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DEFERRED NÖRLUND STATISTICAL RELATIVE UNIFORM CONVERGENCE AND KOROVKIN-TYPE APPROXIMATION THEOREM

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ABSTRACT. In this paper, we define the concept of statistical relative uniform convergence of the deferred Nörlund mean and we prove a general Korovkin-type approximation theorem by using this convergence method. As an application, we use classical Bernstein polynomials for defining an operator that satisfies our new approximation theorem but does not satisfy the theorem given before. Additionally, we estimate the rate of convergence of approximating positive linear operators by means of the modulus of continuity.

1. Introduction and Preliminaries

The notion of statistical convergence for the sequences of real numbers was introduced by Steinhaus [17] and Fast [11] independently in the same year. After that, the concept of statistical convergence in approximation theory has been used by Gadjiev and Orhan [12] to prove Korovkin-type approximation theorem. Recent studies on the statistical approximation may be found in the monograph by Anastassiou and Duman [3]. Later many researchers have investigated the Korovkin-type approximation theorems for various operators defined on different spaces, for example, the space of all continuous functions, modular space, the space of all Bögel continuous functions, etc. ([4–6, 8, 9, 13, 16, 19, 20]). In this paper, we define a new concept of statistical relative uniform convergence of the deferred Nörlund mean that improve the deferred Nörlund statistical uniform convergence. Then, we prove a Korovkin type approximation theorem by using this interesting convergence method and show its importance by giving an example. Finally, we study the rate of convergence.

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First, we begin with some definitions and notations.

The natural density of a set $B \subseteq \mathbb{N}$, the set of natural numbers, is defined by

$$\delta(B) := \lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : k \in B\}\right|,$$

provided that the limit on the right-hand side exists ([15]) where $|B|$ we mean the cardinality of the set B.

A sequence $x = (x_n)$ of real numbers is said to be statistically convergent to L, if for every $\varepsilon > 0$,

$$\delta\left(\{n \in \mathbb{N} : \ |x_n - L| \geq \varepsilon \ \}\right) = 0.$$

It is denoted by $st - \lim_{n\to\infty} x_n = L$.

Now, we remember that the idea of the weighted statistical convergence based upon the deferred Nörlund mean $D_a^b(N, p, q)$.

Let (a_n) and (b_n) be sequences of nonnegative integers and satisfy the following conditions:

(i) $a_n < b_n$,

(ii) $\lim_{n\to\infty} b_n = \infty$.

The above conditions (i) and (ii) are known as the regularity conditions for the deferred Nörlund mean (see [1]).

Assume that (p_n) and (q_n) are the sequences of nonnegative real numbers such that

$$P_n = \sum_{v=a_n+1}^{b_n} p_v \text{ and } Q_n = \sum_{v=a_n+1}^{b_n} q_v.$$

The convolution of the above sequences can be introduced as follows:

$$R_{a_n+1}^{b_n} = (P * Q)_n = \sum_{v=a_n+1}^{b_n} p_v q_{b_n - v}.$$

In order to define the deferred Nörlund mean $D_a^b(N, p, q)$, we first set

$$t_n = \frac{1}{R_{a_n+1}^{b_n}} \sum_{v=a_n+1}^{b_n} p_{b_n - v} q_v x_v.$$

Then we say that a sequence $x = (x_n)$ is deferred Nörlund statistically summable to L or, briefly, statistically summable $D_a^b(N, p, q)$, if

$$st - \lim_{n\to\infty} t_n = L.$$

We denote by S_{t_n} the set of all sequences that are $D_a^b(N, p, q)$ statistically summable.

A sequence $x = (x_n)$ is said to be statistically convergent to L with respect to the deferred Nörlund mean $D_a^b(N, p, q)$ if, for each $\varepsilon > 0$, the following set:

$$\left\{m \leq R_{a_n+1}^{b_n} : p_{b_n - m} q_m \left|x_m - L\right| \geq \varepsilon\right\}$$

has zero deferred Nörlund density, that is,

$$\lim_{n\to\infty}\frac{1}{R_{a_n+1}^{b_n}}\left|\left\{m\leq R_{a_n+1}^{b_n}:p_{b_n-m}q_m\left|x_m-L\right|\geq\varepsilon\right\}\right|=0.$$

In this case, we write $S_{t_n}-\lim x_n=L$.

Let us turn our attention to the definitions deferred Nörlund statistical pointwise and uniform convergence were given in [18]:

Let f and f_n, for $\forall n\in\mathbb{N}$, belong to $C(X)$. We denote by $C(X)$ the space of all continuous functions on X, which is a compact subset of $\mathbb{R}$. This space is equipped with the supremum norm

$$\|f\|=\sup_{z\in X}\left|f(z)\right|,\ \ (f\in C(X)).$$

Definition 1. *([18])(f_n) is said to be deferred Nörlund statistically pointwise convergent to f on X, (i.e., t_n-statistically pointwise convergent) if for every $\varepsilon>0$ and for each $z\in X$,*

$$\lim_{n\to\infty}\frac{\Omega_n(z,\varepsilon)}{R_{a_n+1}^{b_n}}=0$$

where $\Omega_n(z,\varepsilon):=\left|\left\{m\leq R_{a_n+1}^{b_n}:\ p_{b_n-m}q_m\left|f_m(z)-f(z)\right|\geq\varepsilon\right\}\right|$. In this case we write $f_n\to f(t_n\text{-stat-pointwise})$ on X.

In the following, we rewrite the definition of deferred Nörlund statistical uniform convergence given in [18] and we extend this definition to the deferred Nörlund statistical relative uniform convergence:

Definition 2. *([18])(f_n) is said to be deferred Nörlund statistically uniformly convergent to f on X, (i.e., t_n-statistically uniform convergent) if for every $\varepsilon>0$,*

$$\lim_{n\to\infty}\frac{\Phi_n(\varepsilon)}{R_{a_n+1}^{b_n}}=0,$$

where $\Phi_n(\varepsilon):=\left|\left\{m\leq R_{a_n+1}^{b_n}:\ p_{b_n-m}q_m\sup_{z\in X}\left|f_m(z)-f(z)\right|\geq\varepsilon\right\}\right|$. In this case we write $f_n\rightrightarrows f$ (t_n-stat-uniform) on X.

Let us remind the concept of statistical relative uniform convergence. According to Demirci and Orhan [7], this method is defined as follows:

Definition 3. *[7] (f_n) is said to be statistically relatively uniformly convergent to f on X if there exists a scale function $\sigma(z)$, $|\sigma(z)|>0$, such that for every $\varepsilon>0$,*

$$\lim_{n\to\infty}\frac{K_n(\varepsilon)}{n}=0,$$

where $K_n(\varepsilon):=\left|\left\{m\leq n:\ \sup_{z\in X}\left|\frac{f_m(z)-f(z)}{\sigma(z)}\right|\geq\varepsilon\right\}\right|$. In this case we write $f_m\rightrightarrows f$ $(\sigma;st)$ on X.

Actually, the above definitions come from the idea by Duman and Orhan [10] and Karakuş et al. [13]. For the purposes of this paper, we now give our new convergence method with the help of Definition 2 and Definition 3.

Definition 4. *(f_n) is said to be deferred Nörlund statistically relatively uniformly convergent to f on X , (i.e., t_n-statistically relatively uniformly convergent) if there exists a scale function $\sigma(z)$, $|\sigma(z)| > 0$, such that for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{\Phi_n(\varepsilon)}{R_{a_n+1}^{b_n}} = 0,$$

where $\Phi_n(\varepsilon) := \left|\left\{m \leq R_{a_n+1}^{b_n} :\ p_{b_n-m} q_m \sup_{z\in X} \left|\frac{f_m(z)-f(z)}{\sigma(z)}\right| \geq \varepsilon\right\}\right|$. In this case we write $f_n \rightrightarrows f$ $(\sigma; t_n$-stat-uniform$)$ on X.

Here it is important to say that, t_n-statistical uniform convergence is the special case of t_n-statistical relative uniform convergence in which the scale function is a non-zero constant. If the scale function is bounded, then t_n-statistical relative uniform convergence implies t_n-statistical relative uniform convergence. But, t_n-statistical relative uniform convergence does not imply t_n-statistical uniform convergence, when $\sigma(z)$ is unbounded.

Now, we present the following example showing the importance of our new convergence method.

Example 5. *For $p_n = 1$, $q_n = \frac{n+2}{n+1}$, $a_n = 2n$ and $b_n = 4n$, let us define the sequence of continuous real-valued functions $g_n : [0,1] \to \mathbb{R}$ by*

$$g_n(z) = \begin{cases} \frac{2nz}{1+n^2z^2}, & z \in \left[0, \frac{1}{n}\right], \\ 0, & z \in \left(\frac{1}{n}, 1\right], \end{cases}$$

and $g(z) = 0$ on $[0,1]$. Since $\|g_n - g\| = \sup_{z\in X} |g_n(z) - g(z)| = 1$, then $p_{b_n-m} q_m \sup_{z\in X} |g_m(z) - g(z)| = \frac{m+2}{m+1}$. Hence, for $0 < \varepsilon \leq 1$, we obtain

$$\begin{aligned} & \lim_{n\to\infty} \frac{1}{R_{a_n+1}^{b_n}} \left|\left\{m \leq R_{a_n+1}^{b_n} :\ p_{b_n-m} q_m \sup_{z\in X} |g_m(z) - g(z)| \geq \varepsilon\right\}\right| \\ = & \lim_{n\to\infty} \frac{1}{R_{a_n+1}^{b_n}} \left|\left\{m \leq R_{a_n+1}^{b_n} :\ \frac{m+2}{m+1} \geq \varepsilon\right\}\right| \\ = & \ 1 \end{aligned}$$

Therefore, (g_n) is not t_n-statistically uniformly convergent. Now we get the scale function as follows:

$$\sigma(z) = \begin{cases} \frac{1}{z}, & z \in (0,1], \\ 1, & z = 0. \end{cases}$$

As can be seen from, $\left\|\frac{g_m-g}{\sigma}\right\|=\frac{1}{m}$, *it is clear that*

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{R^{b_n}_{a_n+1}}\left|\left\{m\le R^{b_n}_{a_n+1}:\ p_{b_n-m}q_m\left\|\frac{g_m-g}{\sigma}\right\|\ge\varepsilon\right\}\right|\\ =\;&\lim_{n\to\infty}\frac{1}{R^{b_n}_{a_n+1}}\left|\left\{m\le R^{b_n}_{a_n+1}:\frac{m+2}{m+1}.\frac{1}{m}\ge\varepsilon\right\}\right|=0,\end{aligned}$$

i.e.,

$$g_n\rightrightarrows g=0(\sigma;t_n\text{-}stat\text{-}uniform)\ \ on\ \ [0,1]\,.$$

2. Korovkin-type approximation theorem

Let (L_n) be a sequence of positive linear operators on $C\,(X)$. The study of uniform convergence of $(L_n\,(f))$ to a function f by using the test functions defined by 1, z, z^2, was initiated by Korovkin [14] (see, for instance, [2]). Recently, the Korovkin-type theorem has been proved via the concept of statistical convergence in [12] and more recently, the statistical relative Korovkin type approximation theorem has been proved in [7]. In this section, we give the Korovkin-type theorem for sequences of positive linear operators defined on $C\,(X)$ using the concept of deferred Nörlund statistical relative uniform convergence.

Theorem 6. *Let* (L_n) *be a sequence of positive linear operators acting* $C\,(X)$ *into* $C\,(X)$. *Then, we have, for all* $f\in C\,(X)$,

$$L_n\,(f)\rightrightarrows f(\sigma;t_n\text{-}stat\text{-}uniform)\ \ on\ X, \tag{1}$$

if and only if

$$L_n\,(f_i)\rightrightarrows f_i(\sigma_i;t_n\text{-}stat\text{-}uniform)\ \ on\ X,\ \ i=0,1,2, \tag{2}$$

where $|\sigma_i(z)|>0,\ i=0,1,2,\ \sigma(z):=\max\{\sigma_i(z):i=0,1,2\}$ *and* $f_i(z)=z^i, i=0,1,2.$

Proof. Since each of the functions given by $f_i\,(z)=z^i$, $(i=0,1,2)$ belong to $C(X)$, the implication (1)⇒(2) is quite obvious. Now, we assume that (2) holds. Let f belongs to $C(X)$ and $z\in X$ fixed. Then there exists a constant $\kappa>0$ such that,

$$|f(z)|\le\kappa$$

which ensure that

$$|f(u)-f(z)|\le 2\kappa.$$

From the continuity of f, for a given $\varepsilon>0$, there exists $\delta>0$ such that

$$|f(u)-f(z)|<\varepsilon \text{ whenever } |u-z|<\delta \tag{3}$$

for every $u\in X$. Now let $\phi\,(u,z)=(u-z)^2$. If $|u-z|\ge\delta$, $u\in X$, we get

$$|f(u)-f(z)|<\frac{2\kappa}{\delta^2}\phi\,(u,z)\,. \tag{4}$$

From (3) and (4), we can see that

$$|f(u)-f(z)|<\varepsilon+\frac{2\kappa}{\delta^{2}}\phi\left(u,z\right),$$

i.e.,

$$-\varepsilon-\frac{2\kappa}{\delta^{2}}\phi\left(u,z\right)\leq f(u)-f(z)\leq\varepsilon+\frac{2\kappa}{\delta^{2}}\phi\left(u,z\right). \tag{5}$$

Since the positive linear operator $L_n(1;z)$ is monotone, by applying this operator to the inequality in (5), we have

$$\begin{aligned} L_n(1;z)\left(-\varepsilon-\frac{2\kappa}{\delta^{2}}\phi\left(u,z\right)\right) &\leq L_n(1;z)\left(f(u)-f(z)\right) \\ &\leq L_n(1;z)\left(\varepsilon+\frac{2\kappa}{\delta^{2}}\phi\left(u,z\right)\right). \end{aligned} \tag{6}$$

Then, we obtain the following inequality:

$$\begin{aligned} -\varepsilon L_n(1;z)-\frac{2\kappa}{\delta^{2}}L_n\left(\phi\left(u,z\right);z\right) &\leq L_n(f;z)-f(z)L_n(1;z) \\ &\leq \varepsilon L_n(1;z)+\frac{2\kappa}{\delta^{2}}L_n\left(\phi\left(u,z\right);z\right). \end{aligned} \tag{7}$$

Since

$$L_n(f;z)-f(z)=[L_n(f;z)-f(z)L_n(1;z)]+f(z)[L_n(1;z)-1], \tag{8}$$

we apply the equality (8) in (7), it can be easily seen that,

$$L_n(f;z)-f(z)\leq\varepsilon L_n(1;z)+\frac{2\kappa}{\delta^{2}}L_n\left(\phi\left(u,z\right);z\right)+f(z)[L_n(1;z)-1].$$

Now we calculate the term of $L_n\left(\phi\left(u,z\right);z\right)$. We can write the following:

$$\begin{aligned} L_n\left(\phi\left(u,z\right);z\right) &= L_n\left((u-z)^{2};y\right) \\ &= L_n\left(u^{2}-2zu+z^{2};z\right) \\ &= L_n\left(u^{2};z\right)-2zL_n\left(u;z\right)+z^{2}L_n(1;z) \\ &= \left[L_n\left(u^{2};z\right)-z^{2}\right]-2z\left[L_n\left(u;z\right)-z\right]+z^{2}\left[L_n(1;z)-1\right]. \end{aligned} \tag{9}$$

By using (9), we have

$$\begin{aligned} L_n(f;z)-f(z) &\leq \varepsilon L_n(1;z)+f(z)[L_n(1;z)-1] \\ &\quad +\frac{2\kappa}{\delta^{2}}\left\{\left[L_n\left(u^{2};z\right)-z^{2}\right]-2z\left[L_n\left(u;z\right)-z\right]+z^{2}\left[L_n(1;z)-1\right]\right\} \\ &= \varepsilon+\varepsilon\left[L_n(1;z)-1\right]+f(z)[L_n(1;z)-1] \\ &\quad +\frac{2\kappa}{\delta^{2}}\left\{\left[L_n\left(u^{2};z\right)-z\right]-2z\left[L_n\left(u;z\right)-z\right]+z^{2}\left[L_n(1;z)-1\right]\right\}. \end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we can write

$$|L_n(f;z)-f(z)|\leq F\left\{|L_n(1;z)-1|+|L_n\left(u;z\right)-z|+\left|L_n\left(u^{2};z\right)-z^{2}\right|\right\}$$

where $\digamma := \max\left\{\varepsilon + \kappa + \frac{2\kappa}{\delta^2}\left\|f_2\right\|, \frac{4\kappa}{\delta^2}\left\|f_1\right\|, \frac{2\kappa}{\delta^2}\right\}$. Let $\sigma(z) := \max\left\{\sigma_i(z) : i = 0, 1, 2\right\}$ and $|\sigma_i(z)| > 0 : i = 0, 1, 2$. Then, we get

$$\begin{aligned}\left|\frac{L_n(f; z) - f(z)}{\sigma(z)}\right| &\leq \digamma\left\{\left|\frac{L_n(f_0; z) - f_0(z)}{\sigma_0(z)}\right| + \left|\frac{L_n(f_1; z) - f_1(z)}{\sigma_1(z)}\right|\right. \\ &\quad \left. + \left|\frac{L_n(f_2; z) - f_2(z)}{\sigma_2(z)}\right|\right\}\end{aligned}$$

Taking supremum on X, we get

$$\left\|\frac{L_n(f) - f}{\sigma}\right\| \leq \digamma\left\{\left\|\frac{L_n(f_0) - f_0}{\sigma_0}\right\| + \left\|\frac{L_n(f_1) - f_1}{\sigma_1}\right\| + \left\|\frac{L_n(f_2) - f_2}{\sigma_2}\right\|\right\}.$$

Now, for a given $r > 0$, define the following sets:

$$\Phi_n(r) := \left|\left\{m \leq R_{a_n+1}^{b_n} : \ p_{b_n-m}q_m\left\|\frac{L_m(f) - f}{\sigma}\right\| \geq r\right\}\right|$$

and

$$\Phi_{n,i}\left(\frac{r}{3\digamma}\right) := \left|\left\{m \leq R_{a_n+1}^{b_n} : p_{b_n-m}q_m\left\|\frac{L_m(f_i) - f_i}{\sigma_i}\right\| \geq \frac{r}{3\digamma}\right\}\right|, \ (i = 0, 1, 2). \quad (10)$$

It is easy to see that $\frac{\Phi_n(r)}{R_{a_n+1}^{b_n}} \leq \sum_{i=0}^{2}\frac{\Phi_{n,j}\left(\frac{r}{3\digamma}\right)}{R_{a_n+1}^{b_n}}$. Then using the hypothesis (2) and considering Definition 3, the right hand side of (10) tend to zero as $n \to \infty$. Hence, the proof is completed. □

Example 7. *Let $X = [0, 1]$. For $p_n = 1$, $q_n = \frac{n+2}{n+1}$, $a_n = 2n$ and $b_n = 4n$, we first consider the Bernstein polynomials:*

$$B_n(f; z) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}z^k(1-z)^{n-k}, \ f \in C[0, 1].$$

It is known that

$$\begin{aligned}B_n(f_i; z) &= f_i(z), \ i = 0, 1 \\ B_n(f_2; z) &= f_2(z) + \frac{z(1-z)}{n}.\end{aligned}$$

Using this polynomial, we define a sequence of positive linear operators $D_n : C[0, 1] \to C[0, 1]$ as follows:

$$D_n(f; z) = (1 + g_n(z))B_n(f; z), \ z \in [0, 1] \text{ and } f \in C[0, 1]. \quad (11)$$

If we choose the sequence $(g_n(z))$ of functions as we considered in Example 5, then we can see that

$$\begin{aligned}D_n(f_0; z) &= (1 + g_n(z))f_0(z), \\ D_n(f_1; z) &= (1 + g_n(z))f_1(z), \\ D_n(f_2; z)) &= (1 + g_n(z))\left[f_2(z) + \frac{z(1-z)}{n}\right].\end{aligned}$$

Now, we consider the scale function as follows:

$$\sigma(z) = \begin{cases} \frac{1}{z}, & z \in (0,1], \\ 1, & z = 0. \end{cases}$$

Hence, after some simple calculations, it can be easily seen that,

$$\begin{aligned} \left\| \frac{D_n(f_0) - f_0}{\sigma} \right\| &= \left\| \frac{g_n}{\sigma} \right\| \\ \left\| \frac{D_n(f_1) - f_1}{\sigma} \right\| &\leq \left\| \frac{g_n}{\sigma} \right\| \\ \left\| \frac{D_n(f_2) - f_2}{\sigma} \right\| &= \sup_{z \in [0,1]} \left| \frac{(1 + g_n(z)) \left[f_2(z) + \frac{z(1-z)}{n} \right] - f_2(z)}{\sigma(z)} \right| \\ &= \sup_{z \in [0,1]} \left| \frac{\frac{z(1-z)}{n} + g_n(z) f_2(z) + \frac{z(1-z)}{n} g_n(z)}{\sigma(z)} \right| \\ &\leq \left\| \frac{g_n}{\sigma} \right\| + \frac{1}{4n} \left\| \frac{1}{\sigma} \right\| + \frac{1}{4n} \left\| \frac{g_n}{\sigma} \right\| \\ &\leq 2 \left\| \frac{g_n}{\sigma} \right\| + \frac{1}{n}. \end{aligned}$$

Since

$$g_n \rightrightarrows g = 0(\sigma; t_n\text{-stat-uniform}) \text{ on } [0,1] \text{ and } \frac{1}{n} \to 0,$$

we conclude that

$$D_n(f_i) \rightrightarrows f_i(\sigma; t_n\text{-stat-uniform}) \text{ on } X, \; i = 0, 1, 2.$$

Then, by our main theorem, Theorem 6, we have

$$D_n(f) \rightrightarrows f(\sigma; t_n\text{-stat-uniform}) \text{ on } X.$$

Furthermore, since the sequence (g_n) *of functions on* $[0,1]$ *is not* t_n*-statistically uniformly convergent to the function* $g = 0$ *on the interval* $[0,1]$, *we can say that Korovkin-type approximation theorem given via deferred Nörlund statistical uniform convergence does not hold for our operators defined by (11).*

3. Rate of convergence

In this section, we compute the rate of the deferred Nörlund statistical relative uniform convergence of a sequence of positive linear operators defined on $C(X)$ by means of the modulus of continuity.

Now we recall the concept of modulus of continuity. For $f \in C(X)$, the modulus of continuity of f is defined by

$$\omega(f; \delta) = \sup_{|u-z| \leq \delta, \; u,z \in X} |f(u) - f(z)|.$$

It is also well known that for any $\delta > 0$ and each $u, z \in X$

$$|f(u) - f(z)| \leq \omega(f;\delta)\left(\frac{|u-z|}{\delta} + 1\right).$$

Now, we state and prove the following theorem.

Theorem 8. *Let (L_n) be a sequence of positive linear operators acting $C(X)$ into $C(X)$. Assume that the following conditions hold:*
(i) $L_n(f_0) \rightrightarrows f_0(\sigma_0; t_n$-stat-uniform) on X,
(ii) $\omega(f, \alpha_n) \rightrightarrows 0(\sigma_1; t_n$-stat-uniform) on X,
where $\alpha_n(y) = \sqrt{\|L_n(\varphi(.,z))\|}$ with $\varphi(u,z) = (u-z)^2$, $|\sigma_i(z)| > 0$, $i = 0,1$ and $\sigma(z) := \max\{\sigma_i(z) : i = 0,1\}$.
Then we have, for all $f \in C(X)$,

$$L_n(f) \rightrightarrows f(\sigma; t_n\text{-stat-uniform}) \text{ on } X.$$

Proof. Let $f \in C(X)$ and $z \in X$. It is known that ([2]),

$$\begin{aligned}
|L_n(f;z) - f(z)| &\leq L_n(|f(u) - f(z)|;z) + \kappa\,|L_n(f_0;z) - f_0(z)| \\
&\leq L_n(\omega(f;\delta)\left(\frac{|u-z|}{\delta} + 1\right);z) + \kappa\,|L_n(f_0;z) - f_0(z)| \\
&\leq L_n\left(\omega(f;\delta)\left(\frac{(u-z)^2}{\delta^2} + 1\right);z\right) + \kappa\,|L_n(f_0;z) - f_0(z)| \\
&= \kappa\,|L_n(f_0;z) - f_0(z)| + \frac{\omega(f;\delta)}{\delta^2} L_n\left((u-z)^2;z\right) \\
&\quad + \omega(f;\delta)\, L_n(f_0;z) \\
&\leq \kappa\,|L_n(f_0;z) - f_0(z)| + \frac{\omega(f;\delta)}{\delta^2} L_n\left((u-z)^2;z\right) \\
&\quad + \omega(f;\delta)\,|L_n(f_0;z) - f_0(z)| + \omega(f;\delta)
\end{aligned}$$

where $\kappa := \|f\|$ and $\varphi(u,z) = (u-z)^2$. Then, taking supremum on X, we obtain

$$\begin{aligned}
\left\|\frac{L_n(f) - f}{\sigma}\right\| &\leq \kappa\left\|\frac{L_n(f_0) - f_0}{\sigma_0}\right\| + 2\left\|\frac{\omega(f,\alpha_n)}{\sigma_1}\right\| \\
&\quad + \left\|\frac{L_n(f_0) - f_0}{\sigma_0}\right\|\left\|\frac{\omega(f,\alpha_n)}{\sigma_1}\right\|
\end{aligned} \tag{12}$$

where $\delta := \alpha_n(y) = \sqrt{\|L_n(\varphi(.,z))\|}$, $|\sigma_i(z)| > 0$, $i = 0,1$ and $\sigma(z) := \max\{\sigma_i(z) : i = 0,1\}$.

Now, for a given $r > 0$, define the following sets:

$$\Phi_n(r) := \left|\left\{m \leq R_{a_n+1}^{b_n} :\ p_{b_n-m}q_m\left\|\frac{L_m(f) - f}{\sigma}\right\| \geq r\right\}\right|$$

and

$$\Phi_{n,0}(\frac{r}{3}) \quad : \quad = \left|\left\{m \leq R_{a_n+1}^{b_n} : p_{b_n-m}q_m\left\|\frac{L_m(f_0) - f_0}{\sigma_0}\right\|\left\|\frac{\omega(f,\alpha_m)}{\sigma_1}\right\| \geq \frac{r}{3}\right\}\right|,$$

$$\Phi_{n,1}(\frac{r}{3\kappa}) \quad : \quad = \left| \left\{ m \leq R_{a_n+1}^{b_n} : p_{b_n-m} q_m \left\| \frac{L_m(f_0) - f_0}{\sigma_0} \right\| \geq \frac{r}{3\kappa} \right\} \right|,$$

$$\Phi_{n,2}(\frac{r}{6}) \quad : \quad = \left| \left\{ m \leq R_{a_n+1}^{b_n} : p_{b_n-m} q_m \left\| \frac{\omega(f, \alpha_m)}{\sigma_1} \right\| \geq \frac{r}{6} \right\} \right|,$$

Then, we can get

$$\frac{\Phi_n(r)}{R_{a_n+1}^{b_n}} \leq \frac{\Phi_{n,0}(\frac{r}{3})}{R_{a_n+1}^{b_n}} + \frac{\Phi_{n,1}(\frac{r}{3\kappa})}{R_{a_n+1}^{b_n}} + \frac{\Phi_{n,2}(\frac{r}{6})}{R_{a_n+1}^{b_n}}.$$

Now, we also define the following sets:

$$\Phi_{n,3}(\sqrt{\frac{r}{3}}) \quad : \quad = \left| \left\{ m \leq R_{a_n+1}^{b_n} : p_{b_n-m} q_m \left\| \frac{\omega(f, \alpha_m)}{\sigma_1} \right\| \geq \sqrt{\frac{r}{3}} \right\} \right|,$$

$$\Phi_{n,4}(\sqrt{\frac{r}{3}}) \quad : \quad = \left| \left\{ m \leq R_{a_n+1}^{b_n} : p_{b_n-m} q_m \left\| \frac{L_m(f_0) - f_0}{\sigma_0} \right\| \geq \sqrt{\frac{r}{3}} \right\} \right|.$$

We can easily see that

$$\frac{\Phi_{n,0}(\frac{r}{3})}{R_{a_n+1}^{b_n}} \leq \frac{\Phi_{n,3}(\sqrt{\frac{r}{3}})}{R_{a_n+1}^{b_n}} + \frac{\Phi_{n,4}(\sqrt{\frac{r}{3}})}{R_{a_n+1}^{b_n}},$$

which gives

$$\frac{\Phi_n(r)}{R_{a_n+1}^{b_n}} \leq \frac{\Phi_{n,1}(\frac{r}{3\kappa})}{R_{a_n+1}^{b_n}} + \frac{\Phi_{n,2}(\frac{r}{6})}{R_{a_n+1}^{b_n}} + \frac{\Phi_{n,3}(\sqrt{\frac{r}{3}})}{R_{a_n+1}^{b_n}} + \frac{\Phi_{n,4}(\sqrt{\frac{r}{3}})}{R_{a_n+1}^{b_n}}. \tag{13}$$

Then the hypothesis (i) and (ii) leads us to the conclusion that the right hand side of (13) tend to zero as $n \to \infty$. The proof is completed. □

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Authors Contribution Statement All authors have contributed sufficiently in the planning, execution, or analysis of this study to be included as authors. All authors read and approved the final manuscript.

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