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SOLUTION OF FRACTIONAL KINETIC EQUATIONS INVOLVING GENERALIZED HURWITZ-LERCH ZETA FUNCTION USING SUMUDU TRANSFORM

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Abstract. Fractional kinetic equations (FKEs) comprising a large array of special functions have been extensively and successfully applied in specification and solving many significant problems of astrophysics and physics. In this present work, our aim is to demonstrate solutions of (FKEs) of the generalized Hurwitz-Lerch Zeta function by applying the Sumudu transform. In addition to these, solutions of (FKEs) in special conditions of generalised Hurwitz-Lerch Zeta function have been derived.

1. Introduction

The Hurwitz-Lerch Zeta function is defined by [34, 35]:

$$\Phi(\zeta, m, \alpha) = \sum_{n=0}^{\infty} \frac{\zeta^n}{(n+\alpha)^m} \tag{1}$$

$$\big(\alpha \in \mathbb{C} \setminus \mathbb{Z}_0;\ m \in \mathbb{C} \text{ when } |\zeta| < 1;\ \Re(m) > 1 \text{ when } |\zeta| = 1\big).$$

Many researchers studied many different generalisations and extensions of the Hurwitz-Lerch Zeta function by inserting certain additional parameters to the series representation of the Hurwitz-Lerch Zeta function. The interested readers can refer to these earlier publications for further researches and applications [13, 14, 15, 18, 20, 21, 22, 25, 26, 33, 36, 38, 42].

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In 2011, Srivastava et. al [41, p.491, Eq.(1.20)] introduced and studied the following extension of the generalized Hurwitz-Lerch Zeta function:

$$\Phi^{(\sigma,\,\rho,\,\kappa)}_{\lambda,\,\mu;\,\omega}(\zeta, m, a) = \sum_{n=0}^{\infty} \frac{(\lambda)_{\sigma n}\,(\mu)_{\rho n}}{(\omega)_{\kappa n}\, n!} \frac{\zeta^n}{(n+a)^m}, \tag{2}$$

$$(\lambda,\, \mu\, \in\, \mathbb{C};\, a,\, \omega\, \in\, \mathbb{C}\setminus\mathbb{Z}_0^-;\, \sigma,\, \rho,\, \kappa\, \in\, \mathbb{R}^+;\, \kappa-\sigma-\rho > -1\,\text{when}\, m,\, \zeta\, \in\, \mathbb{C};$$
$$\kappa-\sigma-\rho = -1\,\text{and}\, m\, \in\, \mathbb{C}\,\text{when}\, |\zeta| <\, \delta^\star\, =\, \sigma^{-\sigma}\rho^{-\rho}\kappa^{\kappa};$$
$$\kappa-\sigma-\rho = -1\,\text{and}\, \Re(m+\omega-\lambda-\mu) > 1\,\text{when}\, |\zeta| =\, \delta^\star).$$

1.1. **Fractional Kinetic Equations.** In [23] one determinated the fractional differential equation for the rate of change of reaction. The destruction rate and the production rate follow:

$$\frac{d\mathfrak{g}}{d\mathfrak{x}} = -\mathfrak{d}(\mathfrak{g}_{\mathfrak{x}}) + \mathfrak{p}(\mathfrak{g}_{\mathfrak{x}}), \tag{3}$$

where $\mathfrak{g} = \mathfrak{g}(\mathfrak{x})$ the rate of the reaction, $\mathfrak{d} = \mathfrak{d}(\mathfrak{g})$ the rate of destruction, $\mathfrak{p} = \mathfrak{p}(\mathfrak{g})$ the rate of production and $\mathfrak{g}_{\mathfrak{x}}$ denotes the function defined by $\mathfrak{g}_{\mathfrak{x}}(\mathfrak{x}^\star) = \mathfrak{g}(\mathfrak{x}-\mathfrak{x}^\star), \mathfrak{x}^\star > 0$.

The special condition of equation (3) for spatial fluctuations and inhomogeneities in $\mathfrak{g}(\mathfrak{x})$ the quantities are ignored, that is the equation

$$\frac{d\mathfrak{g}}{d\mathfrak{x}} = -\mathfrak{c}_i\, \mathfrak{g}_i(\mathfrak{x}) \tag{4}$$

with the initial condition that $\mathfrak{g}_i(\mathfrak{x} = 0) = \mathfrak{g}_0$ is the number of density of the species i at time $\mathfrak{x} = 0$ and $\mathfrak{c}_i > 0$. If we shift the index i and integrate the standard kinetic equation (4), we have

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0 = -\mathfrak{c}\; {}_0\mathcal{D}_t^{-1}\, \mathfrak{g}(\mathfrak{x}) \tag{5}$$

where ${}_0\mathcal{D}_{\mathfrak{x}}^{-1}$ is the special condition of the Riemann-Liouville integral operator ${}_0\mathcal{D}_{\mathfrak{x}}^{-\xi}$ given as [40],

$${}_0\mathcal{D}_{\mathfrak{x}}^{-\xi} f(\mathfrak{x}) = \frac{1}{\Gamma(\xi)} \int_0^{\mathfrak{x}} (\mathfrak{x}-s)^{\xi-1} f(s)\, ds, \tag{6}$$

$$(\mathfrak{x} > 0,\, \Re(\xi) > 0).$$

The fractional generalisation of the standard kinetic equation (5) is studied by Haubold and Mathai as follows [23]:

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0 = -\mathfrak{c}^{\nu}\; {}_0\mathcal{D}_{\mathfrak{x}}^{-1}\, \mathfrak{g}(\mathfrak{x}) \tag{7}$$

and acquired the solution of (4) as follows:

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\xi k+1)} (\mathfrak{c}\mathfrak{x})^{\xi k}. \tag{8}$$

In addition to that, Saxena and Kalla [30] take into consideration the following fractional kinetic equation:

$$\mathfrak{g}(\mathfrak{r}) - \mathfrak{g}_0 f(\mathfrak{r}) = -\mathfrak{c}^{\xi}\, {}_0\mathcal{D}_{\mathfrak{r}}^{-1}\, \mathfrak{g}(\mathfrak{r}) \;\; (\Re(\xi) > 0), \tag{9}$$

where $\mathfrak{g}(\mathfrak{r})$ denotes the number density of a given species at time $\mathfrak{r}$, $\mathfrak{g}_0 = \mathfrak{g}(0)$ is the number of density of that species at time $\mathfrak{r} = 0$, $\mathfrak{c}$ is a constant and $f \in L(0, \infty)$.

By taking advantage of the Laplace transform [19,37,39] to the equation (9),

$$\mathfrak{L}\{\mathfrak{g}(\mathfrak{r}); p\} = \mathfrak{g}_0 \, \frac{F(p)}{1 + \mathfrak{c}^{\nu}\, p^{-\nu}} = \mathfrak{g}_0 \left(\sum_{n=0}^{\infty} (-\mathfrak{c}^{\nu})^n \, p^{-\nu n} \right) F(p), \tag{10}$$

$$\left(n \in \mathfrak{g}_0, \left| \frac{\mathfrak{c}}{p} \right| < 1 \right).$$

The extension and generalisation of (FKEs) comprising many fractional operators were found in [1,2,3,5,16,17,23,24,28,29,30,31,32,43].

1.2. **Sumudu Transform.** The Sumudu transform is extensively used to solve several type of problems in science and engineering and it was introduced by Watagula [44,45]. For details, the reader is referred to [4,7,8,9,10,11,12].

Suppose that $\mathcal{U}$ be the class of exponentially bounded function $f : \Re \to \Re$, that is,

$$f(\zeta) < \begin{cases} \mathcal{M} \exp\left(-\dfrac{\zeta}{\eta_1} \right) & (\zeta \leqq 0); \\ \mathcal{M} \exp\left(\dfrac{\zeta}{\eta_2} \right) & (\zeta \geqq 0), \end{cases}$$

where $\mathcal{M}$, η_1 and η_2 are positive real constants. The Sumudu transform defined on the set $\mathcal{U}$ is given as follows [44,45]:

$$\mathcal{G}(u) = \mathcal{S}\{f(\zeta); u\} = \int_0^{\infty} e^{-\zeta} f(u\zeta)\, d\zeta \qquad (-\eta_1 < u < \eta_2). \tag{11}$$

The main goal of this work is to demonstrate the generalized (FKEs) involving generalised Hurwitz-Lerch Zeta function (2). Here, we conceive the Sumudu transform methodology to arrive at the solutions.

2. Main Results

Here, we will explain the solution of the generalised (FKEs) which by considering generalized Hurwitz-Lerch Zeta function (2).

Theorem 1. *If $\mathfrak{b} > 0$, $\xi > 0$; λ, μ, $\delta \in \mathbb{C}$, and $\mathfrak{b} \neq \delta$ be such that $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$; σ, ρ, $\kappa \in \mathbb{R}^+$, then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{r}) - \mathfrak{g}_0 \, \Phi_{\lambda,\mu;\omega}^{(\sigma,\rho,\kappa)}(\mathfrak{b}^{\xi}\mathfrak{r}^{\nu}, m, a) = -\delta^{\xi}\, {}_0\mathcal{D}_{\mathfrak{r}}^{-\xi}\, \mathfrak{g}(\mathfrak{r}) \tag{12}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\lambda)_{\sigma n}\,(\mu)_{\rho n}\,\Gamma(\xi n+1)\,\mathfrak{b}^{\xi n}\,t^{\xi n-1}}{(\omega)_{\kappa n}\,n!\,(n+a)^m}\,\mathcal{E}_{\xi,\xi n}(-\delta^{\xi}\,\mathfrak{x}^{\xi}), \tag{13}$$

where $\mathcal{E}_{\xi,\xi n}(.)$ is the Mittag-Leffler function [27].

Proof. The Sumudu transform of the Riemann-Liouville fractional integral operator is defined by [24, p. 460, Eq. (2.10)]:

$$\mathcal{S}\left[{}_0\mathcal{D}_{\mathfrak{x}}^{-\xi} f(\mathfrak{x}); u\right] = \mathcal{S}\left[\frac{\mathfrak{x}^{\xi-1}}{\Gamma(\xi)}; u\right]\cdot \mathcal{S}\left[f(\mathfrak{x}); u\right] = u^{\xi}\,G(u). \tag{14}$$

Now, taking advantage of the Sumudu transform to the both sides of (12), we have

$$\mathcal{S}\{\mathfrak{g}(\mathfrak{x}); u\} = \mathfrak{g}_0\,\mathcal{S}\{\,\Phi_{\lambda,\,\mu;\,\omega}^{(\sigma,\,\rho,\,\kappa)}(\mathfrak{b}^{\nu}\mathfrak{x}^{\nu}, m, a); u\} - \delta^{\xi}\,\mathcal{S}\{{}_0\mathcal{D}_{\mathfrak{x}}^{-\xi}\,\mathfrak{g}(\mathfrak{x}); u\}$$

$$\mathfrak{g}(u) = \mathfrak{g}_0\left\{\int_0^{\infty} e^{-\mathfrak{x}} \sum_{n=0}^{\infty} \frac{(\lambda)_{\sigma n}\,(\mu)_{\rho n}\,(\mathfrak{b}^{\nu}(u\mathfrak{x})^{\xi})^n}{(\omega)_{\kappa n}\,n!\,(n+a)^m}\right\} d\mathfrak{x} - \delta^{\xi} u^{\xi}\,\mathfrak{g}(u)$$

$$\begin{aligned}
&\mathfrak{g}(u) + \delta^{\xi} u^{\xi}\,\mathfrak{g}(u) \\
&\quad = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\lambda)_{\sigma n}\,(\mu)_{\rho n}\,\mathfrak{b}^{\xi n}}{(\omega)_{\kappa n}\,n!\,(n+a)^m}\,u^{\nu n}\int_0^{\infty} e^{-\mathfrak{x}}\,\mathfrak{x}^{\xi n}\,d\mathfrak{x} \\
&\quad = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\lambda)_{\sigma n}\,(\mu)_{\rho n}\,\mathfrak{b}^{\xi n}}{(\omega)_{\kappa n}\,n!\,(n+a)^m}\,u^{\xi n}\,\Gamma(\xi n+1)
\end{aligned} \tag{15}$$

$$N(u) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\lambda)_{\sigma n}\,(\mu)_{\rho n}\,\Gamma(\xi n+1)\,\mathfrak{b}^{\xi n}}{(\omega)_{\kappa n}\,n!\,(n+a)^m}\,u^{\xi n} \sum_{r=0}^{\infty}\left[-(\delta u)^{\xi}\right]^r.$$

Taking the inverse Sumudu transform of (15), and by applying

$$\mathcal{S}^{-1}\{u^{\xi}; \mathfrak{x}\} = \frac{\mathfrak{x}^{\xi-1}}{\Gamma(\xi)}, \quad (\Re(\xi) > 0), \tag{16}$$

we have

$$\begin{aligned}
\mathcal{S}^{-1}\{\mathfrak{g}(u)\} &= \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\lambda)_{\sigma n}\,(\mu)_{\rho n}\,\Gamma(\xi n+1)\,\mathfrak{b}^{\xi n}}{(\omega)_{\kappa n}\,n!\,(n+a)^m} \\
&\quad \times \mathcal{S}^{-1}\left[\sum_{r=0}^{\infty} \delta^{\xi r}\,u^{\xi(n+r)}\right]
\end{aligned} \tag{17}$$

$$\mathfrak{g}(\mathfrak{x}) = \sum_{n=0}^{\infty} \frac{(\lambda)_{\sigma n}\,(\mu)_{\rho n}\,\Gamma(\xi n+1)\,\mathfrak{b}^{\xi n}\,\mathfrak{x}^{\xi n-1}}{(\omega)_{\kappa n}\,n!\,(n+a)^m} \sum_{r=0}^{\infty} (-1)^r \delta^{\xi r}\,\frac{\mathfrak{x}^{\xi r}}{\Gamma(\xi n+\xi r)}.$$

So, we can be yield the required result (13). □

Theorem 2. *If* $\mathfrak{b} > 0$, $\xi > 0$; λ, $\mu \in \mathbb{C}$ *be such that* a, $\omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$; σ, ρ, $\kappa \in \mathbb{R}^+$, *then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0\, \Phi_{\lambda,\, \mu;\, \omega}^{(\sigma,\, \rho,\, \kappa)}(\mathfrak{b}^{\xi}\mathfrak{x}^{\xi}, m, a) = -\mathfrak{b}^{\xi}\, {}_0\mathcal{D}_{\mathfrak{x}}^{-\xi}\, \mathfrak{g}(\mathfrak{x}) \tag{18}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\lambda)_{\sigma n}\, (\mu)_{\rho n}\, \Gamma(\xi n + 1)\, \mathfrak{b}^{\xi n}\, \mathfrak{x}^{\xi n - 1}}{(\omega)_{\kappa n}\, n!\, (n+a)^m}\, \mathcal{E}_{\xi, \xi n}(-\mathfrak{b}^{\xi}\, \mathfrak{x}^{\xi}), \tag{19}$$

where $\mathcal{E}_{\xi,\xi n}(.)$ *is the Mittag-Leffler function [27].*

Proof. The proof of Theorem 2 is parallel to the proof of Theorem 1, thus the details are omitted. □

Theorem 3. *If* $\xi > 0$; λ, μ, $\delta \in \mathbb{C}$ *be such that* a, $\omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$; σ, ρ, $\kappa \in \mathbb{R}^+$, *then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0\, \Phi_{\lambda,\, \mu;\, \omega}^{(\sigma,\, \rho,\, \kappa)}(\mathfrak{x}, m, a) = -\delta^{\xi}\, {}_0\mathcal{D}_{\mathfrak{x}}^{-\xi}\, \mathfrak{g}(\mathfrak{x}) \tag{20}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\lambda)_{\sigma n}\, (\mu)_{\rho n}\, \Gamma(n + 1)\, \mathfrak{x}^{n - 1}}{(\omega)_{\kappa n}\, n!\, (n+a)^m}\, \mathcal{E}_{\xi, n}(-\delta^{\xi}\, \mathfrak{x}^{\xi}), \tag{21}$$

where $\mathcal{E}_{\xi,n}(.)$ *is the Mittag-Leffler function [27].*

Proof. Theorem 3 can be easily acquired from Theorem 1, so the details are omitted. □

2.1. **Special Conditions.** Choosing $\lambda = \sigma = 1$ in the equation (2), which is the generalized Hurwitz-Lerch Zeta function $\Phi_{\mu;\,\omega}^{\rho;\,\kappa}(\zeta, m, a)$ introduced and studied by Lin and Srivastava [25].

Applying $\lambda = \sigma = 1$ in the Theorem 1, Theorem 2, Theorem 3 obtained the following forms:

Corollary 4. *If* $\mathfrak{b} > 0$, $\xi > 0$; μ, $\delta \in \mathbb{C}$, *and* $\mathfrak{b} \neq \delta$ *be such that* a, $\omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$; ρ, $\kappa \in \mathbb{R}^+$, *then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0\, \Phi_{\mu;\, \omega}^{(\rho,\, \kappa)}(\mathfrak{b}^{\xi}\mathfrak{x}^{\xi}, m, a) = -\delta^{\xi}\, {}_0\mathcal{D}_{\mathfrak{x}}^{-\xi}\, \mathfrak{g}(\mathfrak{x}) \tag{22}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}\, \Gamma(\xi n + 1)\, \mathfrak{b}^{\xi n}\mathfrak{x}^{\xi n - 1}}{(\omega)_{\kappa n}\, (n+a)^m}\, \mathcal{E}_{\xi, \xi n}(-\delta^{\xi}\, \mathfrak{x}^{\xi}). \tag{23}$$

Corollary 5. *If* $\mathfrak{b} > 0$, $\xi > 0$; $\mu \in \mathbb{C}$ *be such that* $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $\rho, \kappa \in \mathbb{R}^+$, *then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0 \, \Phi_{\mu;\omega}^{(\rho,\kappa)}(\mathfrak{b}^\xi \mathfrak{x}^\xi, m, a) = -\mathfrak{b}^\xi \, {}_0\mathcal{D}_{\mathfrak{x}}^{-\xi} \, \mathfrak{g}(\mathfrak{x}) \tag{24}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n} \, \Gamma(\xi n + 1) \, \mathfrak{b}^{\xi n} \mathfrak{x}^{\xi n - 1}}{(\omega)_{\kappa n} \, (n+a)^m} \, \mathcal{E}_{\xi, \xi n}(-\mathfrak{b}^\nu \, \mathfrak{x}^\xi). \tag{25}$$

Corollary 6. *If* $\mu, \delta \in \mathbb{C}$ *be such that* $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $\rho, \kappa \in \mathbb{R}^+$, *then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0 \, \Phi_{\mu;\omega}^{(\rho,\kappa)}(\mathfrak{x}, m, a) = -\delta^\xi \, {}_0\mathcal{D}_{\mathfrak{x}}^{-\xi} \, \mathfrak{g}(\mathfrak{x}) \tag{26}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n} \, \Gamma(n + 1) \mathfrak{x}^{n - 1}}{(\omega)_{\kappa n} \, (n+a)^m} \, \mathcal{E}_{\xi, n}(-\delta^\xi \, \mathfrak{x}^\xi). \tag{27}$$

Setting $\sigma = \rho = \kappa = 1$ in the equation (2), which is the generalized Hurwitz-Lerch Zeta function $\Phi_{\lambda,\mu;\omega}(\zeta, m, a)$ introduced and studied by Garg et. all [20].

Applying $\sigma = \rho = \kappa = 1$ in the Theorem 1, Theorem 2, Theorem 3 obtained the following forms:

Corollary 7. *If* $\mathfrak{b} > 0$, $\xi > 0$; $\lambda, \mu, \delta \in \mathbb{C}$, *and* $\mathfrak{b} \neq \delta$ *be such that* $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$, *then the solution of the following given equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0 \, \Phi_{\lambda,\mu;\omega}(\mathfrak{b}^\xi \mathfrak{x}^\xi, m, a) = -\delta^\xi \, {}_0\mathcal{D}_{\mathfrak{x}}^{-\xi} \, \mathfrak{g}(\mathfrak{x}) \tag{28}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\lambda)_n \, (\mu)_n \, \Gamma(\xi n + 1) \, \mathfrak{b}^{\xi n} \mathfrak{x}^{\xi n - 1}}{(\omega)_n \, n! \, (n+a)^m} \, \mathcal{E}_{\xi, \xi n}(-\delta^\xi \, \mathfrak{x}^\xi). \tag{29}$$

Corollary 8. *If* $\mathfrak{b} > 0$, $\xi > 0$; $\lambda, \mu \in \mathbb{C}$ *be such that* $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$, *then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0 \, \Phi_{\lambda,\mu;\omega}(\mathfrak{b}^\xi \mathfrak{x}^\xi, m, a) = -\mathfrak{b}^\xi \, {}_0\mathcal{D}_{\mathfrak{x}}^{-\xi} \, \mathfrak{g}(\mathfrak{x}) \tag{30}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\lambda)_n \, (\mu)_n \, \Gamma(\xi n + 1) \mathfrak{b}^{\xi n} \mathfrak{x}^{\xi n - 1}}{(\omega)_n \, n! \, (n+a)^m} \, \mathcal{E}_{\xi, \xi n}(-\mathfrak{b}^\xi \, \mathfrak{x}^\xi). \tag{31}$$

Corollary 9. *If* $\lambda, \mu, \delta \in \mathbb{C}$ *be such that* $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$, *then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0 \, \Phi_{\lambda,\mu;\omega}(\mathfrak{x}, m, a) = -\delta^\xi \, {}_0\mathcal{D}_{\mathfrak{x}}^{-\xi} \, \mathfrak{g}(\mathfrak{x}) \tag{32}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\lambda)_n\,(\mu)_n\,\Gamma(n+1)\mathfrak{x}^{n-1}}{(\omega)_n\, n!\,(n+a)^m}\,\mathcal{E}_{\xi,n}(-\delta^{\xi}\,\mathfrak{x}^{\xi}). \tag{33}$$

Upon taking $\sigma = \rho = \kappa = 1$ and $\lambda = \omega$ in the equation (2), which is the generalized Hurwitz-Lerch Zeta function $\Phi^{\star}_{\mu}(\zeta, m, a)$ introduced and studied by Goyal and Laddha [21, p.100, Eq.(1.5)].

Applying $\sigma = \rho = \kappa = 1$ and $\lambda = \omega$ in the Theorem 1, Theorem 2, Theorem 3 obtained the following forms:

Corollary 10. *If $\mathfrak{b} > 0$, $\xi > 0$; μ, $\delta \in \mathbb{C}$, and $\mathfrak{b} \neq \delta$ be such that $a \in \mathbb{C}\setminus\mathbb{Z}_0^-$, then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0\,\Phi^{\star}_{\mu}(\mathfrak{b}^{\xi}\mathfrak{x}^{\xi}, m, a) = -\delta^{\xi}\,{}_0\mathcal{D}_{\mathfrak{x}}^{-\xi}\,\mathfrak{g}(\mathfrak{x}) \tag{34}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\mu)_n\,\Gamma(\xi n+1)\,\mathfrak{b}^{\xi n}\mathfrak{x}^{\xi n-1}}{n!\,(n+a)^m}\,\mathcal{E}_{\xi,\xi n}(-\delta^{\xi}\,\mathfrak{x}^{\xi}). \tag{35}$$

Corollary 11. *If $\mathfrak{b} > 0$, $\xi > 0$; $\mu \in \mathbb{C}$ be such that $a \in \mathbb{C}\setminus\mathbb{Z}_0^-$, then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0\,\Phi^{\star}_{\mu}(\mathfrak{b}^{\xi}\mathfrak{x}^{\xi}, m, a) = -\mathfrak{b}^{\xi}\,{}_0\mathcal{D}_{\mathfrak{x}}^{-\xi}\,\mathfrak{g}(\mathfrak{x}) \tag{36}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\mu)_n\,\Gamma(\xi n+1)\,\mathfrak{b}^{\xi n}\mathfrak{x}^{\xi n-1}}{n!\,(n+a)^m}\,\mathcal{E}_{\xi,\xi n}(-\mathfrak{b}^{\xi}\,\mathfrak{x}^{\xi}). \tag{37}$$

Corollary 12. *If λ, μ, $\delta \in \mathbb{C}$ be such that $a \in \mathbb{C}\setminus\mathbb{Z}_0^-$, then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0\,\Phi^{\star}_{\mu}(\mathfrak{x}, m, a) = -\delta^{\xi}\,{}_0\mathcal{D}_{\mathfrak{x}}^{-\xi}\,\mathfrak{g}(\mathfrak{x}) \tag{38}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{(\mu)_n\,\Gamma(n+1)\mathfrak{x}^{n-1}}{n!\,(n+a)^m}\,\mathcal{E}_{\xi,n}(-\delta^{\xi}\,\mathfrak{x}^{\xi}). \tag{39}$$

Upon taking $\sigma = \rho = \mu = 1$ and $\zeta = \frac{\zeta}{\lambda}$. Then, the limit case of (2) when $\lambda \to \infty$, would yield the Mittag-Leffler type function $\mathcal{E}^{(a)}_{\kappa,\,\omega}(m;\mathfrak{x})$ studied by Barnes [6], that is,

$$\mathcal{E}^{(a)}_{\kappa,\,\omega}(m;\zeta) = \sum_{n=0}^{\infty} \frac{\zeta^n}{(n+a)^m\,\Gamma(\omega+\kappa n)}, \tag{40}$$

$$(a,\,\omega \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ \Re(\kappa) > 0;\ m,\,\zeta \in \mathbb{C}).$$

Applying $\sigma = \rho = \mu = 1$ and $\zeta = \frac{\zeta}{\lambda}$. Then, the limit case of (2) when $\lambda \to \infty$ in the Theorem 1, Theorem 2, Theorem 3 obtained the following forms:

Corollary 13. *If* $\mathfrak{b} > 0$, $\xi > 0$; κ, $\delta \in \mathbb{C}$, *and* $\mathfrak{b} \neq \delta$ *be such that* $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$, *then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0\, \mathcal{E}_{\kappa,\omega}^{(a)}(m; \mathfrak{b}^{\xi}\mathfrak{x}^{\xi}) = -\delta^{\xi}\, {}_0\mathcal{D}_{\mathfrak{x}}^{-\xi}\, \mathfrak{g}(\mathfrak{x}) \tag{41}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{\Gamma(\xi n + 1)\, \mathfrak{b}^{\xi n}\mathfrak{x}^{\xi n - 1}}{(n+a)^m\, \Gamma(\omega + \kappa n)}\, \mathcal{E}_{\xi,\xi n}(-\delta^{\xi}\, \mathfrak{x}^{\xi}). \tag{42}$$

Corollary 14. *If* $\mathfrak{b} > 0$, $\xi > 0$; $\kappa \in \mathbb{C}$ *be such that* $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$, *then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0\, \mathcal{E}_{\kappa,\omega}^{(a)}(m; \mathfrak{b}^{\xi}\mathfrak{x}^{\xi}) = -\mathfrak{b}^{\xi}\, {}_0\mathcal{D}_{\mathfrak{x}}^{-\xi}\, \mathfrak{g}(\mathfrak{x}) \tag{43}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{\Gamma(\xi n + 1)\, \mathfrak{b}^{\xi n}\mathfrak{x}^{\xi n - 1}}{(n+a)^m\, \Gamma(\omega + \kappa n)}\, \mathcal{E}_{\xi,\xi n}(-\mathfrak{b}^{\xi}\, \mathfrak{x}^{\xi}). \tag{44}$$

Corollary 15. *If* κ, $\delta \in \mathbb{C}$ *be such that* $a, \omega \in \mathbb{C} \setminus \mathbb{Z}_0^-$, *then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0\, \mathcal{E}_{\kappa,\omega}^{(a)}(m; \mathfrak{x}) = -\delta^{\xi}\, {}_0\mathcal{D}_{\mathfrak{x}}^{-\xi}\, \mathfrak{g}(\mathfrak{x}) \tag{45}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{\Gamma(n + 1)\mathfrak{x}^{n - 1}}{(n+a)^m\, \Gamma(\omega + \kappa n)}\, \mathcal{E}_{\xi,n}(-\delta^{\xi}\, \mathfrak{x}^{\xi}). \tag{46}$$

Finally, upon setting λ, μ, ω, σ, ρ, $\kappa = 1$ in the equation (2), which gives the equation (1) [34, 35].

Choosing λ, μ, ω, σ, ρ, $\kappa = 1$ in the Theorem 1, Theorem 2, Theorem 3 obtained the following forms:

Corollary 16. *If* $\mathfrak{b} > 0$; δ, $\xi \in \mathbb{C}$, $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, *and* $\mathfrak{b} \neq \delta$, *then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0\, \Phi(\mathfrak{b}^{\xi}\mathfrak{x}^{\xi}, m, a) = -\delta^{\xi}\, {}_0\mathcal{D}_{\mathfrak{x}}^{-\xi}\, \mathfrak{g}(\mathfrak{x}) \tag{47}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{\Gamma(\xi n + 1)\, \mathfrak{b}^{\xi n}\mathfrak{x}^{\xi n - 1}}{(n+a)^m}\, \mathcal{E}_{\xi,\xi n}(-\delta^{\xi}\, \mathfrak{x}^{\xi}). \tag{48}$$

Corollary 17. *If* $\mathfrak{b} > 0$; $\xi \in \mathbb{C}$, $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, *then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0\, \Phi(\mathfrak{b}^{\xi}\mathfrak{x}^{\xi}, m, a) = -\mathfrak{b}^{\xi}\, {}_0\mathcal{D}_{\mathfrak{x}}^{-\xi}\, \mathfrak{g}(\mathfrak{x}) \tag{49}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{\Gamma(\xi n + 1)\, \mathfrak{b}^{\xi n} \mathfrak{x}^{\xi n - 1}}{(n+a)^m} \mathcal{E}_{\xi, \xi n}(-\mathfrak{b}^{\xi} \mathfrak{x}^{\xi}). \tag{50}$$

Corollary 18. *If* $\delta \in \mathbb{C}$, $a \in \mathbb{C} \setminus \mathbb{Z}_0^-$, *then the solution of the given fractional equation*

$$\mathfrak{g}(\mathfrak{x}) - \mathfrak{g}_0\, \Phi(\mathfrak{x}, m, a) = -\delta^{\xi}\, {}_0\mathcal{D}_{\mathfrak{x}}^{-\xi}\, \mathfrak{g}(\mathfrak{x}) \tag{51}$$

is derived by

$$\mathfrak{g}(\mathfrak{x}) = \mathfrak{g}_0 \sum_{n=0}^{\infty} \frac{\Gamma(n+1) \mathfrak{x}^{n-1}}{(n+a)^m} \mathcal{E}_{\xi, n}(-\delta^{\xi} \mathfrak{x}^{\xi}). \tag{52}$$

3. Numerical Result and Graphic

In this section, we present the 2D plots of Equation (13) for special values such as: λ, μ, ω, ρ, κ, σ, a, $m = 1$, $\delta = 4$, $\mathfrak{g}_0 = 3$ and $\xi = 0.4, 0.5, 0.6$.

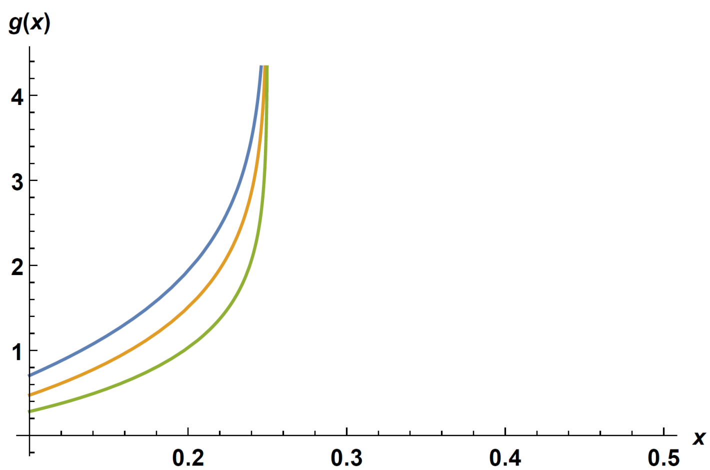

Figure 1. Solution of the FKE for GHLZ

4. Conclusions

The fractional kinetic equation involving the generalized Hurwitz-Lerch Zeta function is studied using the Sumudu transform. The results obtained in this study have remarkable significance as the solution of the equations are general and can be reproduced many new and known solutions of (FKEs) involving various type of special functions.

Authors Contribution Statement All authors have contributed sufficiently in the planning, execution, or analysis of this study to be included as authors. All authors read and approved the final manuscript

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