



On the G-Similarities of two open B-spline curves in \mathbb{R}^3

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ABSTRACT

Let G be a transformation group in \mathbb{R}^3 . Any two vectors x and y in \mathbb{R}^3 are called G -equivalence vectors if there exist a transformation $g \in G$ such that $y = gx$ satisfies. In this paper the transformation group G will be considered as similarity transformations group or its any subgroup. So if given two vectors x and y in \mathbb{R}^3 are G -equivalence vectors then these vectors x and y are called G -similar. i.e. rotational, reflectional, translational or scaling similarity. B-spline curves are used basically in Computer Aided Design (CAD), Computer Aided Geometric Design (CAGD), Computer Aided Modeling (CAM). In determining the invariants of spline curves and surfaces at any point, it is necessary to find the analytical equation of each curve and surface and calculate its invariants such as curvature, torsion, principal curvatures, mean and Gaussian curvatures at the desired point. However, it can be very difficult to find the curve or surface to be designed analytically. For example, when a car is designed, the aerodynamic curves in the car will be different from the known surface equation of the car. It is very difficult to write this equation exactly. For these curves and surfaces we designed, the way to overcome this difficulty is to design them with spline curves and surfaces. In this paper the G -equivalence conditions of given two open B-spline curves are studied in case G is similarity transformations group or its any subgroup.

Keywords: open B-spline curves, similarity groups, G -similar splines

\mathbb{R}^3 de Açık B-Spline Eğrilerinin G -Benzerlikleri

ÖZ

G , \mathbb{R}^3 de bir dönüşüm grubu olsun. \mathbb{R}^3 te herhangi iki x ve y vektörleri verildiğinde eğer bir $g \in G$ dönüşümü $y = gx$ şartını sağlayacak şekilde bulunabilirse bu iki vektöre G - denk vektörler denir. Bu çalışmada G dönüşüm grubu olarak benzerlik dönüşümleri grubu ve bu grubun tüm altgrupları dikkate alınacaktır. Böylece \mathbb{R}^3 te herhangi iki x ve y vektörleri G - denk vektörler ise bu vektörlere G -benzer denir. Döndürülme, yansıtılma, ötelenme, ya da germe benzerliği gibi. B-spline eğrileri temelde Bilgisayar Destekli Tasarım (BDT), Bilgisayar Destekli Geometrik Tasarım (BDGT) ya da Bilgisayar Destekli Modelleme (BDM) alanlarında kullanılır. Herhangi bir noktada spline eğri ve yüzeylerinin invariantslarını belirlemede eğri ve yüzeyin analitik denklemini bulmak ve istenilen noktada eğrilik torsiyon, asal eğrilikler, ortalama ve Gauss eğriliklerini hesaplamak gerekmektedir Oysa ki tasarlanan eğri ve yüzeyde bunu analitik olarak bulmak oldukça zordur. Örneğin, bir araç tasarlandığında, onun aerodinamik yapısından dolayı yüzeyin ve onun üzerindeki eğrilerin analitik denklemini tam olarak bulmak oldukça zordur. Tasarlanan bu eğri ve yüzeyler için bu zorluğun üstesinden gelmenin yolu bunları spline eğri ve yüzeyleri ile tasarlamaktır. Bu çalışmada G , benzerlik dönüşümleri grubu ve onun altgrupları olması durumunda verilen iki B-spline eğrilerinin G - denklik koşulları verilmiştir.

Anahtar Kelimeler: Açık B-spline curves, benzerlik grupları, G -benzer splinelar

INTRODUCTION

Invariant rational functions under any transformation have very important roles to determine any properties which can be formulated by these rational functions and are independent from this transformation. If the generators of invariant rational functions under any transformation known then any properties which are invariant under this transformation can be formulated by the generator functions. So any invariant properties are the functions of generators. Developments in invariant theory at the end of the 19th century have affected different areas of mathematics.

In the 20th century Bridgman [5], Sedov [6], and Langhaar [7] are some contributors in this area. In 1946, Herman Weyl gave the complete invariant system of points for real n dimensional orthogonal group $O(n)$ in [8]. After him, Dj. Khadjiev and R. Aripov generalized this invariants to all Euclidean motions in [9, 10]. Recently Sagioglu [11-13], Oren [14-16, 21] Peksen [13, 16], Incesu and Gursoy [17-23] are some contributors in this area. The best examples of points systems are Bézier curves and Bézier surfaces. The invariants of these curves and surfaces under an affine transformation have the same meaning as the invariants of the control points of these curves and surfaces [22]. Bézier and B-Spline

curves has been studied in many different are of CAD and CAM system. Some of these studies by G. Farin [24], R. Farouki [25, 26], J. Hoschek [27], W. Tiller [28], H. Potmann [29], Incesu and Gursoy [30, 34], Samancı et al. [31, 33, 36, 37], Bulut and Caliskan [32], Erkan and Yuce [35], Baydas and Karakas [38] can be given exemplarily.

MATERIALS and METHODS

G-invariant Functions

Definition 1. Let $(G,*)$ be a group and X be a nonempty set. Let the transformation $\varphi: G \times X \rightarrow X$ be given. If following conditions

- i) $\varphi(g_1; \varphi(g_2; x)) = \varphi(g_1 * g_2; x), \forall g_1, g_2 \in G$ and $\forall x \in X$
- ii) $\varphi(e; x) = x$; for $\forall x \in X$ where $e \in G$ is identity

satisfies then the transformation φ is called Group action of the group G on X . This action is denoted by $G : X$ and the image $(g; x)$ is stated as gx briefly. [19]

Definition 2. Let G be a group and the group action $\varphi = G : X$ and the subset $H \subset X$ be given. the subset H is called "G-invariant subset" if $\varphi(g; h) \in H$ for every $h \in H$ and for every $g \in G$ satisfies [19].

Definition 3. Let G be a transformation group and the action $G:X$ be given. Then The vectors $x_1, x_2 \in X$ are called G-equivalent vectors if there exist any transformation $g \in G$ such that $x_2 = gx_1$ satisfies. The G-equivalent vectors $x_1, x_2 \in X$ is denoted as $x_1 \overset{G}{\approx} x_2$ [23].

Definition 4: Let the point sets $\{x_1, x_2, \dots, x_k\}$, $\{y_1, y_2, \dots, y_k\} \subset X$ and the action $G : X$ be given. Then These sets are called G-equivalent set sif there exist any transformation $g \in G$ such that $y_i = gx_i$ satisfies for every $i=1, 2, \dots, k$. These G-equivalent sets are denoted as $\{x_1, x_2, \dots, x_k\} \overset{G}{\approx} \{y_1, y_2, \dots, y_k\}$ [23].

Definition 5: Let G be a group, $f : X \rightarrow \square$ be a function and the action $G : X$ be given. Then the function f is called G- invariant function if $f(x) = f(y)$ satisfies when $x_1 \overset{G}{\approx} x_2$ satisfied or if $f(gx) = f(x)$ satisfies for $\forall g \in G$ and $\forall x \in X$.[23].

Definition 5: Let G be a group, $H \subset G$ be a subgroup and f , be a real valued function defined in R^3 . If there exist a real valued function $\lambda : H \rightarrow \square$, such that for $\forall h \in H, \forall x \in R^3$

$$f(hx) = \lambda(h)f(x),$$

satisfies then the function f is called proportional G-invariant function and the function λ is called the factor of the function f . [23]

Definition 6: Let f be a proportional invariant function. The function f is called even O(3)-invariant function if $\lambda(g) = 1$ for every $g \in O(3)$.

Open B-Spline Curves

Definition: The B- Spline basis functions of degree d , denoted $N_{i,d}(t)$, defined by the knot vectors t_0, t_1, \dots, t_m are defined recursively as follows

$$N_{i,0}(t) = \begin{cases} 1, & \text{if } t \in [t_i, t_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

and

$$N_{i,d}(t) = \frac{t - t_i}{t_{i+d} - t_i} N_{i,d-1}(t) + \frac{t_{i+d+1} - t}{t_{i+d+1} - t_{i+1}} N_{i+1,d-1}(t)$$

for $i = 0, 1, 2, \dots, n$ and $d \geq 1$. If the knot vector contains a sufficient number of repeated knot values, then a division of the form $N_{i,d-1}(t) / (t_{i+d} - t_i) = 0/0$ (for some i) may be encountered during the execution of the recursion. Whenever this occurs, it is assumed that $0/0 = 0$ [39].

Definition: The B-spline curve of degree d (or order $d + 1$) with control points b_0, \dots, b_n and knots t_0, t_m is defined on the interval $[a, b] = [t_d, t_{m-d}]$ by

$$B(t) = \sum_{i=0}^n b_i N_{i,d}(t) \tag{1}$$

where $N_{i,d}(t)$ are the B-spline basis functions of degree d . [39]

Theorem 1: The B-spline basis functions $N_{i,d}(t)$ satisfy the following properties.

- i) Positivity: $N_{i,d}(t) > 0$ for $t \in (t_i, t_{i+d+1})$.
- ii) Local Support: $N_{i,d}(t) = 0$ for $t \notin (t_i, t_{i+d+1})$.
- iii) Piecewise Polynomial: $N_{i,d}(t)$ are piecewise polynomial functions of degree d .
- iv) Partition of Unity: for $t \in [t_r, t_{r+1})$

$$\sum_{i=r-d}^r N_{i,d}(t) = 1$$

[39].

Theorem 2: The B-spline curves defined as (1) satisfy the following properties.

- i) Local Control: Each segment is determined by $d + 1$ control points. If $t \in [t_r, t_{r+1})$ ($d \leq r \leq m - d - 1$), then

$$B(t) = \sum_{i=r-d}^r b_i N_{i,d}(t) \tag{5}$$

- ii) Convex Hull: If $t \in [t_r, t_{r+1})$

($d \leq r \leq m - d - 1$), then

$$B(t) \in CH\{b_{r-d}, \dots, b_r\}$$

iii) Invariance under Affine Transformations: Let T be an affine transformation. Then

$$T\left(\sum_{i=r-d}^r b_i N_{i,d}(t)\right) = \sum_{i=r-d}^r T(b_i) N_{i,d}(t)$$

[39].

In general, B-spline curves do not interpolate the first and last control points b_0 and b_n . For curves of degree d , end point interpolation and an endpoint tangent condition are obtained by *open B-splines* for which the end knots satisfy $t_0 = t_1 \dots = t_d$ and $t_{m-d} = t_{m-d+1} \dots = t_m$.

Theorem 3: Let the B-spline curves of degree d with control points with control points b_0, \dots, b_n and $t_0 = t_1 \dots = t_d; \dots; t_{m-d} = t_{m-d+1} \dots = t_m$. Be given Then,

$$B(t_d) = b_0 \quad \text{and} \quad B(t_{m-d}) = b_n$$

satisfy. [39].

Theorem 4: Let the B-spline curves of degree d with control points with control points b_0, \dots, b_n and $t_0 = t_1 \dots = t_d; \dots; t_{m-d} = t_{m-d+1} \dots = t_m$. be given Then,

$$B'(t_d) = \frac{d}{t_{d+1} - t_1} (b_1 - b_0)$$

and

$$B'(t_{m-d}) = \frac{d}{t_{d+1} - t_1} (b_n - b_{n-1})$$

Satisfy [39].

The following chapter is cited from [19].

The Similarity Group $G = S(3)$ and its all subgroups in R^3

Orthogonal Transformations' Group $O(3)$

This group is a group of all rotations and reections. Orthogonal Transformations' group is the same of the linear isometries' group. For any $g \in O(3)$; the determinant of g is equal ± 1 . While the totating the rotating frames $\{e_1^*, e_2^*, e_3^*\}$ and fixed frames $\{e_1, e_2, e_3\}$ be given. The rotations are depend on the angles θ_{ij} between the axisses $\{e_i^*, e_j\}$. So this group can be stated as

$$O(3) = \{f: R^3 \rightarrow R^3; f(x) = gx; g^T = g^{-1}; \forall x \in R^3\} \\ = \{g \in M^{3 \times 3}; \det g = \pm 1\} \\ = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta_{11} & \cos\theta_{21} & \cos\theta_{31} \\ \cos\theta_{12} & \cos\theta_{22} & \cos\theta_{32} \\ \cos\theta_{13} & \cos\theta_{23} & \cos\theta_{33} \end{pmatrix} : \theta_{ij} \in R \right\}$$

Special Orthogonal Transformations' Group $SO(3)$

This group is a group of only all rotations. For any $g \in SO(3)$ the determinant of g is equal to $+1$. This group is denoted also as $O^+(3)$. **So this group can be stated as**

$$SO(3) = \{f: R^3 \rightarrow R^3; f(x) = gx; g^T = g^{-1}; \forall x \in R^3\} \\ = \{g \in M^{3 \times 3}; g^T = g^{-1} \text{ and } \det g = 1\} \\ = \left\{ \begin{pmatrix} \cos\theta_{11} & \cos\theta_{21} & \cos\theta_{31} \\ \cos\theta_{12} & \cos\theta_{22} & \cos\theta_{32} \\ \cos\theta_{13} & \cos\theta_{23} & \cos\theta_{33} \end{pmatrix} : \theta_{ij} \in R \right\}$$

Translations' Group $Tr(3)$

This group is a group of all translations.

$$Tr(3) = \{f: R^3 \rightarrow R^3; f(x) = x + b; b \in R^3; \forall x \in R^3\}$$

Euclid Transformations' Group $E(3)$

This group is a group of all translations, rotations, rotations and translations, reections, reections and translations. this group is the same of all isometries' group. Euclid transformations can be stated as a composition of a translation and an orthogonal transformation. So this group can be stated as

$$E(3) = \{f: R^3 \rightarrow R^3; f(x) = gx + b; b \in R^3; g \in O(3)\}$$

Special Euclid Transformations' Group $SE(3)$

This group is a group of all translations, rotations, rotations and translations. Special Euclid transformations can be stated as a composition of a translation and a special orthogonal transformation. So this group can be stated as

$$SE(3) = \{f: R^3 \rightarrow R^3; f(x) = gx + b; b \in R^3; g \in SO(3)\}$$

Linear Homotethies' Group $LH(3)$

This group is a group of all central dilations or radyal transformations. Linear homotethies are the transformations that its center of homotethy are origin. So this group can be stated as

$$LH(3) = \{f: R^3 \rightarrow R^3; f(x) = \lambda x; \lambda \in R^+\}$$

Homotethies' Group $H(3)$

The hometethy transformation can be defined as

$f(x) = a + \lambda(x - a)$ where a is called homotethy center of f . This transformation can be stated also as $f(x) = \lambda x + b$ for $\lambda \neq 1$ where the center of homnotethy is $a = \frac{b}{1-\lambda}$. So this group can be staaed as

$$H(3) = \{f: R^3 \rightarrow R^3; f(x) = \lambda x + b; \lambda \in R^+, b \in R^3\}$$

Linear Similarities' Group $LS(3)$

Linear similarity transformations can be stated as a composition of a linear homotethy transfor-mation and

an orthogonal transformation. So linear similarity transformations can be stated as

$$LS(3) = \{f: R^3 \rightarrow R^3; f(x) = \lambda gx; \lambda \in R^+; g \in O(3)\}$$

Special Linear Similarities' Group SLS(3)

Special linear similarity transformations can be stated as a composition of a linear homotety transformation and a special orthogonal transformation. So linear similarity transformations can be stated as

$$SLS(3) = \{f: R^3 \rightarrow R^3; f(x) = \lambda gx; \lambda \in R^+; g \in SO(3)\}$$

Special Similarities' Group SS(3)

Special similarity transformations can be stated as a composition of a special linear similarity transformation and a translation. In other words a special similarity transformation can be stated as a composition of a linear homotety transformation, a special orthogonal transformation, and a translation. So special similarity transformations can be stated as

$$SS(3) = \{f: R^3 \rightarrow R^3; f(x) = \lambda gx + b; \lambda \in R^+; g \in SO(3), b \in R^3\}$$

All Similarities' Group S(3)

Similarity transformations can be stated as a composition of a linear similarity transformation and a translation. In other words a similarity transformation can be stated as a composition of a linear homotety transformation, an orthogonal transformation, and a translation. So similarity transformations can be stated as

$$S(3) = \{f: R^3 \rightarrow R^3; f(x) = \lambda gx + b; \lambda \in R^+; g \in O(3), b \in R^3\}$$

The Generator Invariants of Points by the group $G = S(3)$ and its Some subgroups in R^3

Let $\{x_1, x_2, \dots, x_k\} \subset R^3$ be given.

Theorem 5: Let $G = O(3) \subset S(3)$ be supposed. In this case;

i) if $k \leq 3$ then the system

$$\langle x_i, x_j \rangle; \quad i, j = 1, \dots, k; \quad i \leq j$$

generate the $O(3)$ - invariant rational functions.

ii) if $k > 3$ then the system

$$\langle x_i, x_j \rangle; \quad i, j = 1, 2, 3; \quad i \leq j$$

$$\langle x_i, x_p \rangle; \quad i, j = 1, 2, 3; \quad p = 4, 5, \dots, k$$

generate the $O(3)$ - invariant rational functions. [22].

Theorem 6: Let $G = SO(3) \subset S(3)$ be supposed. Then;

i) in case $k < 3$ the system

$$\langle x_i, x_j \rangle; \quad i, j = 1, \dots, k; \quad i \leq j$$

generate the $SO(3)$ - invariant rational functions.

ii) in case $k \geq 3$ the system

$$\det [x_1 \ x_2 \ x_3];$$

$$\langle x_i, x_j \rangle; \quad i, j = 1, 2, 3; \quad i \leq j; \quad i + j < 6;$$

$$\langle x_i, x_p \rangle; \quad i, j = 1, 2, 3; \quad p = 4, 5, \dots, k$$

generate the $SO(3)$ - invariant rational functions [1].

Theorem 7: Let $G = LS(3) \subset S(3)$ be supposed. Then;

i) in case $k \leq 3$ the system,

$$\frac{\langle x_i, x_j \rangle}{\langle x_1, x_1 \rangle}; \quad i, j = 1, \dots, k; \quad i \leq j$$

generate the $LS(3)$ - invariant rational functions.

ii) in case $k > 3$ the system

$$\frac{\langle x_i, x_j \rangle}{\langle x_1, x_1 \rangle}; \quad i, j = 1, 2, 3; \quad i \leq j$$

$$\frac{\langle x_i, x_p \rangle}{\langle x_1, x_1 \rangle}; \quad i, j = 1, 2, 3; \quad p = 4, 5, \dots, k$$

generate the $LS(3)$ - invariant rational functions [22].

Theorem 8 [22]: the set of generators of the field of $G = S(3)$ invariant rational functions for k vector variables are

1. if $k \leq 4$ then

$$\left\{ \frac{\langle x_i - x_1, x_j - x_1 \rangle}{\langle x_2 - x_1, x_2 - x_1 \rangle}; \quad i, j = 2, 3, 4; \quad i \leq j \right\}$$

2. if $k \geq 4$ then

$$\left\{ \frac{\langle x_i - x_1, x_j - x_1 \rangle}{\langle x_2 - x_1, x_2 - x_1 \rangle}; \quad i, j = 2, 3, 4; \quad i \leq j \right\} \quad \text{and}$$

$$\left\{ \frac{\langle x_i - x_1, x_p - x_1 \rangle}{\langle x_2 - x_1, x_2 - x_1 \rangle}; \quad i = 2, 3, 4; \quad p = 5, \dots, k \right\}$$

Theorem 9: Let $G = E(3) \subset S(3)$ be given. Then;

i) if $k=1$ then $E(3)$ - invariant rational functions are constant.

ii) if $2 \leq k \leq 4$ then the system

$$\langle x_i - x_1, x_j - x_1 \rangle; \quad i, j = 2, \dots, k; \quad i \leq j$$

generate the $E(3)$ - invariant rational functions.

iii) if $k > 4$ then the system,

$$\langle x_i - x_1, x_j - x_1 \rangle; \quad i, j = 1, 2, 3; \quad i \leq j$$

$$\langle x_i - x_1, x_p - x_1 \rangle; \quad i, j = 1, 2, 3; \quad p = 4, 5, \dots, k$$

generate the $E(3)$ - invariant rational functions. [1]

Theorem 10: Let $G = SE(3) \subset S(3)$ be given. Then;

i) if $k=1$ then $SE(3)$ - invariant rational functions are constant.

ii) if $2 \leq k < 4$ then the system

$$\langle x_i - x_1, x_j - x_1 \rangle; \quad i, j = 1, \dots, k; \quad i \leq j$$

generate the $SE(3)$ - invariant rational functions.

iii) if $k \geq 4$ then the system

$$\det [x_2 - x_1 \ x_3 - x_1 \ x_4 - x_1];$$

$$\langle x_i - x_1, x_j - x_1 \rangle; \quad i, j = 2, 3, 4; \quad i \leq j; \quad i + j < 8;$$

$$\langle x_i - x_1, x_p - x_1 \rangle; \quad i, j = 1, 2, 3; \quad p = 5, \dots, k$$

generate the $SE(3)$ - invariant rational functions. [1]

MAIN RESULTS

The Generator Invariants of Points by the other subgroups of $G = S(3)$ in R^3

Theorem 11: Let $G = Tr(3) \subset S(3)$ be given. Then any $g \in Tr(3)$ - invariant rational function for k vector variables can be written as

$$g(x_1, \dots, x_k) = f(x_2 - x_1, \dots, x_k - x_1)$$

where $x_i \neq x_1$ and f is a linear function.

Proof: since g is a $Tr(3)$ -invariant, $g(x_1 + p, \dots, x_k + p) = g(x_1, \dots, x_k)$ must be satisfied. So

$$\begin{aligned} g(x_1 + p, \dots, x_k + p) &= \\ &= f((x_2 + p) - (x_1 + p), \dots, (x_k + p) - (x_1 + p)) \\ &= f(x_2 - x_1, \dots, x_k - x_1) = g(x_1, \dots, x_k) \end{aligned}$$

satisfies.

Theorem 12: Let $G = LH(3) \subset S(3)$ be given. Then;

The $LH(3)$ -invariant rational function can be written as the proportion of the polynomial functions whose terms have the same degree [1].

Example: For the vector $(x, y, z) \in R^3$, the function $f(x, y, z) = \frac{x^2yz + x^2y^2}{y^2zx + y^2z^2}$ is a $LH(3)$ -invariant. Because;

$$\begin{aligned} f(kx, ky, kz) &= \frac{k^2x^2kykz + k^2x^2k^2y^2}{k^2y^2kzkx + k^2y^2k^2z^2} \\ &= \frac{k^4(x^2yz) + k^4(x^2y^2)}{k^4(y^2zx) + k^4(y^2z^2)} = \frac{x^2yz + x^2y^2}{y^2zx + y^2z^2} \\ &= f(x, y, z) \end{aligned}$$

satisfies. Example 2: For these vectors $X, Y, Z, W \in R^3$ the function $f(X, Y, Z) = \frac{\det[X Y Z]}{\det[X Y W]}$ is a $LH(3)$ -invariant.

Theorem 13: Let $G = SLS(3) \subset S(3)$ be given. Then;

i) if $k < 3$ then the system

$$\frac{\langle x_i, x_j \rangle}{\langle x_1, x_1 \rangle} \quad i, j = 1, \dots, k; \quad i \leq j$$

generate the $SLS(3)$ -invariant rational functions.

ii) if $k \geq 3$ then the system

$$\frac{\det[x_i \ x_j \ x_k]}{\det[x_1 \ x_2 \ x_3]}, \quad i < j < k$$

$$\frac{\langle x_i, x_j \rangle}{\langle x_1, x_1 \rangle} \quad i, j = 1, 2, 3; \quad i \leq j \quad ; \quad i + j < 6;$$

$$\frac{\langle x_i, x_p \rangle}{\langle x_1, x_1 \rangle}; \quad i, j = 1, 2, 3; \quad p = 4, 5, \dots, k$$

generate the $SLS(3)$ -invariant rational functions.

Proof: since the $LS(3)$ -invariant functions are both $O(3)$ -invariant and $LH(3)$ -invariant, it is stated as Theorem 7. Similarly Because the $SLS(3)$ -invariant rational functions will be both $SO(3)$ -invariant and $LH(3)$ -invariant, the proof is completed similarly as Theorem 7.

Theorem 14: The set of $G = SS(3)$ invariant rational functions for k vector variables

1. Contains only one element "1" which is identity of the field if $k = 1$. Indeed in this case $G = SS(3)$ invariant rational functions are constant.
2. Contains the system

$$\frac{\langle x_i - x_1, x_j - x_1 \rangle}{\langle x_2 - x_1, x_2 - x_1 \rangle} \quad i, j = 2, 3, 4; \quad i \leq j$$

if $2 \leq k \leq 4$

3. Contains the system

$$\frac{\det[x_i - x_1 \ x_j - x_1 \ x_k - x_1]}{\det[x_2 - x_1 \ x_3 - x_1 \ x_4 - x_1]}, \quad i < j < k$$

$$\frac{\langle x_i - x_1, x_j - x_1 \rangle}{\langle x_2 - x_1, x_2 - x_1 \rangle} \quad i, j = 2, 3, 4; \quad i \leq j; \quad i + j < 8;$$

$$\frac{\langle x_i - x_1, x_p - x_1 \rangle}{\langle x_2 - x_1, x_2 - x_1 \rangle}; \quad i = 2, 3, 4; \quad p = 5, \dots, k$$

if $k > 4$.

Proof: As the proof of the Theorem 8 with the help of proofs of Theorem 7 and Theorem 11, this theorem can be proved from theorem 13 and theorem 11 similarly.

The G- Equivalence Conditions of Points Systems for $G = S(3)$ and its all subgroups in R^3

Fist of all these propositions can be given for $G = S(3)$ and its all subgroups in R^3

Proposition 1: Let $\{x_1, x_2, \dots, x_k\}, \{y_1, y_2, \dots, y_k\} \subset R^3$ be given and $G \subset S(3)$ be a linear transformation's subgroup (may be $O(3), SO(3), LH(3), LS(3), SLS(3)$). Then

1. If $x_1 = 0$ and $y_1 \neq 0$ (or vice versa) then these vectors is not G -equivalent. i.e. $\{x_1\} \stackrel{G}{\not\approx} \{y_1\}$
2. If $x_1 = 0$ and $y_1 = 0$ then these vectors is always G -equivalent. i.e. $\{x_1\} \stackrel{G}{\approx} \{y_1\}$ and the G -equivalent conditions of two point systems with k vectors $\{x_1, x_2, \dots, x_k\}$ and $\{y_1, y_2, \dots, y_k\}$ is reduced to G -equivalent conditions of two point systems with $k-1$ vectors $\{x_2, \dots, x_k\}$ and $\{y_2, \dots, y_k\}$. (The same theorem can be expressed by subtracting the i th vectors.)

Proof: 1. Let $x_1 = 0$ be given. Then for any transformation g in $S(3)$ or its any subgroup G , the vector gx_1 is equal to zero. So if $\{x_1\} \stackrel{G}{\approx} \{y_1\}$ then $y_1 \neq 0$ must be satisfied. So this is a contradiction. Thus in case $x_1 = 0$ and $y_1 \neq 0$ these vectors must be not G -equivalent. i.e. $\{x_1\} \stackrel{G}{\not\approx} \{y_1\}$. The Other case can be proved similarly.

2. Let $x_1 = 0$ and $y_1 = 0$ be given. Then for any transformation $\in G$, Suppose that $gx_1 = 0$ satisfies. So $\{x_1\} \stackrel{G}{\approx} \{y_1\}$ be proved. Now, $\{x_1, x_2, \dots, x_k\}$ and $\{y_1, y_2, \dots, y_k\}$ be given and $x_1 = 0$ and $y_1 = 0$ be supposed. In this case it must be proved that the G -equivalent conditions of two point systems with k vectors $\{x_1, x_2, \dots, x_k\}$ and $\{y_1, y_2, \dots, y_k\}$ is reduced to G -equivalent conditions of $k-1$ vectors except x_1 and y_1 .

Let $\{x_1, x_2, \dots, x_k\} \stackrel{G}{\approx} \{y_1, y_2, \dots, y_k\}$ be supposed. Then there exist a transformation $g \in G$ such that $y_i = gx_i$ is written for $i = 1, \dots, k$. So $y_i = gx_i$ satisfies for $i = 2, \dots, k$. Thus $\{x_2, \dots, x_k\} \stackrel{G}{\approx} \{y_2, \dots, y_k\}$.

Conversely $\{x_2, \dots, x_k\} \stackrel{G}{\approx} \{y_2, \dots, y_k\}$ be supposed. Then there exist a transformation $g \in G$ such that $y_i = gx_i$ is written for $i = 2, \dots, k$. Since $x_1 = 0$ and $y_1 = 0$, $gx_1 = y_1$ is satisfied ($g0 = 0$) so $y_i = gx_i$ is written for $i = 1, \dots, k$. And then $\{x_1, x_2, \dots, x_k\} \stackrel{G}{\approx} \{y_1, y_2, \dots, y_k\}$ is obtained.

Theorem 15:

$\{x_1, x_2, \dots, x_k\}$ and $\{y_1, y_2, \dots, y_k\}$ be given. Then $\{x_1, x_2, \dots, x_k\} \stackrel{G}{\approx} \{y_1, y_2, \dots, y_k\}$ if and only if if $k \leq 3$ then $\langle x_i, x_j \rangle = \langle y_i, y_j \rangle ; i \leq j ; i, j = 1, \dots, k$ if $k > 3$ then $\langle x_i, x_j \rangle = \langle y_i, y_j \rangle ; i \leq j ; i, j = 1, 2, 3 ;$ and $\langle x_i, x_p \rangle = \langle y_i, y_p \rangle ; i = 1, 2, 3 ; p = 4, \dots, k$ satisfies for $G = O(3) \subset S(3)$.
Proof: see [10].

Theorem 16:

$\{x_1, x_2, \dots, x_k\}$ and $\{y_1, y_2, \dots, y_k\}$ be given. Then $\{x_1, x_2, \dots, x_k\} \stackrel{G}{\approx} \{y_1, y_2, \dots, y_k\}$ if and only if if $k \leq 3$ then $\langle x_i, x_j \rangle = \langle y_i, y_j \rangle ; i \leq j ; i, j = 1, \dots, k$ if $k > 3$ then $\langle x_i, x_j \rangle = \langle y_i, y_j \rangle ; i \leq j ; i + j < 6 ; i, j = 1, 2, 3$ and $\langle x_i, x_p \rangle = \langle y_i, y_p \rangle ; i = 1, 2, 3 ; p = 4, \dots, k$
 $det[x_1 \ x_2 \ x_3] = det[y_1 \ y_2 \ y_3]$
satisfies for $G = SO(3) \subset S(3)$.
Proof: see [10].

Theorem 17:

$\{x_1, x_2, \dots, x_k\}$ and $\{y_1, y_2, \dots, y_k\}$ be given. Then $\{x_1, x_2, \dots, x_k\} \stackrel{G}{\approx} \{y_1, y_2, \dots, y_k\}$ if and only if if $k = 1$ then always $\{x_1\} \stackrel{G}{\approx} \{y_1\}$ satisfies. if $2 \leq k \leq 4$ then $\langle x_i - x_1, x_j - x_1 \rangle = \langle y_i - y_1, y_j - y_1 \rangle ; i \leq j ; i, j = 2, \dots, k$ if $k > 4$ then $\langle x_i - x_1, x_j - x_1 \rangle = \langle y_i - y_1, y_j - y_1 \rangle ; i \leq j ; i + j < 8 ; i, j = 2, 3, 4$ and $\langle x_i - x_1, x_p - x_1 \rangle = \langle y_i - y_1, y_p - y_1 \rangle ; i = 2, 3, 4 ; p = 5, \dots, k$
 $det[x_2 - x_1 \ x_3 - x_1 \ x_4 - x_1] = det[y_2 - y_1 \ y_3 - y_1 \ y_4 - y_1]$
satisfies for $G = SE(3) \subset S(3)$.
Proof: see [10].

Theorem 18:

$\{x_1, x_2, \dots, x_k\}$ and $\{y_1, y_2, \dots, y_k\}$ be given. Then $\{x_1, x_2, \dots, x_k\} \stackrel{G}{\approx} \{y_1, y_2, \dots, y_k\}$ if and only if if $k = 1$ then always $\{x_1\} \stackrel{G}{\approx} \{y_1\}$ satisfies. if $2 \leq k \leq 4$ then $\langle x_i - x_1, x_j - x_1 \rangle = \langle y_i - y_1, y_j - y_1 \rangle ; i \leq j$ if $k > 4$ then $\langle x_i - x_1, x_j - x_1 \rangle = \langle y_i - y_1, y_j - y_1 \rangle ; i \leq j ; i, j = 2, 3, 4$ and $\langle x_i - x_1, x_p - x_1 \rangle = \langle y_i - y_1, y_p - y_1 \rangle ; i = 2, 3, 4 ; p = 5, \dots, k$ satisfies for $G = E(3) \subset S(3)$.
Proof: see [10].

Theorem 19:

$\{x_1, x_2, \dots, x_k\}$ and $\{y_1, y_2, \dots, y_k\}$ be given. the ranks of matrices of these vectors systems $[x_1, x_2, \dots, x_k]$ and $[y_1, y_2, \dots, y_k]$ should be same in order to be equivalent: if the ranks of these matrices, (let denoted it by r), then;

i) in case $r = 3$
 $\{x_1, x_2, \dots, x_k\} \stackrel{G}{\approx} \{y_1, y_2, \dots, y_k\}$ if and only if $\frac{\langle x_i, x_j \rangle}{\langle x_1, x_1 \rangle} = \frac{\langle y_i, y_j \rangle}{\langle y_1, y_1 \rangle} ; i \leq j$ satisfies. (for $k \leq 3$)

or $\frac{\langle x_i, x_j \rangle}{\langle x_1, x_1 \rangle} = \frac{\langle y_i, y_j \rangle}{\langle y_1, y_1 \rangle} ; i \leq j ; i, j = 1, 2, 3$ and

$\frac{\langle x_i, x_p \rangle}{\langle x_1, x_1 \rangle} = \frac{\langle y_i, y_p \rangle}{\langle y_1, y_1 \rangle} ; i = 1, 2, 3 ; p = 4, \dots, k$ satisfies (for $k > 3$)

ii) in case $r = 2$
 $\{x_1, x_2, \dots, x_k\} \stackrel{G}{\approx} \{y_1, y_2, \dots, y_k\}$ if and only if $\frac{\langle x_i, x_j \rangle}{\langle x_1, x_1 \rangle} = \frac{\langle y_i, y_j \rangle}{\langle y_1, y_1 \rangle} ; i \leq j ; i, j = 1, 2$ satisfies. (for $k \leq 2$)

or $\frac{\langle x_i, x_p \rangle}{\langle x_1, x_1 \rangle} = \frac{\langle y_i, y_p \rangle}{\langle y_1, y_1 \rangle} ; i \leq j ; i, j = 1, 2 ; p = 3, \dots, k$ satisfies (for $k > 2$)

iii) in case $r = 1$
 $\{x_1, x_2, \dots, x_k\} \stackrel{G}{\approx} \{y_1, y_2, \dots, y_k\}$ if and only if $\frac{\langle x_1, x_j \rangle}{\langle x_1, x_1 \rangle} = \frac{\langle y_1, y_j \rangle}{\langle y_1, y_1 \rangle} ; j = 2, \dots, k$ satisfies for $G = LS(3) \subset S(3)$.
Proof: see [22].

Theorem 20:

$\{x_1, x_2, \dots, x_k\}$ and $\{y_1, y_2, \dots, y_k\}$ be given. the ranks of matrices of these vectors systems $[x_2 - x_1, x_3 - x_1, \dots, x_k - x_1]$ and $[y_2 - y_1, y_3 - y_1, \dots, y_k - y_1]$ should be same in order to be equivalent: if the ranks of matrices, (let denoted it by r as above), then;

i) in case $r = 3$
 $\{x_1, x_2, \dots, x_k\} \stackrel{G}{\approx} \{y_1, y_2, \dots, y_k\}$ if and only if
 $\frac{\langle x_i - x_1, x_j - x_1 \rangle}{\langle x_2 - x_1, x_2 - x_1 \rangle} = \frac{\langle y_i - y_1, y_j - y_1 \rangle}{\langle y_2 - y_1, y_2 - y_1 \rangle}; i \leq j$ satisfies. (for $k \leq 4$) or
 $\frac{\langle x_i - x_1, x_j - x_1 \rangle}{\langle x_2 - x_1, x_2 - x_1 \rangle} = \frac{\langle y_i - y_1, y_j - y_1 \rangle}{\langle y_2 - y_1, y_2 - y_1 \rangle}; i \leq j; i, j = 2, 3, 4$ and

$\frac{\langle x_i - x_1, x_p - x_1 \rangle}{\langle x_2 - x_1, x_2 - x_1 \rangle} = \frac{\langle y_i - y_1, y_p - y_1 \rangle}{\langle y_2 - y_1, y_2 - y_1 \rangle}; i = 2, 3, 4; p = 5, \dots, k$
 satisfies (for $k > 4$)

ii) in case $r = 2$
 $\{x_1, x_2, \dots, x_k\} \stackrel{G}{\approx} \{y_1, y_2, \dots, y_k\}$ if and only if
 $\frac{\langle x_i - x_1, x_j - x_1 \rangle}{\langle x_2 - x_1, x_2 - x_1 \rangle} = \frac{\langle y_i - y_1, y_j - y_1 \rangle}{\langle y_2 - y_1, y_2 - y_1 \rangle}; i \leq j; i, j = 2, 3$
 satisfies. (for $k \leq 3$) or

$\frac{\langle x_i - x_1, x_p - x_1 \rangle}{\langle x_2 - x_1, x_2 - x_1 \rangle} = \frac{\langle y_i - y_1, y_p - y_1 \rangle}{\langle y_2 - y_1, y_2 - y_1 \rangle}; i = 2, 3; p = 4, \dots, k$
 satisfies (for $k \geq 4$)

iii) in case $r = 1$
 $\{x_1, x_2, \dots, x_k\} \stackrel{G}{\approx} \{y_1, y_2, \dots, y_k\}$ if and only if
 $\frac{\langle x_2 - x_1, x_j - x_1 \rangle}{\langle x_2 - x_1, x_2 - x_1 \rangle} = \frac{\langle y_2 - y_1, y_j - y_1 \rangle}{\langle y_2 - y_1, y_2 - y_1 \rangle}; j = 3, \dots, k$
 satisfies for $G = S(3) \subset S(3)$.

Proof: It can be easily proved like theorem 2.2 in [18].

Theorem 21:

$\{x_1, x_2, \dots, x_k\}$ and $\{y_1, y_2, \dots, y_k\}$ be given.
 the ranks of matrices of these vectors systems $[x_1, x_2, \dots, x_k]$ and $[y_1, y_2, \dots, y_k]$ should be same in order to be equivalent:

if the ranks of these matrices, (let denoted it by r), then;

$\{x_1, x_2, \dots, x_k\} \stackrel{G}{\approx} \{y_1, y_2, \dots, y_k\}$ if and only if
 $\frac{\langle x_i, x_j \rangle}{\langle x_1, x_1 \rangle} = \frac{\langle y_i, y_j \rangle}{\langle y_1, y_1 \rangle}; i \leq j$ satisfies for $k \leq 2$ or for arbitrary k and $r = 1, 2$

or
 $\begin{cases} \frac{\langle x_i, x_j \rangle}{\langle x_1, x_1 \rangle} = \frac{\langle y_i, y_j \rangle}{\langle y_1, y_1 \rangle}; i \leq j; i, j = 1, 2, 3 \text{ and } i + j < 6 \\ \frac{\langle x_i, x_p \rangle}{\langle x_1, x_1 \rangle} = \frac{\langle y_i, y_p \rangle}{\langle y_1, y_1 \rangle}; i = 1, 2, 3; p = 4, \dots, k \\ \frac{\det[x_i \ x_j \ x_m]}{\det[x_1 \ x_2 \ x_3]} = \frac{\det[y_i \ y_j \ y_m]}{\det[y_1 \ y_2 \ y_3]} > 0; i < j < m; i, j, m = 1, \dots, k \end{cases}$
 satisfies for $k \geq 3, r = 3$ for the group $G = SLS(3) \subset S(3)$.

Proof: \Rightarrow Let $\{x_1, x_2, \dots, x_k\} \stackrel{G}{\approx} \{y_1, y_2, \dots, y_k\}$ be given. Then there exist a transformation $g \in G$ such that for every $i, y_i = \lambda g x_i$ satisfies. So $\frac{\langle y_i, y_j \rangle}{\langle y_1, y_1 \rangle} = \frac{\langle \lambda g x_i, \lambda g x_j \rangle}{\langle \lambda g x_1, \lambda g x_1 \rangle} = \frac{\langle x_i, x_j \rangle}{\langle x_1, x_1 \rangle}$ is written for every i and j . In addition for $r = 3$, $\frac{\det[y_i \ y_j \ y_m]}{\det[y_1 \ y_2 \ y_3]} = \frac{\det[\lambda g x_i \ \lambda g x_j \ \lambda g x_m]}{\det[\lambda g x_1 \ \lambda g x_2 \ \lambda g x_3]} = \frac{\det[x_i \ x_j \ x_m]}{\det[x_1 \ x_2 \ x_3]}$ satisfies.

\Leftarrow : Let $k \geq 3, r = 3$ be supposed and

$$\begin{cases} \frac{\langle x_i, x_j \rangle}{\langle x_1, x_1 \rangle} = \frac{\langle y_i, y_j \rangle}{\langle y_1, y_1 \rangle}; i \leq j; i, j = 1, 2, 3 \text{ and } i + j < 6 \\ \frac{\langle x_i, x_p \rangle}{\langle x_1, x_1 \rangle} = \frac{\langle y_i, y_p \rangle}{\langle y_1, y_1 \rangle}; i = 1, 2, 3; p = 4, \dots, k \\ \frac{\det[x_i \ x_j \ x_m]}{\det[x_1 \ x_2 \ x_3]} = \frac{\det[y_i \ y_j \ y_m]}{\det[y_1 \ y_2 \ y_3]}; i < j < m; i, j, m = 1, \dots, k \end{cases}$$

be given. Then, $\langle y_i, y_j \rangle = \frac{\langle y_1, y_1 \rangle}{\langle x_1, x_1 \rangle} \langle x_i, x_j \rangle$ can be written.

This equations can be rewritten as

$$\langle y_i, y_j \rangle = \langle \lambda x_i, \lambda x_j \rangle \tag{1}$$

if $\frac{\langle y_1, y_1 \rangle}{\langle x_1, x_1 \rangle}$ is denoted by λ . Since $r = 3$ then there are three linear independent vectors in each systems. So, without anything in general, these vectors let be choosen as $x_1 \ x_2 \ x_3$ and $y_1 \ y_2 \ y_3$. In this case for every i, j, m

$$\begin{cases} x_i = \alpha_{i1} x_1 + \alpha_{i2} x_2 + \alpha_{i3} x_3 \\ x_j = \alpha_{j1} x_1 + \alpha_{j2} x_2 + \alpha_{j3} x_3 \\ x_m = \alpha_{m1} x_1 + \alpha_{m2} x_2 + \alpha_{m3} x_3 \end{cases} \tag{2}$$

and

$$\begin{cases} y_i = \beta_{i1} y_1 + \beta_{i2} y_2 + \beta_{i3} y_3 \\ y_j = \beta_{j1} y_1 + \beta_{j2} y_2 + \beta_{j3} y_3 \\ y_m = \beta_{m1} y_1 + \beta_{m2} y_2 + \beta_{m3} y_3 \end{cases} \tag{3}$$

can be written. After this the proof can be completed as the proof of theorem 19 in [22]. So,

$$y_i = \lambda g x_i \tag{4}$$

is obtained. However, the thing to note here is that the transformation g satisfying the condition (4) may be from SLS(3) or it may be from reflections whose determinant is -1. The way to ensure that this is from SLS(3) is to satisfy that the third condition, determinant ratios are equal and possitive. Now let it be proved. From (2) and (3)

$$\frac{\det[y_i \ y_j \ y_m]}{\det[x_i \ x_j \ x_m]} = \frac{\det B \cdot \det[y_1 \ y_2 \ y_3]}{\det A \cdot \det[x_1 \ x_2 \ x_3]}$$

is obtained where

$$A = \begin{pmatrix} \alpha_{i1} & \alpha_{i2} & \alpha_{i3} \\ \alpha_{j1} & \alpha_{j2} & \alpha_{j3} \\ \alpha_{m1} & \alpha_{m2} & \alpha_{m3} \end{pmatrix} \text{ and } B = \begin{pmatrix} \beta_{i1} & \beta_{i2} & \beta_{i3} \\ \beta_{j1} & \beta_{j2} & \beta_{j3} \\ \beta_{m1} & \beta_{m2} & \beta_{m3} \end{pmatrix}$$

As can be seen in the proof of theorem 19 in [22], A and B are equal. So $\det A = \det B$ is written. Then,

$$\frac{\det[y_1 \ y_2 \ y_3]}{\det[x_1 \ x_2 \ x_3]} = \frac{\det[\lambda g x_1 \ \lambda g x_2 \ \lambda g x_3]}{\det[x_1 \ x_2 \ x_3]} = \lambda^3 \det g$$

is obtained. Since $\lambda > 0$, the signal of proportion $\frac{\det[y_1 \ y_2 \ y_3]}{\det[x_1 \ x_2 \ x_3]}$ depend on the signal of g . So it must be possitive. Then $\det g = 1$ and from (4) $\{x_1, x_2, \dots, x_k\} \stackrel{G}{\approx} \{y_1, y_2, \dots, y_k\}$ is obtained for the group $G = SLS(3)$.

In case $r = 2$ or $r = 1$, since the determinant of the matrices of any three vectors must be zero. i.e.

$$\det[x_1 \ x_2 \ x_3] = \det[y_1 \ y_2 \ y_3] = 0$$

and the proportion of determinant is undefined, the conditions in the first are sufficient. This means that every

two lines (or two planes) passing through origine can be rotated to transform one into the other. So the first conditions are sufficient.

Theorem 22:

$\{x_1, x_2, \dots, x_k\}$ and $\{y_1, y_2, \dots, y_k\}$ be given. the ranks of matrices of these vectors systems $[(x_2 - x_1) (x_3 - x_1) \dots (x_k - x_1)]$ and $[(y_2 - y_1) (y_3 - y_1) \dots (y_k - y_1)]$ should be same in order to be equivalent:

if the ranks of matrices, (let denoted it by r as above), then; $\{x_1, x_2, \dots, x_k\} \stackrel{G}{\approx} \{y_1, y_2, \dots, y_k\}$ if and only if

$$\frac{\langle x_i - x_1, x_j - x_1 \rangle}{\langle x_2 - x_1, x_2 - x_1 \rangle} = \frac{\langle y_i - y_1, y_j - y_1 \rangle}{\langle y_2 - y_1, y_2 - y_1 \rangle}; i \leq j \text{ satisfies for } k \leq 3$$

or for arbitrary k and $r=1,2$

$$\begin{cases} \frac{\langle x_i - x_1, x_j - x_1 \rangle}{\langle x_2 - x_1, x_2 - x_1 \rangle} = \frac{\langle y_i - y_1, y_j - y_1 \rangle}{\langle y_2 - y_1, y_2 - y_1 \rangle}; i \leq j; i, j = 2,3,4 \text{ and } i + j < 8; \\ \frac{\langle x_i - x_1, x_p - x_1 \rangle}{\langle x_2 - x_1, x_2 - x_1 \rangle} = \frac{\langle y_i - y_1, y_p - y_1 \rangle}{\langle y_2 - y_1, y_2 - y_1 \rangle}; i = 2,3,4; p = 5, \dots, k \\ \frac{\det[x_i - x_1 \ x_j - x_1 \ x_m - x_1]}{\det[x_2 - x_1 \ x_3 - x_1 \ x_4 - x_1]} = \frac{\det[y_i - y_1 \ y_j - y_1 \ y_m - y_1]}{\det[y_2 - y_1 \ y_3 - y_1 \ y_4 - y_1]} > 0; \\ i < j < m; i, j, m = 2, \dots, k \end{cases}$$

satisfies (for $k \geq 4$) for $G = SS(3) \subset S(3)$.

Proof: Since $SS(3)$ - invariant rational function is both $SLS(3)$ - invariant and $Tr(3)$ - invariant, , this theorem can easily proved from Theorem 11 and Theorem 21.

The G- Equivalence Conditions of Open B-Spline Curves for $G= S(3)$ and its all subgroups in R^3

One of the most important properties of bezier and b-spline curves, even their surfaces, is the Invariance under Affine Transformations property expressed in theorem 2. According to this property, when we want to transform a bezier or b-spline curve with any affine transformation, it is sufficient to transform only control points instead of transforming each point of the curve. In other words, when we transform a B-spline curve $B(t)$ of degree d with control points b_i by an affine transformation T , the curve we will obtain is also a B-spline curve of degree d whose control points are $T(b_i)$.

So these theorems can be stated as follows:

Theorem 23:Let $\mathfrak{B}_1, \mathfrak{B}_2$ are open B-spline curves of degree d with control points b_0, b_1, \dots, b_n and c_0, c_1, \dots, c_n respectively and knot vektors $t_0 = t_1 = \dots = t_d; t_{d+1}, t_{d+2}, \dots, t_{m-d-1}; t_{m-d} = t_{m-d+1} = \dots = t_m$ be given. Then

$$\mathfrak{B}_1 \stackrel{G}{\approx} \mathfrak{B}_2 \text{ if and only if } \begin{cases} \langle b_i, b_j \rangle = \langle c_i, c_j \rangle; i \leq j; i, j = 0,1,2; \\ \langle b_i, b_p \rangle = \langle c_i, c_p \rangle; i = 0,1,2; p = 3, \dots, n \end{cases}$$

satisfies for $G = O(3) \subset S(3)$.

Proof: This theorem is a result of theorem 15.

Theorem 24:Let $\mathfrak{B}_1, \mathfrak{B}_2$ are open B-spline curves of degree d with control points b_0, b_1, \dots, b_n and c_0, c_1, \dots, c_n respectively and knot vektors $t_0 = t_1 = \dots = t_d; t_{d+1}, t_{d+2}, \dots, t_{m-d-1}; t_{m-d} = t_{m-d+1} = \dots = t_m$ be given. Then

$$\mathfrak{B}_1 \stackrel{G}{\approx} \mathfrak{B}_2 \text{ if and only if } \begin{cases} \langle b_i, b_j \rangle = \langle c_i, c_j \rangle; i \leq j; i, j = 0,1,2; i + j < 4 \\ \langle b_i, b_p \rangle = \langle c_i, c_p \rangle; i = 0,1,2; p = 3, \dots, n \\ \det[b_0 \ b_1 \ b_2] = \det[c_0 \ c_1 \ c_2] \end{cases}$$

satisfies for $G = SO(3) \subset S(3)$.

Proof: This theorem is a result of theorem 16.

Theorem 25:Let $\mathfrak{B}_1, \mathfrak{B}_2$ are open B-spline curves of degree d with control points b_0, b_1, \dots, b_n and c_0, c_1, \dots, c_n respectively and knot vektors $t_0 = t_1 = \dots = t_d; t_{d+1}, t_{d+2}, \dots, t_{m-d-1}; t_{m-d} = t_{m-d+1} = \dots = t_m$ be given. Then

$$\mathfrak{B}_1 \stackrel{G}{\approx} \mathfrak{B}_2 \text{ if and only if } \begin{cases} \langle b_i - b_0, b_j - b_0 \rangle = \langle c_i - c_0, c_j - c_0 \rangle; i \leq j; i, j = 1,2,3; i + j < 6 \\ \langle b_i - b_0, b_p - b_0 \rangle = \langle c_i - c_0, c_p - c_0 \rangle; i = 1,2,3; p = 4, \dots, n \\ \det[b_1 - b_0 \ b_2 - b_0 \ b_3 - b_0] = \det[c_1 - c_0 \ c_2 - c_0 \ c_3 - c_0] \end{cases}$$

satisfies for $G = SE(3) \subset S(3)$.

Proof: This theorem is a result of theorem 17.

Theorem 26:Let $\mathfrak{B}_1, \mathfrak{B}_2$ are open B-spline curves of degree d with control points b_0, b_1, \dots, b_n and c_0, c_1, \dots, c_n respectively and knot vektors $t_0 = t_1 = \dots = t_d; t_{d+1}, t_{d+2}, \dots, t_{m-d-1}; t_{m-d} = t_{m-d+1} = \dots = t_m$ be given. Then

$$\mathfrak{B}_1 \stackrel{G}{\approx} \mathfrak{B}_2 \text{ if and only if } \begin{cases} \langle b_i - b_0, b_j - b_0 \rangle = \langle c_i - c_0, c_j - c_0 \rangle; i \leq j; i, j = 1,2,3 \ i + j < 6 \\ \langle b_i - b_0, b_p - b_0 \rangle = \langle c_i - c_0, c_p - c_0 \rangle; i = 1,2,3; p = 4, \dots, n \end{cases}$$

satisfies for $G = E(3) \subset S(3)$.

Proof: This theorem is a result of theorem 18.

Theorem 27:Let $\mathfrak{B}_1, \mathfrak{B}_2$ are open B-spline curves of degree d with control points b_0, b_1, \dots, b_n and c_0, c_1, \dots, c_n respectively and knot vektors $t_0 = t_1 = \dots = t_d; t_{d+1}, t_{d+2}, \dots, t_{m-d-1}; t_{m-d} = t_{m-d+1} = \dots = t_m$ be given and $\text{rank}[b_0 \ b_1 \ \dots \ b_n] = \text{rank}[c_0 \ c_1 \ \dots \ c_n] = r$ be supposed. Then

$$\mathfrak{B}_1 \stackrel{G}{\approx} \mathfrak{B}_2 \text{ if and only if } \begin{cases} \frac{\langle b_i, b_j \rangle}{\langle b_0, b_0 \rangle} = \frac{\langle c_i, c_j \rangle}{\langle c_0, c_0 \rangle}; i \leq j; i, j = 0,1,2; \\ \frac{\langle b_i, b_p \rangle}{\langle b_0, b_0 \rangle} = \frac{\langle c_i, c_p \rangle}{\langle c_0, c_0 \rangle}; i = 0,1,2; p = 3, \dots, n \end{cases}$$

satisfies for $r = 3$ or

$$\frac{\langle b_i, b_p \rangle}{\langle b_0, b_0 \rangle} = \frac{\langle c_i, c_p \rangle}{\langle c_0, c_0 \rangle}; i = 0,1; p = 2, \dots, n$$

satisfies for $r = 2$ or

$$\frac{\langle b_0, b_p \rangle}{\langle b_0, b_0 \rangle} = \frac{\langle c_0, c_p \rangle}{\langle c_0, c_0 \rangle}; p = 1, \dots, n$$

satisfies for $r = 1$

for $G = LS(3) \subset S(3)$.

Proof: This theorem is a result of theorem 19.

Theorem 28:Let $\mathfrak{B}_1, \mathfrak{B}_2$ are open B-spline curves of degree d with control points b_0, b_1, \dots, b_n and c_0, c_1, \dots, c_n respectively and knot vektors $t_0 = t_1 = \dots = t_d; t_{d+1}, t_{d+2}, \dots, t_{m-d-1}; t_{m-d} = t_{m-d+1} = \dots = t_m$ be given and $\text{rank}[b_1 - b_0 \dots b_n - b_0] = \text{rank}[c_1 - c_0 \dots c_n - c_0] = r$ be supposed. Then

$\mathfrak{B}_1 \stackrel{G}{\approx} \mathfrak{B}_2$ if and only if

$$\begin{cases} \frac{\langle b_i - b_0, b_j - b_0 \rangle}{\langle b_1 - b_0, b_1 - b_0 \rangle} = \frac{\langle c_i - c_0, c_j - c_0 \rangle}{\langle c_1 - c_0, c_1 - c_0 \rangle}; i \leq j; i, j = 1, 2, 3; \\ \frac{\langle b_i - b_0, b_p - b_0 \rangle}{\langle b_1 - b_0, b_1 - b_0 \rangle} = \frac{\langle c_i - c_0, c_p - c_0 \rangle}{\langle c_1 - c_0, c_1 - c_0 \rangle}; i = 1, 2, 3; p = 4, \dots, n \end{cases}$$

satisfies for $r = 3$ or

$$\frac{\langle b_i - b_0, b_p - b_0 \rangle}{\langle b_1 - b_0, b_1 - b_0 \rangle} = \frac{\langle c_i - c_0, c_p - c_0 \rangle}{\langle c_1 - c_0, c_1 - c_0 \rangle}; i = 1, 2; p = 3, \dots, n$$

satisfies for $r = 2$ or

$$\frac{\langle b_1 - b_0, b_p - b_0 \rangle}{\langle b_1 - b_0, b_1 - b_0 \rangle} = \frac{\langle c_1 - c_0, c_p - c_0 \rangle}{\langle c_1 - c_0, c_1 - c_0 \rangle}; p = 1, \dots, n$$

satisfies for $r = 1$

for $G = S(3) \subset S(3)$.

Proof: This theorem is a result of theorem 20.

Theorem 29:Let $\mathfrak{B}_1, \mathfrak{B}_2$ are open B-spline curves of degree d with control points b_0, b_1, \dots, b_n and c_0, c_1, \dots, c_n respectively and knot vektors $t_0 = t_1 = \dots = t_d; t_{d+1}, t_{d+2}, \dots, t_{m-d-1}; t_{m-d} = t_{m-d+1} = \dots = t_m$ be given and $\text{rank}[b_0 \ b_1 \dots b_n] = \text{rank}[c_0 \ c_1 \dots c_n] = r$ be supposed. Then

$\mathfrak{B}_1 \stackrel{G}{\approx} \mathfrak{B}_2$ if and only if

$$\begin{cases} \frac{\langle b_i, b_j \rangle}{\langle b_0, b_0 \rangle} = \frac{\langle c_i, c_j \rangle}{\langle c_0, c_0 \rangle}; i \leq j; i, j = 0, 1, 2; i + j < 4 \\ \frac{\langle b_i, b_p \rangle}{\langle b_0, b_0 \rangle} = \frac{\langle c_i, c_p \rangle}{\langle c_0, c_0 \rangle}; i = 0, 1, 2; p = 3, \dots, n \\ \frac{\det[b_i \ b_j \ b_m]}{\det[b_0 \ b_1 \ b_2]} = \frac{\det[c_i \ c_j \ c_m]}{\det[c_0 \ c_1 \ c_2]} > 0; i < j < m; i, j, m = 0, \dots, n \end{cases}$$

satisfies for $r = 3$ or

$$\frac{\langle b_i, b_p \rangle}{\langle b_0, b_0 \rangle} = \frac{\langle c_i, c_p \rangle}{\langle c_0, c_0 \rangle}; i = 0, 1; p = 2, \dots, n$$

satisfies for $r = 2$ or

$$\frac{\langle b_0, b_p \rangle}{\langle b_0, b_0 \rangle} = \frac{\langle c_0, c_p \rangle}{\langle c_0, c_0 \rangle}; p = 1, \dots, n$$

satisfies for $r = 1$

for $G = SLS(3) \subset S(3)$.

Proof: This theorem is a result of theorem 21.

Theorem 30:Let $\mathfrak{B}_1, \mathfrak{B}_2$ are open B-spline curves of degree d with control points b_0, b_1, \dots, b_n and c_0, c_1, \dots, c_n respectively and knot vektors $t_0 = t_1 = \dots = t_d; t_{d+1}, t_{d+2}, \dots, t_{m-d-1}; t_{m-d} = t_{m-d+1} = \dots = t_m$ be given and $\text{rank}[b_1 - b_0 \dots b_n - b_0] = \text{rank}[c_1 - c_0 \dots c_n - c_0] = r$ be supposed. Then

$\mathfrak{B}_1 \stackrel{G}{\approx} \mathfrak{B}_2$ if and only if

$$\begin{cases} \frac{\langle b_i - b_0, b_j - b_0 \rangle}{\langle b_1 - b_0, b_1 - b_0 \rangle} = \frac{\langle c_i - c_0, c_j - c_0 \rangle}{\langle c_1 - c_0, c_1 - c_0 \rangle}; i \leq j; i, j = 1, 2, 3; \\ \frac{\langle b_i - b_0, b_p - b_0 \rangle}{\langle b_1 - b_0, b_1 - b_0 \rangle} = \frac{\langle c_i - c_0, c_p - c_0 \rangle}{\langle c_1 - c_0, c_1 - c_0 \rangle}; i = 1, 2, 3; p = 4, \dots, n \\ \frac{\det[b_i - b_0 \ b_j - b_0 \ b_m - b_0]}{\det[b_1 - b_0 \ b_2 - b_0 \ b_3 - b_0]} = \frac{\det[c_i - c_0 \ c_j - c_0 \ c_m - c_0]}{\det[c_1 - c_0 \ c_2 - c_0 \ c_3 - c_0]} > 0; \\ i < j < m; i, j, m = 1, \dots, n \end{cases}$$

satisfies for $r = 3$ or

$$\frac{\langle b_i - b_0, b_p - b_0 \rangle}{\langle b_1 - b_0, b_1 - b_0 \rangle} = \frac{\langle c_i - c_0, c_p - c_0 \rangle}{\langle c_1 - c_0, c_1 - c_0 \rangle}; i = 1, 2; p = 3, \dots, n$$

satisfies for $r = 2$, or

$$\frac{\langle b_1 - b_0, b_p - b_0 \rangle}{\langle b_1 - b_0, b_1 - b_0 \rangle} = \frac{\langle c_1 - c_0, c_p - c_0 \rangle}{\langle c_1 - c_0, c_1 - c_0 \rangle}; p = 1, \dots, n$$

satisfies for $r = 1$

for $G = SS(3) \subset S(3)$.

Proof: This theorem is a result of theorem 22.

Example:

Let γ be a cubic B-spline curve of degree 3 with control points $b_0 = (4, 2, 2), b_1 = (2, 1, 4), b_2 = (3, 4, 1), b_3 = (3, 5, 5)$ and $t_0 = t_1 = \dots = t_3 = 0; \dots; t_4 = t_5 = \dots = t_7 = 1$. be given.

The spline basis functions:

$$N_{0,0} = 0, \quad N_{1,0} = 0, \quad N_{2,0} = 0, \quad N_{3,0} = \begin{cases} 1, & t \in [0, 1] \\ 0, & \text{other case} \end{cases}$$

$$N_{4,0} = 0, \quad N_{5,0} = 0, \quad N_{6,0} = 0$$

$$N_{0,1} = 0, \quad N_{1,1} = 0, \quad N_{2,1} = \begin{cases} 1-t, & t \in [0, 1] \\ 0, & \text{other case} \end{cases}$$

$$N_{3,1} = \begin{cases} t, & t \in [0, 1] \\ 0, & \text{other case} \end{cases}, \quad N_{4,1} = 0, \quad N_{5,1} = 0$$

$$N_{0,2} = 0, \quad N_{1,2} = \begin{cases} (1-t)^2, & t \in [0, 1] \\ 0, & \text{other case} \end{cases}, \quad N_{4,2} = 0$$

$$N_{2,2} = \begin{cases} 2t(1-t), & t \in [0, 1] \\ 0, & \text{other case} \end{cases}, \quad N_{3,2} = \begin{cases} t^2, & t \in [0, 1] \\ 0, & \text{other case} \end{cases}$$

$$N_{0,3} = \begin{cases} (1-t)^3, & t \in [0, 1] \\ 0, & \text{other case} \end{cases}, \quad N_{1,3} = \begin{cases} 3t(1-t)^2, & t \in [0, 1] \\ 0, & \text{other case} \end{cases}$$

$$N_{2,3} = \begin{cases} 3t^2(1-t), & t \in [0, 1] \\ 0, & \text{other case} \end{cases}, \quad N_{3,3} = \begin{cases} t^3, & t \in [0, 1] \\ 0, & \text{other case} \end{cases}$$

Then

$$\begin{aligned} \gamma(t) &= N_{0,3}b_0 + N_{1,3}b_1 + N_{2,3}b_2 + N_{3,3}b_3 \\ &= \begin{cases} (1-t)^3b_0 + 3t(1-t)^2b_1 + 3t^2(1-t)b_2 + t^3b_3, & t \in [0, 1] \\ 0, & \text{other case} \end{cases} \end{aligned}$$

This means: for $t \in [0, 1]$

$$\gamma(t) = \begin{pmatrix} -4t^3 + 9t^2 - 6t + 4, & -6t^3 + 12t^2 - 3t + 2, \\ & 12t^3 - 15t^2 + 6t + 2 \end{pmatrix}$$

can be written.

Let these transformations be chosen as element of $S(3)$ and its subgroups:

For $x \in R^3$,

$$g_1 \in Tr(3) \Rightarrow g_1(x) = x + (2,3,5)$$

$$g_2 \in O(3) \Rightarrow g_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$g_3 \in SO(3) \Rightarrow g_3 = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

$$g_4 \in SE(3) \Rightarrow g_4(x) = g_3(x) + (2,3,5)$$

$$g_5 \in E(3) \Rightarrow g_5(x) = g_2(x) + (2,3,5)$$

$$g_6 \in LH(3) \Rightarrow g_6 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$g_7 \in SLS(3) \Rightarrow g_7 = \frac{3\sqrt{2}}{2} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

$$g_8 \in LS(3) \Rightarrow g_8 = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 3 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

$$g_9 \in SS(3) \Rightarrow g_9(x) = g_7(x) + (2,3,5)$$

$$g_{10} \in S(3) \Rightarrow g_{10}(x) = g_8(x) + (2,3,5)$$

Let the images of the curve γ under the transformations g_i be denoted by γ_i respectively.

If we denote the subgroup of $S(3)$ containing the transformation g_i with G_i then $\gamma \stackrel{G_i}{\approx} \gamma_i$. (see Fig.1)

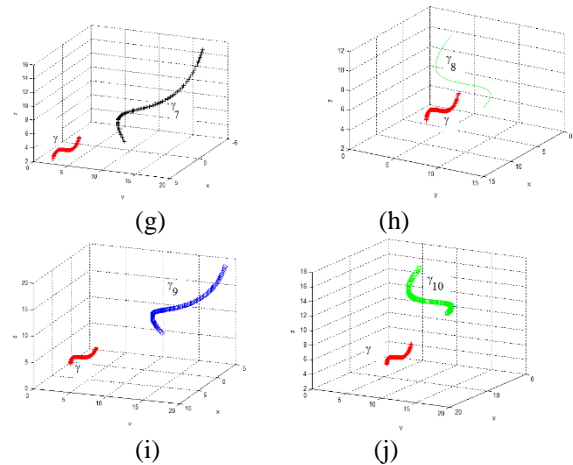
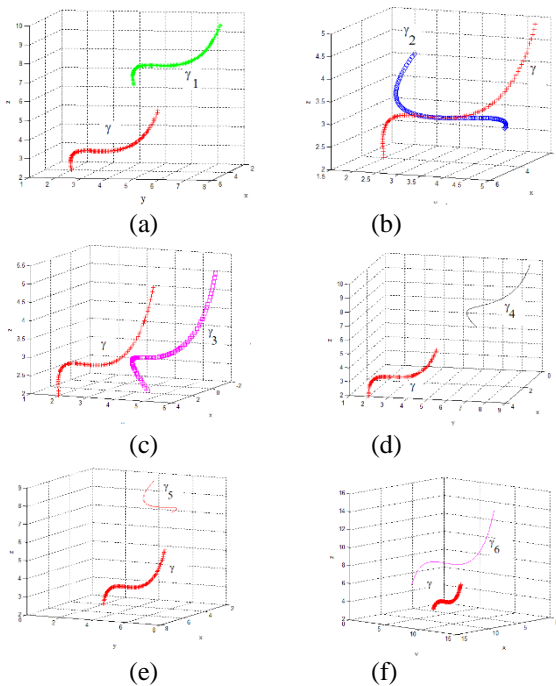


Figure 1: The G_i -Equivalent open B-spline Curves:

- (a): $\gamma \stackrel{G_1}{\approx} \gamma_1$, (b): $\gamma \stackrel{G_2}{\approx} \gamma_2$, (c): $\gamma \stackrel{G_3}{\approx} \gamma_3$, (d): $\gamma \stackrel{G_4}{\approx} \gamma_4$
 (e): $\gamma \stackrel{G_5}{\approx} \gamma_5$, (f): $\gamma \stackrel{G_6}{\approx} \gamma_6$, (g): $\gamma \stackrel{G_7}{\approx} \gamma_7$, (h): $\gamma \stackrel{G_8}{\approx} \gamma_8$
 (i): $\gamma \stackrel{G_9}{\approx} \gamma_9$, (j): $\gamma \stackrel{G_{10}}{\approx} \gamma_{10}$

REFERENCES

- [1] Incesu M. The Complete System of Point Invariants in the Similarity Geometry, Phd. Thesis, Karadeniz Technical University, Trabzon, 2008.
- [2] Nagata M., Local Rings, Interscience, NY, 1962.
- [3] Hadjiev D. Some questions in the theory of vector invariants, Math. USSR-Sbornic. 1 383-396, 1967.
- [4] Grosshans F. Obsevable groups and Hilbert’s problem, Am. J. Math., 95 229-253, 1973.
- [5] Bridgman P.W. Dimensional Analysis, 2 Eds., Yale University Press, New Heaven, 1931.
- [6] Sedov L.I. Similarity and Dimensional Method in Mechanics, English Tr. By V. Kisin, Mir Publishers, USSR, 1982.
- [7] Langhaar H.L. Dimensional Analysis and Theory of Models, Wiley, 1951.
- [8] Weyl H. The Classical Groups, Their Invariants and Representations, 2 Eds., with suppl., Princeton, Princeton University Press, 1946.
- [9] Aripov R.G., Khadzhiev D. The complete system of differential and integral invariants of a curve in Euclidean geometry, Russian Math., 51:7 1-14, 2007.
- [10] Khadjev D. An Application of the Invariant Theory to the Differential Geometry of Curves, Fan, Tashkent, 1988.
- [11] Sagirolgu Y. The equivalence problem for parametric curves in one-dimensional affine space, Int. Math. Forum, 6 177-184, 2011.
- [12] Sagirolgu Y. Equi-affine differential invariants of a pair of curves, TWMS J. Pure Appl. Math., 6 238-245, 2015.
- [13] Sagirolgu Y., Peksen O. The equivalence Of Centro-Equiaffine curves, Turk. J. Math., 34 95-104, 2010.
- [14] Oren I. Complete system of invariants of subspaces of Lorentzian space, Iran. J. Sci. Technol. A., 40 1-8, 2016.
- [15] Oren I. On invariants of m-vectors in Lorentzian geometry, Int. Electron. J. Geom., 9 38-44, 2016.
- [16] Khadjev D., Oren I., Peksen O. Generating systems of differential invariants and the theorem on existence for curves in the pseudo-Euclidean geometry, Turk. J. Math., 37 80-94, 2013.

- [17] İncesu M., Gursoy O. LS (2)-Equivalence conditions of control points and application to planar Bézier curves, *New Trends Math. Sci.*, 5:3 70-84, 2017.
- [18] İncesu M., Gursoy O. Düzlemsel Bézier Eğrilerinin S(2)-Denklik Sartları, *MSU. J. Sci.*, 5:2 471-477, 2017.
- [19] İncesu M. The Similarity Orbits in R^3 , *Math: Modelling Application Theory*, 2:1 28-37, 2017.
- [20] İncesu M., Gürsoy O. $G=S(1)$, $G=S(2)$ ve alt Grubları için G-Yörüngeler, *MSU. J. Sci.*, 6:2 595-602, 2018.
- [21] Ören I., İncesu M. Detecting Similarities of Bézier Curves for the Groups $LSim(E2)$, $LSim+(E2)$, *Conference Proceedings of Science and Technology*, 2:2 129-133, 2019.
- [22] İncesu M. LS (3)-equivalence conditions of control points and application to spatial Bézier curves and surfaces. *AIMS Mathematics*, 5:2 1216-1246, 2020.
- [23] İncesu M. R^3 de k- Vektör İçin $R(x_1, x_2, \dots, x_k)^S(3)$ Cisminin Üreteçleri. *International Journal of Advances in Engineering and Pure Sciences.*, 32:3 239-250, 2020.
- [24] Farin G. Curvature continuity and offsets for piecewise conics, *ACM T. Graphic*, 8 89-99, 1989.
- [25] Farouki R. Exact offsets procedures for simple solids, *Comput. Aided. Geom. D.*, 2 257-279, 1985.
- [26] Farouki R., Rajan V.T. On the numerical condition of polynomials in Bernstein form, *Comput. Aided Geom. D.*, 4 191-216, 1987.
- [27] Hoschek J. Offset curves in the plane, *Comput. Aided. Des.*, 17 77-82, 1985.
- [28] Tiller W., Hanson E. Offsets of two-dimensional profiles, *IEEE Comput. Graph.*, 4 36-46, 1984.
- [29] Potmann H. Rational curves and surfaces with rational offsets, *Comput. Aided. Geom. D.*, 12 175-192, 1995.
- [30] İncesu M., Gursoy O. The similarity invariants of Bézier curves and surfaces, *Nat. Math. Symp.*, Ataturk University, Erzurum, Turkey, 2007.
- [31] Samancı H.K., Celik S., İncesu M. The Bishop Frame of Bézier Curves, *Life Sci. J.*, 12:6 175-180, 2015.
- [32] Bulut V., Caliskan A. The Exchange Variations Between Bézier Directrix Curves of Two Developable Ruled Surfaces, *J. Dynamical Sys. Geom. Theories*, 13 103-114, 2015.
- [33] Samancı H.K. Generalized dual-variable Bernstein polynomials, *Konuralp J. Math.*, 5 56-67, 2016.
- [34] İncesu M., Gursoy O. The similarity invariants of integral B-splines. *International scientific conference Algebraic and geometric methods of analysis*, May 31-June 5, Odesa, 2017.
- [35] Erkan E., Yuce S. Serret-Frenet frame and curvatures of Bézier curves, *Mathematics*, 6 321-351, 2018.
- [36] Samancı H.K. Some geometric properties of the spacelike Bézier curve with a timelike principal normal in Minkowski 3-space, *Cumhuriyet Sci. J.*, 39 71-79, 2018.
- [37] Samancı H.K., Kalkan O., Celik S. The timelike Bézier spline in Minkowski 3-space, *J. Sci. Arts*, 19 357-374, 2019.
- [38] Baydas S., Karakas B. Detecting a curve as a Bézier curve, *J. Taibah Univ. Sci.*, 13 522-528, 2019.
- [39] Marsh D. *Applied geometry for computer graphics and CAD*, Springer-Verlag London Berlin Heidelberg, London, 1999.