}-L|\leq\varepsilon\right\}\in\mathcal{F}$.

The statistical limit superior and limit inferior for single sequences have been first given by Fridy and Orhan in [12]. Then, Demirci [9] has generalized these concepts to ideal limit superior and limit inferior. Now, in view of these studies, we can give the concepts of $\mathcal{F}-$limit inferior and limit superior for double sequences.

For any real double sequence $x=(x_{mn})$ the $\mathcal{F}-$limit superior of x is defined by

$$\mathcal{F}-\limsup_{m,n} x_{mn}=\begin{cases}\sup A_x, & \text{if } A_x\neq\emptyset,\\ -\infty, & \text{if } A_x=\emptyset,\end{cases}$$

where $A_x=\left\{a\in\mathbb{R}:\left\{(m,n)\in\mathbb{N}^2:x_{mn}\leq a\right\}\notin\mathcal{F}\right\}$. Similarly, the $\mathcal{F}-$limit inferior of x is defined by

$$\mathcal{F}-\liminf_{m,n} x_{mn}=\begin{cases}\inf B_x, & \text{if } B_x\neq\emptyset,\\ +\infty, & \text{if } B_x=\emptyset,\end{cases}$$

where $B_x=\left\{b\in\mathbb{R}:\left\{(m,n)\in\mathbb{N}^2:x_{mn}\geq b\right\}\notin\mathcal{F}\right\}$ (see also [8]).

Let (x_{mn}) and (y_{mn}) be two real double sequences with $\mathcal{F}-\lim_{m,n} x_{mn}=\mathcal{F}-\lim_{m,n} y_{mn}=0$ and $x_{mn}\neq 0$, $y_{mn}\neq 0$ for every $(m,n)\in\mathbb{N}^2$. We say that $x_{mn}=o_{\mathcal{F}}(y_{mn})$ iff $\mathcal{F}-\lim_{m,n}\frac{|x_{mn}|}{|y_{mn}|}=0$, and that $x_{mn}=O_{\mathcal{F}}(y_{mn})$ iff $\mathcal{F}-\limsup_{m,n}\frac{|x_{mn}|}{|y_{mn}|}\in\mathbb{R}$.

Let's recall some notations related to the summability theory.

Let $A=[a_{klmn}]$, $k,l,m,n\in\mathbb{N}$, be a four-dimensional infinite matrix. For a given double sequence $x=(x_{mn})$, the $A-$transform of x, denoted by $Ax:=((Ax)_{kl})$, is given by

$$(Ax)_{kl}=\sum_{(m,n)\in\mathbb{N}^2}a_{klmn}x_{mn},\quad k,l\in\mathbb{N},$$

provided the double series converges in Pringsheim's sense for every $(k,l)\in\mathbb{N}^2$. We say that a sequence x is $A-$summable to L if the $A-$transform of x exists for all $k,l\in\mathbb{N}$ and convergent in the Pringsheim's sense i.e.,

$$P-\lim_{p,q}\sum_{m=1}^{p}\sum_{n=1}^{q}a_{klmn}x_{mn}=y_{kl}\text{ and }\ P-\lim_{k,l}y_{kl}=L.$$

In summability theory, a two-dimensional matrix transformation is called *regular* if it maps every convergent sequence in to a convergent sequence with the same limit.

Now let $\mathcal{A}:=\left(A^{(i,j)}\right)=\left(a_{klmn}^{(i,j)}\right)$ be a sequence of four-dimensional infinite matrices with non-negative real entries. For a given double sequence of real numbers, $x=(x_{mn})$ is said to be $\mathcal{A}-$*summable* to L if

$$P-\lim_{k,l}\sum_{(m,n)\in\mathbb{N}^2}a_{klmn}^{(i,j)}x_{mn}=L$$

uniformly in i and j.

If $A^{(i,j)} = A$, four-dimensional infinite matrix, then $\mathcal{A}-$summability is the $A-$summability for four-dimensional infinite matrix. Some results regarding matrix summability method for double sequences may be found in the papers [32, 34].

Now let's give basic concepts and facts of modular spaces.

Assume that X be a locally compact Hausdorff topological space with a uniform structure $\mathcal{U} \subset 2^{X \times X}$ that generates the topology of X (see, [20]). Let $\mathcal{B}$ be the $\sigma-$algebra of all Borel subsets of X and $\mu : \mathcal{B} \to \mathbb{R}$ is a positive $\sigma-$finite regular measure. Let $L^0(X)$ be the space of all real valued $\mu-$measurable functions on X provided with equality almost everywhere, $C_b(X)$ be the space of all continuous real valued and bounded functions on X and $C_c(X)$ be the subspace of $C_b(X)$ of all functions with compact support on X. In this case, we say that a functional $\rho : L^0(X) \to [0, \infty]$ is a modular on $L^0(X)$ if it satisfies the following conditions:

(i) $\rho(h) = 0$ if and only if $h = 0$ $\mu-$almost everywhere on X,
(ii) $\rho(-h) = \rho(h)$ for every $h \in L^0(X)$,
(iii) $\rho(\alpha h + \beta g) \leq \rho(h) + \rho(g)$ for every $h, g \in L^0(X)$ and for any $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

A modular ρ is $N-$quasi convex if there exists a constant $N \geq 1$ such that the inequality $\rho(\alpha h + \beta g) \leq N\alpha\rho(Nh) + N\beta\rho(Ng)$ holds for every $h, g \in L^0(X)$, $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Note that if $N = 1$, then ρ is called convex. Furthermore, a modular ρ is $N-$quasi semiconvex if there exists a constant $N \geq 1$ such that $\rho(\alpha h) \leq N\alpha\rho(Nh)$ holds for every $h \in L^0(X)$ and $\alpha \in (0, 1]$.

The modular space X_ρ generated by modular ρ, given by

$$X_\rho := \left\{ h \in L^0(X) : \lim_{\lambda \to 0^+} \rho(\lambda h) = 0 \right\}$$

and the space of the finite elements of X_ρ, given by

$$X_\rho^* := \{h \in X_\rho : \rho(\lambda h) < \infty \text{ for all } \lambda > 0\}.$$

Also, note that if ρ is $N-$quasi semiconvex, then the space $\{h \in L^0(X) : \rho(\lambda h) < \infty \text{ for some } \lambda > 0\}$ coincides with X_ρ.

Now we give the $\mathcal{F}-$relative modular and $\mathcal{F}-$strong convergence for double sequences.

Definition 1.1. Let (h_{mn}) be a double function sequence whose terms belong to X_ρ. Then, (h_{mn}) is said to be $\mathcal{F}-$*relatively modularly convergent* to a function $h \in X_\rho$ if there exists a function $\sigma(u)$, called a scale function $\sigma \in L^0(X)$, $|\sigma(u)| \neq 0$ such that

$$\mathcal{F} - \lim_{m,n} \rho\left(\lambda_0 \left(\frac{h_{mn} - h}{\sigma}\right)\right) = 0 \quad \text{for some } \lambda_0 > 0.$$

Also, (h_{mn}) is $\mathcal{F}-$relatively $F-$*norm convergent* (or, $\mathcal{F}-$relatively *strongly convergent*) to h iff

$$\mathcal{F} - \lim_{m,n} \rho\left(\lambda \left(\frac{h_{mn} - h}{\sigma}\right)\right) = 0 \quad \text{for every } \lambda > 0.$$

The two notions of convergence are equivalent if and only if the modular satisfies a Δ_2-condition, i.e. there exists a constant $M > 0$ such that $\rho(2h) \leq M\rho(h)$ for every $h \in L^0(X)$, see [25].

Note that if the scale function is selected a non-zero constant, then $\mathcal{F}-$modular convergence is the special case of $\mathcal{F}-$relative modular convergence. Moreover, if $\sigma(u)$ is bounded, $\mathcal{F}-$relative modular convergence implies $\mathcal{F}-$modular convergence. However, if $\sigma(u)$ is unbounded, then $\mathcal{F}-$relative modular convergence does not imply $\mathcal{F}-$modular convergence.

Recently, Orhan and Kolay ([31]) presented $\mathcal{A}-$summation process for double sequences on a modular space and more recently, Demirci, Orhan and Kolay ([10]) introduced the notion of *relative modular $\mathcal{A}-$summation process* for double sequences as follows:

A sequence $\mathbb{T} := (T_{mn})$ of positive linear operators from D into $L^0(X)$ with $C_b(X) \subset D \subset L^0(X)$ is called a *relative $\mathcal{A}-$summation process* on D if $(T_{mn}h)$ is relatively $\mathcal{A}-$summable to h (with respect to modular ρ) for every $h \in D$, i.e.,

$$P - \lim_{k,l} \rho\left[\lambda\left(\frac{A^{\mathbb{T}}_{klij}h - h}{\sigma}\right)\right] = 0, \text{ uniformly in } i,j, \text{ for some } \lambda > 0,$$

where for all $k,l,i,j \in \mathbb{N}$, $h \in D$ the series

$$A^{\mathbb{T}}_{klij}h := \sum_{(m,n)\in\mathbb{N}^2} a^{(i,j)}_{klmn} T_{mn}h$$

is absolutely convergent almost everywhere with respect to Lebesgue measure and we denote the value of $T_{mn}h$ at a point $u \in X$ by $T_{mn}(h(v);u)$ or briefly, $T_{mn}(h;u)$. It will be observed that *$\mathcal{A}-$summation process* is the special case of *relative $\mathcal{A}-$summation process* in which the scale function is a non-zero constant.

In this regard, some results on this new convergence method can be obtained by applying some Korovkin type theorems for double sequences of linear operators on a modular space.

In the present paper, we consider the following assumptions:

- A modular ρ is said to be monotone if $\rho(h) \le \rho(g)$ for $|h| \le |g|$.
- A modular ρ is finite if $\chi_A \in X_\rho$ whenever $A \in \mathcal{B}$ with $\mu(A) < \infty$.
- A modular ρ is strongly finite if $\chi_A \in X^*_\rho$ for all $A \in \mathcal{B}$ such that $\mu(A) < \infty$.
- A modular ρ is said to be absolutely continuous if there exists an $\alpha > 0$ such that, for every $h \in L^0(X)$ with $\rho(h) < \infty$, the following conditions hold:
  - for each $\varepsilon > 0$ there exists a set $A \in \mathcal{B}$ such that $\mu(A) < \infty$ and $\rho\left(\alpha h \chi_{X \setminus A}\right) \le \varepsilon$,
  - for every $\varepsilon > 0$ there is a $\delta > 0$ with $\rho(\alpha h \chi_B) \le \varepsilon$ for every $B \in \mathcal{B}$ with $\mu(B) < \delta$.

If a modular ρ is monotone and finite, then $C(X) \subset X_\rho$. If ρ is monotone and strongly finite, then $C(X) \subset X^*_\rho$. Also, if ρ is monotone, strongly finite and absolutely continuous, $\overline{C_c(X)} = X_\rho$ with respect to the modular convergence in the ordinary sense (see [19, 21, 26]).

2. Main results

We now prove some Korovkin type theorems with respect to an abstract finite set of test functions $h_0, h_1, ..., h_q$ in the sense of $\mathcal{F}-$relative $\mathcal{A}-$Summation process in modular spaces.

Let $\mathbb{T} = (T_{mn})$ be a double sequence of positive linear operators from D into $L^0(X)$ with $C_b(X) \subset D \subset L^0(X)$. Let ρ be monotone and finite modular on $L^0(X)$. Assume further that the double sequence $\mathbb{T}$, together with modular ρ, satisfies the following property:

there exists a subset $X_{\mathbb{T}} \subset D \cap X_\rho$ with $C_b(X) \subset X_{\mathbb{T}}$ and $\sigma \in L^0(X)$ is an unbounded function satisfying $\sigma(u) \neq 0$ such that the inequality

$$\mathcal{F} - \lim\sup_{k,l} \rho\left(\lambda\left(\frac{A^{\mathbb{T}}_{klij}h}{\sigma}\right)\right) \le R\rho(\lambda h), \text{ uniformly in } i,j, \tag{2.1}$$

holds for every $h \in X_{\mathbb{T}}$, $\lambda > 0$ and for an absolute positive constant R.

Set $h_0(v) \equiv 1$ for all $v \in X$, let h_r, $r = 1,2,...,q$ and a_r, $r = 0,1,2,...,q$, be functions in $C_b(X)$. Put

$$P_u(v) = \sum_{r=0}^{q} a_r(u) h_r(v), \ u,v \in X, \tag{2.2}$$

and suppose that $P_u(v)$, $u,v \in X$, satisfies the following properties:

$(P1)$ $P_u(u) = 0$, for all $u \in X$,

$(P2)$ for every neighbourhood $U \in \mathfrak{U}$ there is a positive real number η with $P_u(v) \ge \eta$ whenever $u,v \in X$, $(u,v) \notin U$ (see [6] for examples).

In order to obtain our main theorem, we first give the following result.

Theorem 2.1. *Let $\mathcal{A}=\left(A^{(i,j)}\right)$ be a sequence of four dimensional infinite non-negative real matrices and let ρ be a monotone, strongly finite and $N-$quasi semiconvex modular. Suppose that h_r and a_r, $r=0,1,2,...,q$, satisfy properties $(P1)$ and $(P2)$. Let $\mathbb{T}=(T_{mn})$ be a double sequence of positive linear operators from D into $L^0(X)$ and assume that $\sigma_r(u)$ is an unbounded function satisfying $|\sigma_r(u)|\geq b_r>0$ $(r=0,1,2,...,q)$. If*

$$\mathcal{F}-\lim_{k,l}\rho\left(\lambda_0\left(\frac{A^{\mathbb{T}}_{klij}h_r-h_r}{\sigma_r}\right)\right)=0,\ \textit{uniformly in } i,j, \tag{2.3}$$

for some $\lambda_0>0$, $r=0,1,2,...,q$, in X_ρ then for every $h\in C_c(X)$

$$\mathcal{F}-\lim_{k,l}\rho\left(\gamma\left(\frac{A^{\mathbb{T}}_{klij}h-h}{\sigma}\right)\right)=0,\ \textit{uniformly in } i,j, \tag{2.4}$$

for some $\gamma>0$, in X_ρ where $\sigma(u)=\max\{|\sigma_r(u)|:r=0,1,2,...,q\}$. If

$$\mathcal{F}-\lim_{k,l}\rho\left(\lambda\left(\frac{A^{\mathbb{T}}_{klij}h_r-h_r}{\sigma_r}\right)\right)=0,\ \textit{uniformly in } i,j,$$

for every $\lambda>0$, $r=0,1,2,...,q$, in X_ρ then for every $h\in C_c(X)$

$$\mathcal{F}-\lim_{k,l}\rho\left(\lambda\left(\frac{A^{\mathbb{T}}_{klij}h-h}{\sigma}\right)\right)=0,\ \textit{uniformly in } i,j,$$

for every $\lambda>0$, in X_ρ where $\sigma(u)=\max\{|\sigma_r(u)|:r=0,1,2,...,q\}$.

Proof. We first claim that, for every $h\in C_c(X)$,

$$\mathcal{F}-\lim_{k,l}\rho\left(\gamma\left(\frac{A^{\mathbb{T}}_{klij}h-h}{\sigma}\right)\right)=0,\ \text{uniformly in } i,j, \tag{2.5}$$

for some $\gamma>0$. To see this, assume that $h\in C_c(X)$. Then, since X is endowed with the uniformity $\mathcal{U}$, h is uniformly continuous and bounded on X. By the uniform continuity of h, choose $\varepsilon\in(0,1]$, there exists a set $U\in\mathcal{U}$ such that $|h(u)-h(v)|\leq\varepsilon$ whenever $u,v\in X$, $(u,v)\in U$.

For all $u,v\in X$ let $P_u(v)$ be as in (2.2), and $\eta>0$ satisfy condition $(P2)$. Then for $u,v\in X$, $(u,v)\notin U$, we have $|h(u)-h(v)|\leq\frac{2M}{\eta}P_u(v)$ where $M:=\sup_{v\in X}|h(v)|$. Therefore, in any case we get $|h(u)-h(v)|\leq\varepsilon+\frac{2M}{\eta}P_u(v)$ for all $u,v\in X$, namely,

$$-\varepsilon-\frac{2M}{\eta}P_u(v)\leq h(u)-h(v)\leq\varepsilon+\frac{2M}{\eta}P_u(v). \tag{2.6}$$

Since T_{mn} is linear and positive, by applying T_{mn} to (2.6) we have

$$\begin{aligned}-\varepsilon A^{\mathbb{T}}_{klij}(h_0;u)-\frac{2M}{\eta}A^{\mathbb{T}}_{klij}(P_u;u) &\leq h(u)A^{\mathbb{T}}_{klij}(h_0;u)-A^{\mathbb{T}}_{klij}(h;u)\\ &\leq \varepsilon A^{\mathbb{T}}_{klij}(h_0;u)+\frac{2M}{\eta}A^{\mathbb{T}}_{klij}(P_u;u).\end{aligned}$$

Hence

$$\begin{aligned}\left|A^{\mathbb{T}}_{klij}(h;u)-h(u)\right| &\le \left|A^{\mathbb{T}}_{klij}(h;u)-h(u)A^{\mathbb{T}}_{klij}(h_0;u)\right|+|h(u)|\left|A^{\mathbb{T}}_{klij}(h_0;u)-h_0(u)\right| \\ &\le \varepsilon A^{\mathbb{T}}_{klij}(h_0;u)+\frac{2M}{\eta}A^{\mathbb{T}}_{klij}(P_u;u)+M\left|A^{\mathbb{T}}_{klij}(h_0;u)-h_0(u)\right| \\ &\le \varepsilon+(\varepsilon+M)\left|A^{\mathbb{T}}_{klij}(h_0;u)-h_0(u)\right| \\ &\quad +\frac{2M}{\eta}\sum_{r=0}^{q}a_r(u)\left|A^{\mathbb{T}}_{klij}(h_r;u)-h_r(u)\right|.\end{aligned}$$

Let $\gamma>0$. Now for each $r=0,1,2,...,q,\ u\in X$ and $i,j\in\mathbb{N}$, choose $M_0>0$ such that $|a_r(u)|\le M_0$ and multiplying the both sides of the above inequality by $\frac{1}{|\sigma(u)|}$, the last inequality gives that

$$\gamma\left|\frac{A^{\mathbb{T}}_{klij}(h;u)-h(u)}{\sigma(u)}\right|\le\frac{\gamma\varepsilon}{|\sigma(u)|}+K\gamma\sum_{r=0}^{q}\left|\frac{A^{\mathbb{T}}_{klij}(h_r;u)-h_r(u)}{\sigma(u)}\right|$$

where $K:=\varepsilon+M+\frac{2M}{\eta}M_0$. Now, applying the modular ρ to both sides of the above inequality, since ρ is monotone and $\sigma(u)=\max\{|\sigma_r(u)|:r=0,1,2,...,q\}$, we get

$$\rho\left(\gamma\left(\frac{A^{\mathbb{T}}_{klij}h-h}{\sigma}\right)\right)\le\rho\left(\frac{\gamma\varepsilon}{|\sigma|}+K\gamma\sum_{r=0}^{q}\frac{A^{\mathbb{T}}_{klij}h_r-h_r}{\sigma_r}\right).$$

Thus, we can see that

$$\rho\left(\gamma\left(\frac{A^{\mathbb{T}}_{klij}h-h}{\sigma}\right)\right)\le\rho\left(\frac{(q+2)\gamma\varepsilon}{\sigma}\right)+\sum_{r=0}^{q}\rho\left((q+2)K\gamma\left(\frac{A^{\mathbb{T}}_{klij}h_r-h_r}{\sigma_r}\right)\right).$$

Since ρ is $N-$quasi semiconvex and strongly finite, we have,

$$\rho\left(\gamma\left(\frac{A^{\mathbb{T}}_{klij}h-h}{\sigma}\right)\right)\le N\varepsilon\rho\left(\frac{(q+2)\gamma N}{\sigma}\right)+\sum_{r=0}^{q}\rho\left((q+2)K\gamma\left(\frac{A^{\mathbb{T}}_{klij}h_r-h_r}{\sigma_r}\right)\right), \quad (2.7)$$

which proves our claim (2.5).

The last part of theorem can be proved similarly to the first one. □

Now, we can give our main theorem of this paper.

Theorem 2.2. *Let $\mathcal{A}=\left(A^{(i,j)}\right)$ be a sequence of four dimensional infinite non-negative real matrices and let ρ be a monotone, strongly finite, absolutely continuous and $N-$quasi semiconvex modular. Suppose that h_r and a_r, $r=0,1,2,...,q$, satisfy properties $(P1)$ and $(P2)$. Let $\mathbb{T}=(T_{mn})$ be a double sequence of positive linear operators satisfying (2.1) and assume that $\sigma_r(u)$ is an unbounded function satisfying $|\sigma_r(u)|\ge b_r>0$ $(r=0,1,2,...,q)$. If*

$$\mathcal{F}-\lim_{k,l}\rho\left(\lambda\left(\frac{A^{\mathbb{T}}_{klij}h_r-h_r}{\sigma_r}\right)\right)=0,\ \textit{uniformly in } i,j,$$

for every $\lambda>0$, $r=0,1,2,...,q$, in X_ρ, then for every $h\in D\cap X_\rho$ with $h-C_b(X)\subset X_{\mathbb{T}}$,

$$\mathcal{F}-\lim_{k,l}\rho\left(\lambda_0\left(\frac{A^{\mathbb{T}}_{klij}h-h}{\sigma}\right)\right)=0,\ \textit{uniformly in } i,j,$$

for some $\lambda_0>0$, in X_ρ where $\sigma(u)=\max\{|\sigma_r(u)|:r=0,1,2,...,q\}$ and D, $X_{\mathbb{T}}$ are as before.

Proof. Let $h \in D \cap X_\rho$ with $h - C_b(X) \subset X_{\mathbb{T}}$. It is known from [5, 21] that there exists a sequence $(g_{kl}) \subset C_c(X)$ such that $\rho(3\lambda_0^* h) < \infty$ and $P - \lim_{k,l} \rho(3\lambda_0^*(g_{kl} - h)) = 0$ for some $\lambda_0^* > 0$. This means that, for every $\varepsilon > 0$, there is a positive number $l_0 = l_0(\varepsilon)$ with

$$\rho(3\lambda_0^*(g_{kl} - h)) < \varepsilon \text{ for every } k, l \geq l_0. \tag{2.8}$$

By linearity and positivity of the operators T_{mn}, we have

$$\begin{aligned} \lambda_0^* \left| A_{klij}^{\mathbb{T}}(h; u) - h(u) \right| &\leq \lambda_0^* \left| A_{klij}^{\mathbb{T}}(h - g_{l_0 l_0}; u) \right| + \lambda_0^* \left| A_{klij}^{\mathbb{T}}(g_{l_0 l_0}; u) - g_{l_0 l_0}(u) \right| \\ &\quad + \lambda_0^* \left| g_{l_0 l_0}(u) - h(u) \right| \end{aligned}$$

holds for every $u \in X$ and $i, j \in \mathbb{N}$. Now, applying modular ρ in the last inequality and using the monotonicity of ρ and moreover multiplying both sides of the above inequality by $\frac{1}{|\sigma(u)|}$, we get

$$\begin{aligned} \rho\left(\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}} h - h}{\sigma}\right)\right) &\leq \rho\left(3\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}}(h - g_{l_0 l_0})}{\sigma}\right)\right) + \rho\left(3\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}} g_{l_0 l_0} - g_{l_0 l_0}}{\sigma}\right)\right) \\ &\quad + \rho\left(3\lambda_0^*\left(\frac{g_{l_0 l_0} - h}{\sigma}\right)\right). \end{aligned}$$

Hence, observing $|\sigma| \geq b > 0$, $(b = \max\{b_r : r = 0, 1, 2, ..., q\})$, we can write that

$$\begin{aligned} \rho\left(\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}} h - h}{\sigma}\right)\right) &\leq \rho\left(3\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}}(h - g_{l_0 l_0})}{\sigma}\right)\right) + \rho\left(3\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}} g_{l_0, l_0} - g_{l_0 l_0}}{\sigma}\right)\right) \\ &\quad + \rho\left(\frac{3\lambda_0^*}{b}(g_{l_0 l_0} - h)\right). \end{aligned} \tag{2.9}$$

Then using the (2.8) in (2.9), we have

$$\rho\left(\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}} h - h}{\sigma}\right)\right) \leq \varepsilon + \rho\left(3\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}}(h - g_{l_0 l_0})}{\sigma}\right)\right) + \rho\left(3\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}} g_{l_0 l_0} - g_{l_0 l_0}}{\sigma}\right)\right).$$

By property (2.1) and also using the facts that $g_{l_0 l_0} \in C_c(G)$ and $h - g_{l_0 l_0} \in X_{\mathbb{T}}$, we obtain

$$\begin{aligned} &\mathcal{F} - \limsup_{k,l} \rho\left(\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}} h - h}{\sigma}\right)\right) \\ \leq\ & \varepsilon + R\rho(3\lambda_0^*(h - g_{l_0 l_0})) + \mathcal{F} - \limsup_{k,l} \rho\left(3\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}} g_{l_0 l_0} - g_{l_0 l_0}}{\sigma}\right)\right) \\ \leq\ & \varepsilon(1 + R) + \mathcal{F} - \limsup_{k,l} \rho\left(3\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}} g_{l_0 l_0} - g_{l_0 l_0}}{\sigma}\right)\right) \end{aligned}$$

also, resulting from previous theorem,

$$\begin{aligned} 0 &= \mathcal{F} - \lim_{k,l} \rho\left(3\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}} g_{l_0 l_0} - g_{l_0 l_0}}{\sigma}\right)\right) \\ &= \mathcal{F} - \limsup_{k,l} \rho\left(3\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}} g_{l_0 l_0} - g_{l_0 l_0}}{\sigma}\right)\right) \end{aligned}$$

which gives

$$0 \leq \mathcal{F} - \limsup_{k,l} \rho\left(\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}} h - h}{\sigma}\right)\right) \leq \varepsilon(1 + R).$$

From arbitrariety of $\varepsilon > 0$, it follows that

$$\mathcal{F} - \limsup_{k,l} \rho\left(\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}} h - h}{\sigma}\right)\right) = 0.$$

Furthermore,

$$\mathcal{F}-\lim_{k,l}\rho\left(\lambda_0^*\left(\frac{A_{klij}^{\mathbb{T}}h-h}{\sigma}\right)\right)=0,$$

this completes the proof. □

Remark 2.3. Note that, in Theorem 2.2, in general it is not possible to obtain $\mathcal{F}-$relative strong convergence unless the modular ρ satisfies the Δ_2-condition.

If one replaces the scale function by a nonzero constant, then the condition (2.1) reduces to

$$\mathcal{F}-\limsup_{k,l}\rho\left(\lambda\left(A_{klij}^{\mathbb{T}}h\right)\right)\leq R\rho\left(\lambda h\right), \text{ uniformly in } i,j, \tag{2.10}$$

for every $h\in X_{\mathbb{T}}$, $\lambda>0$ and for an absolute positive constant R. In this case, the next result immediately follows from our Theorem 2.2.

Corollary 2.4. *Let $\mathcal{A}=\left(A^{(i,j)}\right)$ be a sequence of four dimensional infinite non-negative real matrices and let ρ be a monotone, strongly finite, absolutely continuous and $N-$quasi semiconvex modular. Suppose that h_r and a_r, $r=0,1,2,...,q$, satisfy properties $(P1)$ and $(P2)$. Let $\mathbb{T}=(T_{mn})$ be a double sequence of positive linear operators satisfying (2.10). If $\left(A_{klij}^{\mathbb{T}}h_r\right)$ is $\mathcal{F}-$strongly convergent to h_r, uniformly in i,j, $r=0,1,2,...,q$, in X_ρ, then $\left(A_{klij}^{\mathbb{T}}h\right)$ is $\mathcal{F}-$modularly convergent to h, uniformly in i,j, in X_ρ such that h is any function belonging to $D\cap X_\rho$ with $h-C_b(X)\subset X_{\mathbb{T}}$.*

If one replaces the matrices $A^{(i,j)}$ by the identity matrix and take the scale function as a non-zero constant, then the condition (2.1) reduces to

$$\mathcal{F}-\limsup_{m,n}\rho\left(\lambda\left(T_{mn}h\right)\right)\leq R\rho\left(\lambda h\right) \tag{2.11}$$

for every $h\in X_{\mathbb{T}}$, $\lambda>0$ and for an absolute positive constant R. In this case, the following result immediately follows from our Theorem 2.2.

Corollary 2.5. *Let ρ be a monotone, strongly finite, absolutely continuous and $N-$quasi semiconvex modular. Suppose that h_r and a_r, $r=0,1,2,...,q$, satisfy properties $(P1)$ and $(P2)$. Let $\mathbb{T}=(T_{mn})$ be a double sequence of positive linear operators satisfying (2.11) and assume that $\sigma_r(u)$ is an unbounded function satisfying $|\sigma_r(u)|\geq b_r>0$ $(r=0,1,2,...,q)$. If $(T_{mn}h_r)$ is $\mathcal{F}-$strongly convergent to h_r to the scale function σ_r, $r=0,1,2,...,q$, in X_ρ then $(T_{mn}h)$ is $\mathcal{F}-$modularly convergent to h to the scale function σ in X_ρ where $\sigma(u)=\max\{|\sigma_r(u)|:r=0,1,2,...,q\}$ and h is any function belonging to $D\cap X_\rho$ with $h-C_b(X)\subset X_{\mathbb{T}}$.*

3. Application

In this section, we display an example such that our Korovkin-type approximation results in modular spaces are stronger than the classical ones.

Example 3.1. Let us consider $X=[0,1]^2\subset\mathbb{R}^2$, $\mathcal{F}$ be a free filter of $\mathbb{N}^2$, containing a subset $F\subset\mathbb{N}^2$ such that $\mathbb{N}^2\backslash F$ is infinite and let $\varphi:[0,\infty)\to[0,\infty)$ is a continuous function with φ is convex, $\varphi(0)=0$, $\varphi(x)>0$ *for any* $x>0$ *and* $\lim_{x\to\infty}\varphi(x)=\infty$. Then, the functional ρ^φ defined by

$$\rho^\varphi(h):=\int_0^1\int_0^1\varphi\left(|h(x,y)|\right)dxdy \quad \text{for } h\in L^0(X),$$

is a convex modular on $L^0(X)$ and

$$X_{\rho^\varphi}:=\left\{h\in L^0(X):\rho^\varphi(\lambda h)<+\infty \quad \text{for some } \lambda>0\right\}$$

is the Orlicz space generated by φ.

For every $(x,y) \in X$, let $h_0(x,y) = a_3(x,y) = 1$, $h_1(x,y) = x$, $h_2(x,y) = y$, $h_3(x,y) = a_0(x,y) = x^2+y^2$, $a_1(x,y) = -2x$, $a_2(x,y) = -2y$. For every $m,n \in \mathbb{N}$, $u_1,u_2 \in [0,1]$, let $K_{mn}(u_1,u_2) = (m+1)(n+1)u_1^m u_2^n$ and for $h \in C(X)$ and $(x,y) \in X$, set

$$M_{mn}(h;x,y) = \int_0^1 \int_0^1 K_{mn}(u_1,u_2)\, h(u_1 x, u_2 y)\, du_1 du_2.$$

Then we get

$$\begin{aligned} & \int_0^1 \int_0^1 K_{mn}(u_1,u_2)\, du_1 du_2 \\ = \;& (m+1)\left(\int_0^1 u_1^m du_1\right)(n+1)\left(\int_0^1 u_2^n du_2\right) = 1, \end{aligned}$$

and hence, $M_{mn}(h_0;x,y) = h_0(x,y) = 1$. Also, we know from [7] that

$$\begin{aligned} |M_{mn}(h_1;x,y) - h_1(x,y)| &\leq \frac{1}{m+2}, \quad |M_{mn}(h_2;x,y) - h_2(x,y)| \leq \frac{1}{n+2}, \\ \left|M_{mn}\left(h_1^2;x,y\right) - h_1^2(x,y)\right| &\leq \frac{2}{m+3}, \quad \left|M_{mn}\left(h_2^2;x,y\right) - h_2^2(x,y)\right| \leq \frac{2}{n+3}, \end{aligned}$$

and for each $m,n \geq 2$, $h \in X_{\rho^\varphi}$ we get $\rho^\varphi(M_{mn}h) \leq 32\rho^\varphi(h)$. Moreover, (M_{mn}) satisfies the condition (14) in [30] with $X_{\mathbb{M}} = X_{\rho^\varphi}$ and $(M_{mn}h)$ is modular convergent to $h \in X_{\rho^\varphi}$. Using the operators $\mathbb{M} = (M_{mn})$, we define the double sequence of positive linear operators $\mathbb{L} = (L_{mn})$ on X_{ρ^φ} as follows:

$$L_{mn}(h;x,y) = (1 + g_{mn}(x,y))\, M_{mn}(h;x,y), \text{ for } h \in X_{\rho^\varphi},$$

$x,y \in [0,1]$ and $m,n \in \mathbb{N}$, where $g_{mn} : X \to \mathbb{R}$ defined by

$$\begin{aligned} & g_{mn}(x,y) \\ = & \begin{cases} 1, & (m,n) \notin F, \\ (m^2+1)\, n^3 xy, & (x,y) \in \left(0,\frac{1}{m}\right) \times \left(0,\frac{1}{n}\right); (m,n) \in F, \\ 0, & (x,y) \notin \left(0,\frac{1}{m}\right) \times \left(0,\frac{1}{n}\right); \ (m,n) \in F. \end{cases} \end{aligned}$$

If $\varphi(x) = x^p$ for $1 \leq p < \infty$, $x \geq 0$ then $X_{\rho^\varphi} = L_p(X)$ and we have for any function $h \in X_{\rho^\varphi}$, $\rho^\varphi(h) = \|h\|_p^p$. Choose $p = 1$.

It is clear that

$$\begin{aligned} \rho(\lambda_0(g_{mn} - g)) &= \|\lambda_0(g_{mn} - g)\|_1 \\ &= \lambda_0 \begin{cases} 1, & (m,n) \notin F, \\ \frac{(m^2+1)n}{4m^2}, & (x,y) \in \left(0,\frac{1}{m}\right) \times \left(0,\frac{1}{n}\right); (m,n) \in F, \\ 0, & (x,y) \notin \left(0,\frac{1}{m}\right) \times \left(0,\frac{1}{n}\right); \ (m,n) \in F. \end{cases} \end{aligned}$$

where $g = 0$, then (g_{mn}) does not converge $\mathcal{F}-$modularly to $g = 0$. Now, we choose $\sigma_r(x,y) = \sigma(x,y)$ $(r = 0,1,2,3)$ where $\sigma(x,y) = \begin{cases} \frac{1}{x^2 y}, & (x,y) \in (0,1] \times (0,1] \\ 1, & \text{otherwise} \end{cases}$ on $L_1(X)$. Also, assume that $\mathcal{A} := \left(A^{(i,j)}\right) = \left(a_{klmn}^{(i,j)}\right)$ is a sequence of four dimensional infinite matrices defined by $a_{klmn}^{(i,j)} = \frac{1}{kl}$ if $i \leq m \leq i+k-1$, $j \leq n \leq j+l-1$, $(i,j = 1,2,\ldots)$ and $a_{klmn}^{(i,j)} = 0$ otherwise. In this case $\mathcal{A}-$summability method reduces to almost convergence of double sequences introduced by Moricz and Rhoades [24]. Then, it can be seen that, for every $h \in L_1(X)$, $\lambda > 0$ and for positive constant R_0 that

$$\mathcal{F} - \limsup_{k,l} \left\| \lambda \left(\frac{A^{\mathbb{L}}_{klij} h}{\sigma} \right) \right\|_1 \leq R_0 \left\| \lambda h \right\|_1, \text{ uniformly in } i, j.$$

Now, observe that

$$\begin{aligned}
L_{mn}(h_0; x, y) - h_0(x, y) &= g_{mn}(x, y), \\
L_{mn}(h_1; x, y) - h_1(x, y) &\leq \frac{1 + g_{mn}(x, y)}{m+2} + g_{mn}(x, y), \\
L_{mn}(h_2; x, y) - h_2(x, y) &\leq \frac{1 + g_{mn}(x, y)}{n+2} + g_{mn}(x, y), \\
L_{mn}(h_3; x, y) - h_3(x, y) &\leq (1 + g_{mn}(x, y)) \left(\frac{2}{m+3} + \frac{2}{n+3} \right) + 2 g_{mn}(x, y).
\end{aligned}$$

Hence, we can see, for any $\lambda > 0$, that

$$\begin{aligned}
&\left\| \lambda \left(\frac{A^{\mathbb{L}}_{klij} h_0 - h_0}{\sigma} \right) \right\|_1 \qquad (3.1) \\
&= \left\| \frac{\lambda}{\sigma} \left(\frac{1}{kl} \sum_{m=i}^{i+k-1} \sum_{n=j}^{j+l-1} (1 + g_{mn}) - 1 \right) \right\|_1 \leq \frac{1}{kl} \sum_{m=i}^{i+k-1} \sum_{n=j}^{j+l-1} \left\| \lambda \frac{g_{mn}}{\sigma} \right\|_1 \\
&= \lambda \begin{cases} 1, & (m,n) \notin F, \\ \frac{1}{kl} \sum\limits_{m=i}^{i+k-1} \sum\limits_{n=j}^{j+l-1} \frac{m^2+1}{12m^4}, & (x,y) \in \left(0, \frac{1}{m}\right) \times \left(0, \frac{1}{n}\right); (m,n) \in F, \\ 0, & (x,y) \notin \left(0, \frac{1}{m}\right) \times \left(0, \frac{1}{n}\right); \ (m,n) \in F. \end{cases}
\end{aligned}$$

Then we can easily see that

$$\mathcal{F} - \lim_{k,l} \left\| \lambda \left(\frac{A^{\mathbb{L}}_{klij} h_0 - h_0}{\sigma} \right) \right\|_1 = 0, \text{ uniformly in } i, j.$$

Also, we have

$$\begin{aligned}
\left\| \lambda \left(\frac{A^{\mathbb{L}}_{klij} h_1 - h_1}{\sigma} \right) \right\|_1 &\leq \left\| \frac{\lambda}{\sigma} \sum_{m=i}^{i+k-1} \sum_{n=j}^{j+l-1} \frac{1}{kl} \left(\frac{1 + g_{mn}}{m+2} + g_{mn} \right) \right\|_1 \\
&\leq \frac{1}{kl} \sum_{m=i}^{i+k-1} \sum_{n=j}^{j+l-1} \frac{1}{m+2} \left\| x^2 y \right\|_1 \\
&\quad + \frac{2}{kl} \sum_{m=i}^{i+k-1} \sum_{n=j}^{j+l-1} \left\| \lambda \frac{g_{mn}}{\sigma} \right\|_1
\end{aligned}$$

from above inequality, since $\mathcal{F} - \lim\limits_{k,l} \left(\sup\limits_{i,j} \frac{1}{kl} \sum\limits_{m=i}^{i+k-1} \sum\limits_{n=j}^{j+l-1} \frac{1}{m+2} \right) = 0$ and from the inequality (3.1), we have

$$\mathcal{F} - \lim_{k,l} \left\| \lambda \left(\frac{A^{\mathbb{L}}_{klij} h_1 - h_1}{\sigma} \right) \right\|_1 = 0, \text{ uniformly in } i, j.$$

Similarly, we get

$$\mathcal{F} - \lim_{k,l} \left\| \lambda \left(\frac{A^{\mathbb{L}}_{klij} h_2 - h_2}{\sigma} \right) \right\|_1 = 0, \text{ uniformly in } i, j.$$

Finally, since

$$\begin{aligned} &\left\|\lambda\left(\frac{A_{klij}^{\mathbb{L}}h_3-h_3}{\sigma}\right)\right\|_1 \\ \leq\ & \left\|\frac{\lambda}{\sigma}\sum_{m=i}^{i+k-1}\sum_{n=j}^{j+l-1}\frac{1}{kl}\left((1+g_{mn}(x,y))\left(\frac{2}{m+3}+\frac{2}{n+3}\right)+2g_{mn}(x,y)\right)\right\|_1 \\ \leq\ & \frac{1}{kl}\sum_{m=i}^{i+k-1}\sum_{n=j}^{j+l-1}\left(\frac{2}{m+3}+\frac{2}{n+3}\right)\left\|x^2y\right\|_1+\frac{4}{kl}\sum_{m=i}^{i+k-1}\sum_{n=j}^{j+l-1}\left\|\lambda\frac{g_{mn}}{\sigma}\right\|_1. \end{aligned}$$

Hence we can easily see that

$$\mathcal{F}-\lim_{k,l}\left\|\lambda\left(\frac{A_{klij}^{\mathbb{L}}h_3-h_3}{\sigma}\right)\right\|_1=0,\ \text{uniformly in } i,j.$$

So, our new operator $\mathbb{L}=(L_{mn})$ satisfies all conditions of Theorem 2.2 and therefore we obtain

$$\mathcal{F}-\lim_{k,l}\left\|\lambda_0\left(\frac{A_{klij}^{\mathbb{L}}h-h}{\sigma}\right)\right\|_1=0,\ \text{uniformly in } i,j.$$

for some $\lambda_0>0$, for any $h\in L_1(X)$. However, $(L_{mn}h_0)$ is neither $\mathcal{F}-\mathcal{A}-$summable nor $\mathcal{F}-$modularly convergent to h_0. Thus (L_{mn}) does not fulfil the Corollary 2.4 and Corollary 2.5. Also, since $(L_{mn}h_0)$ does not converge modularly to h_0, the double modular Korovkin type theorem does not work for the sequence $\mathbb{L}=(L_{mn})$ ([30]).

4. Rates of approximation for abstract Korovkin theorems

Now we present some estimates of rates of convergence for abstract Korovkin-type theorems. Let $X=(X,d)$ is a metric space, satisfying the following property:

(P^*) For every $n\in\mathbb{N}$ and $u,v\in X$ with $u\neq v$, there are $n+1$ points x_i, $i=0,1,...,n+1$, such that $u=x_0$, $v=x_{n+1}$ and $d(x_i,x_{i+1})\leq\frac{1}{n}d(u,v)$ for each $i=0,1,...,n$.

Some classical examples of these spaces satisfying condition (P^*) are the Euclidean multidimensional spaces equipped with the usual metric and the space $\mathbb{R}^\Lambda$ endowed with the sup-norm, where Λ is any abstract nonempty set (see also [7]).

For $h\in C_b(X)$ and $\delta>0$, we consider the following usual modulus of continuity:

$$\omega(h,\delta):=\sup_{u,v\in X:\ d(u,v)\leq\delta}|h(u)-h(v)|.$$

It is readily seen that, $\omega(h,\delta)$ is an increasing function of δ, $|h(u)-h(v)|\leq\omega(h,d(u,v))$ for each $u,v\in X$, $\omega(h,\delta)\leq 2M$ for every $\delta>0$, where $M:=\sup_{v\in X}|h(v)|$, and $\omega(h,\gamma\delta)\leq(1+\gamma)\omega(h,\delta)$ for every $\gamma,\delta>0$.

Theorem 4.1. *Let $\mathcal{A}=\left(A^{(i,j)}\right)$ be a sequence of four dimensional infinite non-negative real matrices, ρ be a monotone, strongly finite, $\mathbb{T}=(T_{mn})$ be a double sequence of positive linear operators, and (X,d) satisfy condition (P^*). Let (α_{mn}) and (β_{mn}) be two double sequences of positive real numbers with $\mathcal{F}-\lim_{m,n}\alpha_{mn}=0$ and $\mathcal{F}-\lim_{m,n}\beta_{mn}=0$ and $\sigma_r(u)$ be an unbounded function, $|\sigma_r(u)|>0$ $(r=0,1)$. For each $u,v\in X$, set $\psi(u)(v)=d(u,v)$ and for every $h\in C_c(X)$, put $\delta_{klij}=\left\|A_{klij}^{\mathbb{T}}\psi\right\|$, where $\|.\|$ is the sup-norm and the supremum is taken with respect to the support of h. If there exist $\zeta>0$ with*

(i) $\rho\left(\zeta\left(\frac{A_{klij}^{\mathbb{T}}h_0-h_0}{\sigma_0}\right)\right)=o_{\mathcal{F}}(\alpha_{kl})$, *uniformly in* i,j,

(ii) $\rho\left(\zeta\left(\frac{\omega(h,\delta_{klij})}{\sigma_1}\right)\right)=o_{\mathcal{F}}(\beta_{kl})$, *uniformly in* i,j.

Then for every $h \in C_c(X)$ we get

$$\rho\left(\zeta\left(\frac{A^{\mathbb{T}}_{klij}h-h}{\sigma}\right)\right)=o_{\mathcal{F}}(\gamma_{kl}),\ \textit{uniformly in } i,j,$$

where $\gamma_{kl}:=\max\{\alpha_{kl},\beta_{kl}\}$ and $\sigma(u)=\max\{|\sigma_r(u)|:r=0,1\}$.

A similar result holds also when $o_{\mathcal{F}}$ replaced by $O_{\mathcal{F}}$.

Proof. Let $h \in C_c(X)$, $M := \sup_{v\in X}|h(v)|$. Observe that $\omega(h,\delta)\le 2M$ for each $\delta>0$. Using the properties of the usual modulus of continuity, we get

$$|h(u)-h(v)|\le\omega(h,d(u,v))\le\left(1+\frac{d(u,v)}{\delta}\right)\omega(h,\delta) \tag{4.1}$$

for every $\delta>0$ and $u,v\in X$.Also, the support of h is (totally) bounded and so, $\sup_{u,v\in X} d(u,v)<\infty$ and by applying $A^{\mathbb{T}}_{klij}$, we find $K_{klij}>0$ with $\sup_{u,v\in X} A^{\mathbb{T}}_{klij}d(u,v)\le K_{klij}A^{\mathbb{T}}_{klij}h_0$. Hence, $\delta_{klij}\in\mathbb{R}$ for each $k,l,i,j\in\mathbb{N}$. Let now $\delta=\delta_{klij}$. Since T_{mn} is linear and positive, by applying T_{mn} to (4.1), we have

$$\begin{aligned}\left|A^{\mathbb{T}}_{klij}(h;u)-h(u)\right| &\le A^{\mathbb{T}}_{klij}\left(\left(1+\frac{d(u,v)}{\delta}\right)\omega(h,\delta);u\right)+M\left|A^{\mathbb{T}}_{klij}(h_0;u)-h_0(u)\right|\\ &\le \omega(h,\delta)\left|A^{\mathbb{T}}_{klij}(h_0;u)-h_0(u)\right|+\frac{\omega(h,\delta)}{\delta}A^{\mathbb{T}}_{klij}(\psi;u)\\ &\quad+\omega(h,\delta)+M\left|A^{\mathbb{T}}_{klij}(h_0;u)-h_0(u)\right|\\ &\le 3M\left(\left|A^{\mathbb{T}}_{klij}(h_0;u)-h_0(u)\right|+\omega(h,\delta)\right).\end{aligned}$$

Let $\zeta>0$. By multiplying the both sides of the above inequality by $\frac{1}{|\sigma(u)|}$ the last inequality gives that

$$\zeta\left|\frac{A^{\mathbb{T}}_{klij}(h;u)-h(u)}{\sigma(u)}\right|\le 3M\zeta\left(\left|\frac{A^{\mathbb{T}}_{klij}(h_0;u)-h_0(u)}{\sigma(u)}\right|+\frac{\omega(h,\delta)}{|\sigma(u)|}\right).$$

Now, applying the modular ρ to both sides of the above inequality, since ρ is monotone and $\sigma(u)=\max\{|\sigma_r(u)|:r=0,1\}$, we get

$$\rho\left(\zeta\left(\frac{A^{\mathbb{T}}_{klij}h-h}{\sigma}\right)\right)\le\rho\left(6M\zeta\left(\frac{A^{\mathbb{T}}_{klij}h_0-h_0}{\sigma_0}\right)\right)+\rho\left(6M\zeta\frac{\omega(h,\delta)}{\sigma_1}\right).$$

Now considering the above inequality, the hypotheses (i) and (ii) we have

$$\mathcal{F}-\lim_{k,l}\frac{\rho\left(\zeta\left(\frac{A^{\mathbb{T}}_{klij}h-h}{\sigma}\right)\right)}{\gamma_{kl}}=0,$$

hence, the proof is completed at once. □

If one replaces the matrices $A^{(i,j)}$ by the identity matrix, then we get the following:

Corollary 4.2. *Let ρ be a monotone, strongly finite, $\mathbb{T}=(T_{mn})$ be a double sequence of positive linear operators, and (X,d) satisfy condition (P^*). Let (α_{mn}) and (β_{mn}) be two double sequences of positive real numbers with $\mathcal{F}-\lim_{m,n}\alpha_{mn}=0$ and $\mathcal{F}-\lim_{m,n}\beta_{mn}=0$ and $\sigma_r(u)$ be an unbounded function, $|\sigma_r(u)|>0$ $(r=0,1)$. For each $u,v\in X$, set $\psi(u)(v)=d(u,v)$ and for every $h\in C_c(X)$, put $\delta_{mn}=\|T_{mn}\psi\|$, where $\|.\|$ is the sup-norm and the supremum is taken with respect to the support of h. If there exist $\zeta>0$ with*

(i) $\rho\left(\zeta\left(\frac{T_{mn}h_0-h_0}{\sigma_0}\right)\right)=o_{\mathcal{F}}(\alpha_{mn})$, *(ii)* $\rho\left(\zeta\left(\frac{\omega(h,\delta_{mn})}{\sigma_1}\right)\right) = o_{\mathcal{F}}\left(\beta_{mn}\right).$

Then for every $h \in C_c(X)$ *we get*

$$\rho\left(\zeta\left(\frac{T_{mn}h - h}{\sigma}\right)\right) = o_{\mathcal{F}}\left(\gamma_{mn}\right),$$

where $\gamma_{mn} := \max\{\alpha_{mn}, \beta_{mn}\}$ *and* $\sigma(u) = \max\{|\sigma_r(u)| : r = 0,1\}.$

A similar results hold also when $o_{\mathcal{F}}$ *replaced by* $O_{\mathcal{F}}$.

5. An extension to non-positive linear operators

In this section, we relax the positivity condition of linear operators in the Korovkin theorems. In [1, 6, 7] there are some positive answers. Following this approach, we give some positive answers also for $\mathcal{F}-$relative $\mathcal{A}-$Summation process in modular spaces and prove a Korovkin type approximation theorem.

Let G be a bounded interval of $\mathbb{R}^2$, $C^2(G)$ (resp. $C_b^2(G)$) be the space of all functions defined on G, (resp. bounded and) continuous together with their first and second derivatives, $C_+ := \{h \in C_b^2(G) : h \geq 0\}$, $C_+^2 := \{h \in C_b^2(G) : h'' \geq 0\}$.

Let h_r, $r = 1,2,...,q$, and a_r, $r = 0,1,2,...,q$, be functions in $C_b^2(G)$, $P_u(v)$, $u,v \in G$, be as in (2.2), and suppose that $P_u(v)$ satisfies the properties $(P1)$, $(P2)$ and

$(P3)$ there is a positive real constant S_0 such that $P''_u(v) \geq S_0$ for all $u,v \in G$ (Here the second derivative is intended with respect to v).

Now we prove the following Korovkin type approximation theorem for not necessarily positive linear operators.

Theorem 5.1. *Let* $\mathcal{A}$, ρ *and* σ_r *be as in Theorem 2.1 and* h_r, a_r, $r = 0,1,2,...,q$ *and* $P_u(v)$, $u,v \in G$, *satisfies the properties* $(P1)$, $(P2)$ *and* $(P3)$. *Assume that* $\mathbb{T} = (T_{mn})$ *be a double sequence of linear operators and suppose that* $T_{mn}\left(C_+ \cap C_+^2\right) \subset C_+$, *for all* $m,n \in \mathbb{N}$. *If* $\left(A^{\mathbb{T}}_{klij}h_r\right)$ *is* $\mathcal{F}-$*relatively modularly convergent to* h_r *to the scale function* σ_r, *uniformly in* i,j, *in* X_ρ *for each* $r = 0,1,2,...,q$, *then* $\left(A^{\mathbb{T}}_{klij}h\right)$ *is* $\mathcal{F}-$*relatively modularly convergent to* h *to the scale function* σ, *uniformly in* i,j, *in* X_ρ *for every* $h \in C_b^2(G)$ *where* $\sigma(u) = \max\{|\sigma_r(u)| : r = 0,1,2,...,q\}.$

If $\left(A^{\mathbb{T}}_{klij}h_r\right)$ *is* $\mathcal{F}-$*relatively strongly convergent to* h_r *to the scale function* σ_r, *uniformly in* i,j, $r = 0,1,2,...,q$, *in* X_ρ *then* $\left(A^{\mathbb{T}}_{klij}h\right)$ *is* $\mathcal{F}-$*relatively strongly convergent to* h *to the scale function* σ, *uniformly in* i,j, *in* X_ρ *for every* $h \in C_b^2(G)$ *where* $\sigma(u) = \max\{|\sigma_r(u)| : r = 0,1,2,...,q\}.$

Furthermore, if ρ *is absolutely continuous,* $\mathbb{T}$ *satisfies the property (2.1) and* $\left(A^{\mathbb{T}}_{klij}h_r\right)$ *is* $\mathcal{F}-$*relatively strongly convergent to* h_r *to the scale function* σ_r, $r = 0,1,2,...,q$, *in* X_ρ *then* $\left(A^{\mathbb{T}}_{klij}h\right)$ *is* $\mathcal{F}-$*relatively modularly convergent to* h *to the scale function* σ *in* X_ρ *for every* $h \in D \cap X_\rho$ *with* $h - C_b(G) \subset X_{\mathbb{T}}$ *where* $\sigma(u) = \max\{|\sigma_r(u)| : r = 0,1,2,...,q\}.$

Proof. Let $h \in C_b^2(G)$. Since h is uniformly continuous and bounded on G, given $\varepsilon > 0$ with $0 < \varepsilon \leq 1$, there exists a $\delta > 0$ such that $|h(u) - h(v)| \leq \varepsilon$ for all $u,v \in G$, $|u - v| \leq \delta$. Let $P_u(v)$, $u,v \in G$, be as in (2.2) and let $\eta > 0$ be associated with δ, satisfying $(P2)$. As in Theorem 2.1, for every $\alpha \geq 1$ and $u,v \in G$, we have

$$-\varepsilon - \frac{2M\alpha}{\eta}P_u(v) \leq h(u) - h(v) \leq \varepsilon + \frac{2M\alpha}{\eta}P_u(v) \tag{5.1}$$

where $M = \sup_{v\in G}|h(v)|$. From (5.1) it follows that

$$h_{1,\alpha}(v) := \varepsilon + \frac{2M\alpha}{\eta}P_u(v) + h(v) - h(u) \geq 0, \tag{5.2}$$

$$h_{2,\alpha}(v) := \varepsilon + \frac{2M\alpha}{\eta} P_u(v) - h(v) + h(u) \geq 0. \tag{5.3}$$

Let H_0 satisfy $(P3)$. For each $v \in G$, we get

$$h''_{1,\alpha}(v) \geq \frac{2M\alpha H_0}{\eta} + h''(v), \; h''_{2,\alpha}(v) \geq \frac{2M\alpha H_0}{\eta} - h''(v).$$

Because of h'' is bounded on G, we can choose $\alpha \geq 1$ in such a way that $h''_{1,\alpha}(v) \geq 0$, $h''_{2,\alpha}(v) \geq 0$ for each $v \in G$. Hence $h_{1,\alpha}, h_{2,\alpha} \in C_+ \cap C^2_+$. Then, by hypothesis we get

$$A^{\mathbb{T}}_{klij}(h_{r,\alpha}; u) \geq 0 \text{ for all } k, l \in \mathbb{N}, \; u \in G \text{ and } r = 1, 2. \tag{5.4}$$

From (5.2)-(5.4) and the linearity of T_{mn}, for all $k, l \in \mathbb{N}$, we get

$$\begin{aligned} \varepsilon A^{\mathbb{T}}_{klij}(h_0; u) + \frac{2M\alpha}{\eta} A^{\mathbb{T}}_{klij}(P_u; u) + A^{\mathbb{T}}_{klij}(h; u) - h(u) A^{\mathbb{T}}_{klij}(h_0; u) &\geq 0, \\ \varepsilon A^{\mathbb{T}}_{klij}(h_0; u) + \frac{2M\alpha}{\eta} A^{\mathbb{T}}_{klij}(P_u; u) - A^{\mathbb{T}}_{klij}(h; u) + h(u) A^{\mathbb{T}}_{klij}(h_0; u) &\geq 0, \end{aligned}$$

thus,

$$\begin{aligned} -\varepsilon A^{\mathbb{T}}_{klij}(h_0; u) - \frac{2M\alpha}{\eta} A^{\mathbb{T}}_{klij}(P_u; u) &\leq h(u) A^{\mathbb{T}}_{klij}(h_0; u) - A^{\mathbb{T}}_{klij}(h; u) \\ &\leq \varepsilon A^{\mathbb{T}}_{klij}(h_0; u) + \frac{2M\alpha}{\eta} A^{\mathbb{T}}_{klij}(P_u; u). \end{aligned}$$

By arguing similarly as in the proof of Theorem 2.1, multiplying the inequality by $\frac{1}{|\sigma(u)|}$, using the modular ρ and for $i, j \in \mathbb{N}$, we have the assertion of the first part.

The other parts can be proved similarly as in the proofs of Theorem 2.1 and Theorem 2.2. □

References

[1] G.A. Anastassiou and O. Duman, *Towads intelligent modeling: Statistical approximation theory*, Intelligent System Reference Library 14, Springer-Verlag, Berlin, Heidelberg, New York, 2011.

[2] Ö.G. Atlıhan and C. Orhan, *Matrix summability and positive linear operators*, Positivity **11**, 387–389, 2007.

[3] Ö.G. Atlıhan and C. Orhan, *Summation process of positive linear operators*, Comput. Math. Appl. **56**, 1188–1195, 2008.

[4] C. Bardaro and I. Mantellini, *Korovkin's theorem in modular spaces*, Comment. Math. **47**, 239–253, 2007.

[5] C. Bardaro, J. Musielak and G. Vinti, *Nonlinear Integral Operators and Applications*, Walter de Gruyter, Berlin, Germany, 2003.

[6] C. Bardaro, A. Boccuto, X. Dimitriou and I. Mantellini, *Abstract Korovkin type theorems in modular spaces and applications*, Cent. Eur. J. Math. **11** (10), 1774–1784, 2013.

[7] C. Bardaro, A. Boccuto, K. Demirci, I. Mantellini and S. Orhan, *Korovkin-Type Theorems for Modular $\Psi - A-$Statistical Convergence*, J. Funct. Spaces, **2015**, 1–11, 2015.

[8] A. Boccuto and X. Dimitriou, *Korovkin-type theorems for abstract modular convergence*, Results Math. **69** (3-4), 477–495, 2016.

[9] K. Demirci, *$I-$limit superior and limit inferior*, Math. Commun. **6**, 165-172, 2001.

[10] K. Demirci, S. Orhan and B. Kolay, *Statistical Relative $A-$Summation Process for Double Sequences on Modular Spaces*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM **112**, 1249–1264, 2018.

[11] F. Dirik and K. Demirci, *Korovkin type approximation theorem for functions of two variables in statistical sense*, Turk. J. Math. **34**, 73–83, 2010.

[12] J. Fridy and C. Orhan, *Statistical limit superior and limit inferior*, Proc. Amer. Math. Soc. **125** (12), 3625–3631, 1997.

[13] A.D. Gadjiev and C. Orhan, *Some approximation theorems via statistical convergence*, Rocky Mountain J. Math. **32**, 129–138, 2002.

[14] U. Kadak, *Relative weighted almost convergence based on fractional-order difference operator in multivariate modular function spaces*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM **113** (3), 2201–2220, 2019.

[15] U. Kadak and S.A. Mohiuddine, *Generalized statistically almost convergence based on the difference operator which includes the (p, q)-gamma function and related approximation theorems*, Results Math. **73** (1), 1–31, 2018.

[16] S. Karakuş, K. Demirci and O. Duman, *Statistical approximation by positive linear operators on modular spaces*, Positivity **14**, 321–334, 2010.

[17] B. Kolay, S. Orhan, and K. Demirci, *Statistical Relative $A-$Summation Process and Korovkin-Type Approximation Theorem on Modular Spaces*, Iran. J. Sci. Technol. Trans. A Sci. **42** (2), 683–692, 2018.

[18] P.P. Korovkin, *Linear Operators and Approximation Theory*, Hindustan Publ. Co., Delhi, 1960.

[19] W.M. Kozlowski, *Modular function spaces*, Pure Appl. Math. **122**, Marcel Dekker, Inc., New York, 1988.

[20] K. Kuratowski, *Topology I-II*, Academic Press/PWN, New York-London/Warsaw, 1966/1968.

[21] I. Mantellini, *Generalized sampling operators in modular spaces*, Comment. Math. **38**, 77–92, 1998.

[22] S.A. Mohiuddine and B.A.S. Alamri, *Generalization of equi-statistical convergence via weighted lacunary sequence with associated Korovkin and Voronovskaya type approximation theorems*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM, **113** (3), 1955–1973, 2019.

[23] S.A. Mohiuddine, B. Hazarika and M.A. Alghamdi, *Ideal relatively uniform convergence with Korovkin and Voronovskaya types approximation theorems*, Filomat, **33** (14), 4549–4560, 2019.

[24] F. Moricz and B.E. Rhoades, *Almost convergence of double sequences and strong regularity of summability matrices*, Math. Proc. Camb. Phil. Soc. **104**, 283–294, 1988.

[25] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics **1034**, Springer-Verlag, Berlin, 1983.

[26] J. Musielak, *Nonlinear approximation in some modular function spaces I*, Math. Japon. **38**, 83–90, 1993.

[27] T. Nishishiraho, *Quantitative theorems on linear approximation processes of convolution operators in Banach spaces*, Tŏhoku Math. J. **33**, 109–126, 1981.

[28] T. Nishishiraho, *Convergence of positive linear approximation processes*, Tŏhoku Math. J. **35**, 441–458, 1983.

[29] S. Orhan and K. Demirci, *Statistical $A-$summation process and Korovkin type approximation theorem on modular spaces*, Positivity, **18** (4), 669–686, 2014.

[30] S. Orhan and K. Demirci, *Statistical approximation by double sequences of positive linear operators on modular spaces*, Positivity **19**, 23–36, 2015.

[31] S. Orhan and B. Kolay, *Korovkin type approximation for double sequences via statistical $A-$summation process on modular spaces*, Stud. Univ. Babeş-Bolyai Math. **63** (1), 125–140, 2018.

[32] R.F. Patterson and E. Savaş, *Uniformly summable double sequences*, Studia Sci. Math. Hungar. **44**, 147–158, 2007.

[33] A. Pringsheim, *Zur theorie der zweifach unendlichen zahlenfolgen*, Math. Ann. **53**, 289–321, 1900.

[34] E. Savaş and B.E. Rhoades, *Double summability factor theorems and applications*, Math. Inequal. Appl. **10**, 125–149, 2007.

[35] E. Taş, *Abstract Korovkin type theorems on modular spaces by A-summability*, Math. Bohem. **143** (4), 419–430, 2018.

[36] S. Yıldız, *Abstract versions of Korovkin theorems on modular spaces via statistical relative summation process for double sequences*, Tbilisi Math. J. **13** (1), 139–156, 2020.