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Equation Including Local Fractional Derivative and Neumann Boundary Conditions

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Abstract

The aim of this study to discuss the construction of solution of fractional partial differential equations (FPDEs) with initial and boundary conditions. Since the homogenous initial boundary value problem involves local fractional-order derivative, it has classical initial and boundary conditions. By means of the separation of variables method (SVM) and the inner product on $L^2[0,l]$, we construct the solution in this series form in terms of eigenfunctions of related Sturm-Liouville problem. An illustrative example presents the applicability and influence of separation of variables method on fractional mathematical problems.

1. Introduction

Since mathematical models including fractional derivatives play a vital role fractional derivatives draw the growing attention of many researchers in various branches of sciences. Therefore there are many different fractional derivatives such as Caputo, Riemann-Liouville, Atangana-Baleanu. However these fractional derivatives do not satisfy the most important properties of ordinary derivative which leads to many difficulties to analyze or obtain the solution of fractional mathematical models. As a result many scientists focus on defining new fractional derivatives to cover the setbacks of the defined ones. Moreover, the success of mathematical modelling of systems or processes depends on the fractional derivative, it involves, since the correct choice of the fractional derivative allows us to model the real data of systems or processes accurately.

In order to define new fractional derivatives, various methods exist and these ones are classified based on their features and formation such as nonlocal fractional derivatives and local fractional derivatives. The proportional derivative is a newly defined fractional derivative which is generally defined as

- ${}^{P}D_{\alpha}f(t) = K_1(\alpha,t)\, f(t) + K_0(\alpha,t) f'(t)$

where the functions K_0 and K_1 satisfy certain properties in terms of limit [1] and f is a differentiable function. Notice that this derivative can be regarded as an extension of conformable derivative and is used in control theory.

In this study we focus on obtaining the solution of the following fractional diffusion equation including various proportional derivative operator by making use of the separation of variables method:

- ${}^{P}D_t^{\alpha}u(x,t) = \gamma^2 u_{xx}(x,t),$
- $u_x(0,t) = u_x(l,t) = 0,$
- $u(x,0) = f(x)$

where $0 < \alpha < 1, 0 \le x \le l, 0 \le t \le T, \gamma \in R$. Here we use the following forms of the proportional derivatives:

$${}^{P}D_t^{\alpha}f(t) = K_1(\alpha)\, f(t) + K_0(\alpha) f'(t). \tag{1}$$

Especially we consider the following ones:

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- ${}_1^P D_t^\alpha f(t) = (1 - \alpha) f(t) + \alpha f'(t)$ and
- ${}_2^P D_t^\alpha f(t) = (1 - \alpha^2) f(t) + \alpha^2 f'(t)$

From a physical aspect, the intrinsic nature of the physical system can be reflected in the mathematical model of the system by using fractional derivatives. Therefore the solution of the fractional mathematical model is in excellent agreement with the predictions and experimental measurement of it. The systems whose behaviour is non-local can be modelled better by fractional mathematical models. Moreover, the degree of its non-locality can be arranged by the order of fractional derivative. In order to analyze the diffusion in a non-homogenous medium that has memory effects, it is better to analyze the solution of the fractional mathematical model for this diffusion. As a result in order to model a process, the correct choices of fractional derivative and its order must be determined.

In this study, local fractional derivative is used to model diffusion problems as in the case of non-local fractional derivative, models including local fractional derivatives give better results than models including integer order derivatives. In the mathematical modelling of diffusion problem for different matters such as liquid, gas and temperature, the suitable fractional order α is chosen, since the diffusion coefficient γ^2 depends on the order α of fractional derivative [2]. This mathematical modelling describes the behaviour of matter in a phase. There are many published works on the diffusion of various matters in science especially in fluid mechanics and gas dynamics [3-14]. From this aspect, analysis of this problem plays an important role in application. Moreover, sub-diffusion cases for which $0 < \alpha < 1$ are under consideration. The solution of the fractional mathematical model of sub-diffusion cases behaves much slower than the solution of the integer-order mathematical model unlike the fractional mathematical model for super-diffusion.

2. Main Results

Let us consider the following problem including the proportional derivative in Eq. (1)

$$ {}^P D_t^\alpha u(x,t) = \gamma^2 u_{xx}(x,t), \quad (2)$$

$$ u_x(0,t) = u_x(l,t) = 0, \quad (3)$$

$$ u(x,0) = f(x) \quad (4)$$

where $0 < \alpha < 1, 0 \le x \le l, 0 \le t \le T, \gamma \in R$.

By employing SVM, the solution of the problem in Eqs. (2-4) can be written in the following form:

$$ u(x,t;\alpha) = X(x)\, T(t;\alpha) \quad (5)$$

where $0 \le x \le l,\ 0 \le t \le T$. Plugging Eq. (5) into Eq. (2) and arranging it, we have

$$ \frac{{}^P D_t^\alpha (T(t;\alpha))}{T(t;\alpha)} = \gamma^2 \frac{X''(x)}{X(x)} = -\lambda^2 \quad (6)$$

By taking the last equality from Eq. (6) and utilizing the boundary conditions in Eq. (3), we have the following problem:

$$ X''(x) + \lambda^2 X(x) = 0, \quad (7)$$

$$ X'(0) = X'(l) = 0. \quad (8)$$

The problem in Eq. (7-8) is the well known Sturm-Liouville problem. The solution to this problem is as in Eq. (9):

$$ X_n(x) = \cos\left(w_n\left(\frac{x}{l}\right)\right), n = 0,1,2,3,\ldots \quad (9)$$

where $w_n = n\pi, n = 0,1,2,3,\ldots$ satisfy the equation $sin(w_n) = 0$ and $\lambda_n = \frac{w_n}{l}, \lambda_1 < \lambda_2 < \lambda_3 < \cdots, n = 0,1,2,3, \ldots.$

The second equation in Eq. (6) for eigenvalue λ_n yields the fractional differential equation below:

- $\frac{{}^P D_t^\alpha (T(t;\alpha))}{T(t;\alpha)} = -\gamma^2 \lambda_n^2,$
- $\frac{K_1(\alpha)\, T_n(t;\alpha) + K_0(\alpha) T_n^1(t;\alpha)}{T_n(t;\alpha)} = -\gamma^2 \lambda_n^2,$
- $K_0(\alpha) T_n^1(t;\alpha) + (\gamma^2 \lambda_n^2 + K_1(\alpha)) T_n(t;\alpha) = 0,$

which yields the following solution

- $T_n(t;\alpha) = exp\left(-\frac{\gamma^2\lambda_n^2 + K_1(\alpha)}{K_0(\alpha)} t\right), n = 0,1,2,3,$

The solution for every eigenvalue λ_n is constructed as

- $u_n(x,t;\alpha) = X_n(x) T_n(t;\alpha) =$
  $exp\left(-\frac{\gamma^2\lambda_n^2 + K_1(\alpha)}{K_0(\alpha)} t\right) \cos\left(w_n\left(\frac{x}{l}\right)\right), n = 0,1,2,3,$

which leads to the following general solution

$$ u(x,t;\alpha) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(w_n\left(\frac{x}{l}\right)\right) exp\left(-\frac{\gamma^2\lambda_n^2 + K_1(\alpha)}{K_0(\alpha)} t\right). \quad (10)$$

Note that it satisfies boundary conditions and fractional differential equation. If γ^2 is replaced by the fractional diffusion coefficient $c^2\tau_\alpha^{1-\alpha}$ where c^2 is ordinary diffusion coefficient and τ_α is a time constant the solution takes the following form:

- $u(x,t;\alpha) =$
  $A_0 + \sum_{n=1}^{\infty} A_n \cos\left(w_n\left(\frac{x}{l}\right)\right) exp\left(-\frac{c^2\tau_\alpha^{1-\alpha}\lambda_n^2+K_1(\alpha)}{K_0(\alpha)}t\right).$

The coefficients of the general solution are established by taking the following initial condition into account:

- $u(x,0) = f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(w_n\left(\frac{x}{l}\right)\right).$

The coefficients A_n for $n = 0,1,2,3,\ldots$ determined by the help of inner product defined on $L^2[0,l]$:

- $A_0 = \frac{1}{l}\int_0^l f(x)dx,$
- $A_n = \frac{2}{l}\int_0^l f(x)\cos\left(w_n\left(\frac{x}{l}\right)\right).$

3. Illustrative Examples

Now the obtained results are illustrated by two examples in this part. Let the Neumann boundary value problem below be taken into account:

$$u_t(x,t) = u_{xx}(x,t), \tag{11}$$

$$u_x(0,t) = 0,\ u_x(1,t) = 0, \tag{12}$$

$$u(x,0) = \cos(\pi x) \tag{13}$$

whose solution is established as:

- $u(x,t) = \cos(\pi x)\, e^{-\pi^2 t}$

where $0 \le x \le 1, 0 \le t \le T$.

Example 1. Now let the following problem called fractional heat-like problem be taken into consideration:

$$\begin{aligned} &{}^P_1D_t^\alpha u(x,t) = u_{xx}(x,t),\\ &u_x(0,t) = 0,\ u_x(1,t) = 0,\\ &u(x,0) = \cos(\pi x) \end{aligned} \tag{14}$$

where $0 < \alpha < 1, 0 \le x \le 1,\ 0 \le t \le T$. It is clear from Eq. (10) that the solution to the above problem can be obtained in the following form:

$$u(x,t;\alpha) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x)\, exp\left(-\frac{n^2\pi^2+1-\alpha}{\alpha}t\right) \tag{15}$$

Plugging $t = 0$ into the solution in Eq. (15) and utilizing the initial condition in Eq. (14) leads to:

- $\cos(\pi x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x)$

The coefficients A_n for $n = 0,1,2,3,\ldots$ are determined by the help of the inner product as follows:

- $A_0 = 0.$

For $n \neq 1, A_n = 0. n = 1$ we get

- $A_1 = 1.$

Thus

$$u(x,t;\alpha) = \cos(\pi x)\, exp\left(-\frac{\pi^2+1-\alpha}{\alpha}t\right). \tag{16}$$

It is clear that taking $\alpha = 1$ in Eq. (16) leads to the solution of the problem (Eqs. (11-13)) which implies the correctness of the method employed in this study.

Example 2. Now let the following problem called fractional heat-like problem be taken into consideration:

- ${}^P_2D_t^\alpha u(x,t) = u_{xx}(x,t),$
- $u_x(0,t) = 0,\ u_x(1,t) = 0,$
- $u(x,0) = \cos(\pi x)$

where $0 < \alpha < 1, 0 \le x \le 1,\ 0 \le t \le T$. It is clear from Eq. (10) that the solution to the above problem can be obtained in the following form:

- $u(x,t;\alpha) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x)\, exp\left(-\frac{\gamma^2\lambda_n^2+1-\alpha^2}{\alpha^2}t\right).$

As in Example 1, after similar computations the solution can be constructed as follows:

- $u(x,t;\alpha) = \cos(\pi x)\, exp\left(-\frac{\pi^2+1-\alpha^2}{\alpha^2}t\right).$

The graphics of solutions for Example 1, Example 2, and the problem (Eqs. (11-13)) in 2D and 3D are given in Figure 1 and Figure 2 respectively.

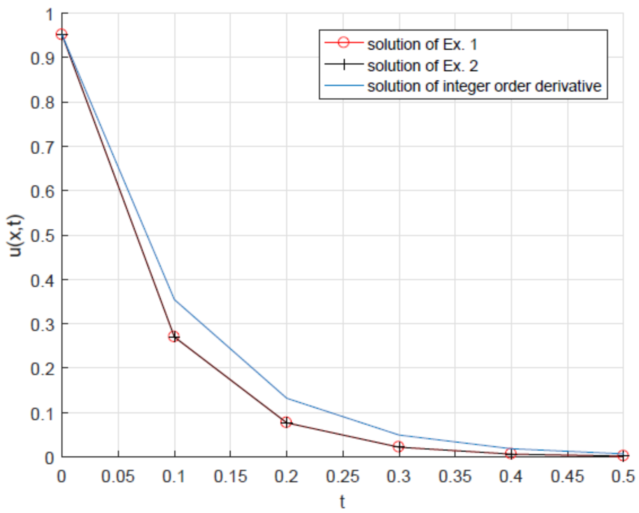

Figure 1. The graphics of solutions for Example 1 and Example 2 in 2D at $x = 0.1$ for $\alpha = 0.8$.

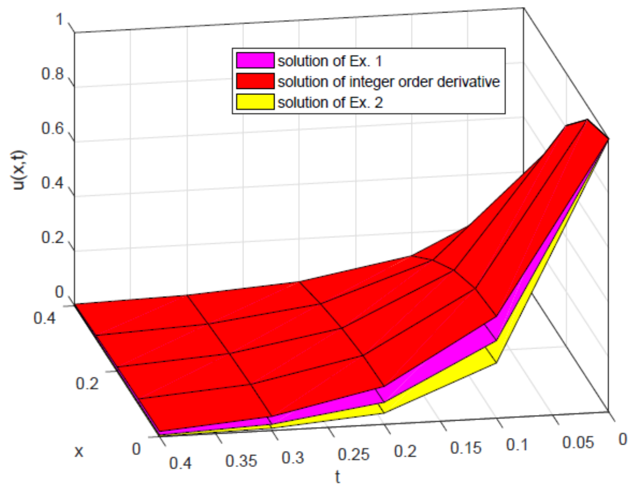

Figure 2. The graphics of solutions for Example 1 and Example 2 in 3D for $\alpha = 0.8$.

4. Conclusion

In this investigation, the solution of the diffusion problem including local time-fractional derivatives in one dimension is constructed analytically in the series form. Taking the SVM into account, the solution is formed in the form of a series in terms of the eigenfunctions of a related Sturm-Liouville problem including fractional derivative in a proportional sense.

Based on the analytic solution, we reach the conclusion that diffusion processes decay exponential with time until initial condition is reached. As α tends to 0, the rate of decay increases. This implies that in the mathematical model for diffusion of the matter which has a small diffusion rate the value of α must be close to 0. This model can account for various diffusion processes of various methods.

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