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ON THE DUS-KUMARASWAMY DISTRIBUTION

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Abstract: Kumaraswamy distribution is introduced by [7] and it is particularly useful for many natural phenomena whose outcomes have lower and upper bounds or bounded outcomes in biomedical and epidemiological research (see [12]). In this paper, a new statistical distribution called DUS-Kumaraswamy is introduced by using DUS transformation (which is recently introduced by [6]) on Kumaraswamy distribution. The proposed distribution has the same domain as Kumaraswamy and it can be used as an alternative model to describe the natural phenomena mentioned above. Several distributional properties such as mean, variance, skewness, kurtosis, Lorenz and Bonferroni curves are studied. The statistical inference on the parameters of Dus-Kumaraswamy is discussed by maximum likelihood methodology. A simulation study is conducted to observe the behaviors of maximum likelihood estimates under different conditions. A numerical example is also presented.

Key words: Data analysis, Kumaraswamy distribution, maximum likelihood estimator, monte carlo simulation

1. Introduction

In this study, the DUS transformation of [7] is used to introduce a new distribution bounded within (0,1). The Kumaraswamy distribution is considered as a baseline distribution in their DUS transformation. Let $F(x)$ and $f(x)$ denote respectively the cumulative distribution function (cdf) and probability density function (pdf) of baseline distribution. Then the pdf and cdf of the DUS family are given, respectively, by

$$f_{DUS}(x) = \frac{1}{e-1} f(x) e^{F(x)},\ x \in D \tag{1.1}$$

and

$$F_{DUS}(x) = \frac{1}{e-1}\left(e^{F(x)} - 1\right), \tag{1.2}$$

where D is a domain of the baseline distribution with cdf F.

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The DUS transformation with a exponential cdf is considered by [7]. Using DUS transformation, the DUS-Lomax distribution is proposed by [4]. [5] introduced a new lifetime distribution called DUS-Weibull distribution by using same mechanism. Recently, [10] generalized the DUS transformation and they studied the exponential baseline in their generalized DUS transformation.

It was reported in these studies that the DUS transformation increases the distribution flexibility. It can also be said that the DUS transformation has become a center of attraction in recent years. In this paper, the DUS transformation is applied to the Kumaraswamy cdf to get new distribution. The paper is organized as follows. In Section 2, moments, hazard rate, survival and quantile functions are obtained. The maximum likelihood method is discussed in Section 3. In Section 4, a simulation study is also performed to observe the performance of the maximum likelihood estimate. A numerical example is given to illustrate the capability of the proposed distribution for modeling a real data in Section 5. In Section 6, concluding remarks are provided.

2. DUS-Kumaraswamy distribution

In this section, DUS transformation is applied to Kumaraswamy cdf. The pdf and cdf of the Kumaraswamy distribution are given, respectively, by

$$f_K(x) = \alpha\beta x^{\alpha-1}\left(1-x^{\alpha}\right)^{\beta-1}, \; 0<x<1, \tag{2.1}$$

and

$$F_K(x) = 1-\left(1-x^{\alpha}\right)^{\beta}, \tag{2.2}$$

where $\alpha, \beta > 0$ are the parameters.

Using Eqs. (2.1) and (2.2) in Eqs. (1.1) and (1.2), respectively, the pdf and cdf of the new distribution are given, respectively, by

$$f_{DUS-K}(x) = \tfrac{1}{e-1}\alpha\beta x^{\alpha-1}\left(1-x^{\alpha}\right)^{\beta-1} e^{\left(1-\left(1-x^{\alpha}\right)^{\beta}\right)}, \; 0<x<1 \tag{2.3}$$

and

$$F_{DUS-K}(x) = \frac{1}{e-1}\left(e^{\left(1-\left(1-x^{\alpha}\right)^{\beta}\right)}-1\right), \tag{2.4}$$

with parameters $\alpha > 0$ and $\beta > 0$. The random variable X with cdf (2.2) is said to have two-parameter DUS-Kumaraswamy distribution and it is denoted by $DUS-K(\alpha,\beta)$.

Fig. 1 presents the plots of the $DUS-K(\alpha,\beta)$ probability density function for some choices of α and β. From Fig. 1, it is concluded that the pdf of $DUS-K(\alpha,\beta)$ can be unimodal as well as have decreasing and increasing forms.

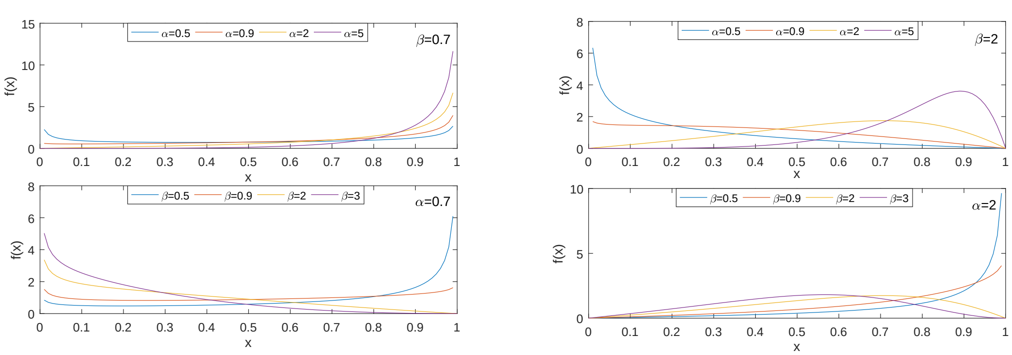

FIGURE 1. Pdf plots of the $DUS-K$ distribution for selected parameters values

The survival function, $S(x)$, and the hazard rate function, $h(x)$, for $DUS-K(\alpha,\beta)$ distribution are in the following forms:

$$S_{DUS-K(\alpha,\beta)}(x)=\frac{e-e^{\left(1-(1-x^{\alpha})^{\beta}\right)}}{e-1} \tag{2.5}$$

and

$$h_{DUS-K(\alpha,\beta)}(x)=\frac{\alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}e^{\left(1-(1-x^{\alpha})^{\beta}\right)}}{e-e^{\left(1-(1-x^{\alpha})^{\beta}\right)}}. \tag{2.6}$$

Fig. 2 presents plots of the hazard rate function of $DUS-K(\alpha,\beta)$ for some selected values of α and β. From Fig. 2, it is observed that the hazard function of introduced distribution is increasing.

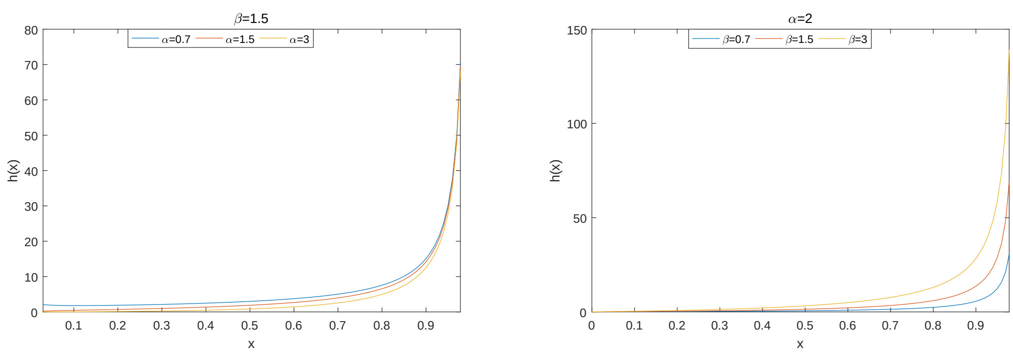

FIGURE 2. Hazard rate function plots of the $DUS-K$ distribution for selected parameters values

The quantile function of the $DUS-K(\alpha,\beta)$ distribution is given by

$$Q(u;\alpha,\beta)=\left\{1-\left[1-\log\left(1+u(e-1)\right)\right]^{\frac{1}{\beta}}\right\}^{\frac{1}{\alpha}},\ u\in(0,1). \tag{2.7}$$

Using Eq. (2.7), the median is obtained as

$$Q(0.5;\alpha,\beta)=\left\{1-\left[1+\log(2)-\log(1+e)\right]^{\frac{1}{\beta}}\right\}^{\frac{1}{\alpha}}.$$

Let X be an absolutely continuous random variable with distribution function F. Then, by using probability integral transformation, we can write

$$E(X_{i:n})\simeq F^{-1}\left(\frac{i}{n+1}\right),\ i=1,2,\ldots,m, \tag{2.8}$$

where $X_{i:n}$ is the ith order statistic of the sample of size n and F^{-1} is the inverse of F.

Let X be a random variable from DUS family with a baseline cdf F in (1.2). Using Taylor series expansion, the distribution function of X can be written as

$$\begin{aligned} G(x) &= \frac{1}{e-1}\left(e^{F(x)}-1\right) \\ &= \frac{1}{e-1}\sum_{i=0}^{\infty}\frac{(F(x))^{i}}{i!}-1 \\ &= \frac{1}{e-1}\sum_{i=0}^{\infty}\frac{F_{Y_{i:i}}(x)}{i!}-1, \end{aligned} \tag{2.9}$$

where $Y_{i:i}$ is the ith order statistic of sample of size i from $K(\alpha,\beta)$ with cdf $F_{Y_{i:i}}(x)$. Using (2.9), the expected value of $DUS-K(\alpha,\beta)$ can be represented by

$$E(X)=\frac{1}{e-1}\sum_{i=0}^{\infty}\frac{1}{i!}E(Y_{i:i}), \tag{2.10}$$

where $f_{i:i}(y)$ is pdf of $Y_{i:i}$. Using (2.8) in (2.10) and quantile function of Kumaraswamy distribution, the approximate expected value of $DUS-K(\alpha,\beta)$ is obtained as

$$E(X)\simeq\frac{1}{e-1}\sum_{i=0}^{\infty}\frac{1}{i!}\left(1-\left(\frac{i}{n+1}\right)^{1/\beta}\right)^{1/\alpha}. \tag{2.11}$$

Let X be the $DUS-K(\alpha,\beta)$ random variable with pdf (2.3). Then, for $r=1,2,...$, the approximate r-th moment of X is given by

$$\begin{aligned}\mu_r &= \int_0^1 x^r f(x)\,dx\\ &= \int_{-1}^1 \frac{1}{2}\left(\frac{y_\ell+1}{2}\right)^r f\left(\frac{y_\ell+1}{2}\right)dy\\ &\simeq \sum_{l=1}^{N}\varpi_\ell\frac{1}{2}\left(\frac{y_\ell+1}{2}\right)^r f\left(\frac{y_\ell+1}{2}\right),\end{aligned} \tag{2.12}$$

where $f(\cdot)$ is the pdf given in Eq. (2.3), y_ℓ and ϖ_ℓ are the zeros and the corresponding Christoffel numbers of the Legendre-Gauss quadrature formula on the interval $(-1,1)$, respectively, see [2]. It is also noticed here, ϖ_ℓ is given by

$$\varpi_\ell=\frac{2}{(1-y_\ell)^2\left[L'_{N+1}(y_\ell)\right]^2}, \tag{2.13}$$

where

$$L'_{N+1}(y_\ell)=\frac{dL_{N+1}(y)}{dy} \tag{2.14}$$

at $y=y_\ell$ and $L_{N+1}(\cdot)$ is the Legendre polynomial of degree N.

The relation between the approximation of the mean and degree (N) of Legendre polynomial is presented in Fig. 3. It can be observed that $N=30$ is enough to reach the acceptable approximation to the true mean.

For some selected parameters, the mean, variance, skewness and kurtosis of $DUS-K$ distribution are presented in Table 1. The values of N has been taken to be $N=30$ in the numerical calculations.

The Bonferroni and Lorenz curves are introduced by [1]. They have applications in economics and insurance. In the following, we give the Bonferroni and Lorenz curves of $DUS-K(\alpha,\beta)$ distribution.

Let the random variable X have $DUS-K(\alpha,\beta)$ distribution with pdf (2.3). Then, the Bonferroni and Lorenz curves are given, respectively, by

$$BC(\xi)=\frac{q^2}{4\xi\mu_1}\sum_{\ell=0}^{N}\varpi_\ell\left(y_\ell+1\right)f\left(\frac{q}{2}\left(y_\ell+1\right)\right), \tag{2.15}$$

and

$$LC(\xi)=\frac{q^2}{4\mu_1}\sum_{\ell=0}^{N}\varpi_\ell\left(y_\ell+1\right)f\left(\frac{q}{2}\left(y_\ell+1\right)\right), \tag{2.16}$$

where ϖ_ℓ is given by (2.13) and $q=F^{-1}(\xi)$.

In the following, we compute the well-known stress-strength reliability $R=P(Y<X)$ for the model under concern.

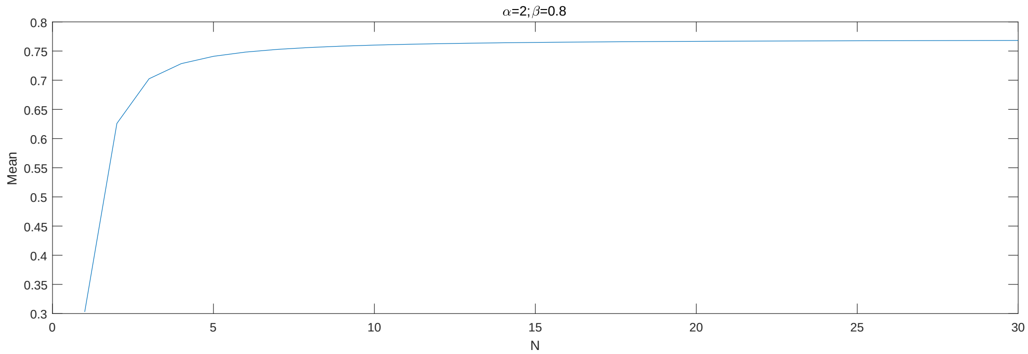

FIGURE 3. The relationship between approximated mean and degree (N) of Legendre polynomial

TABLE 1. The mean, variance, skewness and kurtosis of $DUS-K(\alpha,\beta)$ distribution for different values of α and β

α	β	Mean	Variance	Skewness	Kurtosis
0.5	0.5	0.6251	0.1134	-0.5366	1.8555
	1.5	0.2951	0.0680	0.7450	2.4860
	3	0.1341	0.0237	1.6482	5.6597
1.5	0.5	0.7948	0.0592	-1.4293	4.2872
	1.5	0.5822	0.0548	-0.3473	2.1865
	3	0.4268	0.0432	0.1011	2.2450
3	0.5	0.8698	0.0341	-2.4082	9.7298
	1.5	0.7411	0.0328	-0.9021	3.3317
	3	0.6294	0.0307	-0.5212	2.7864

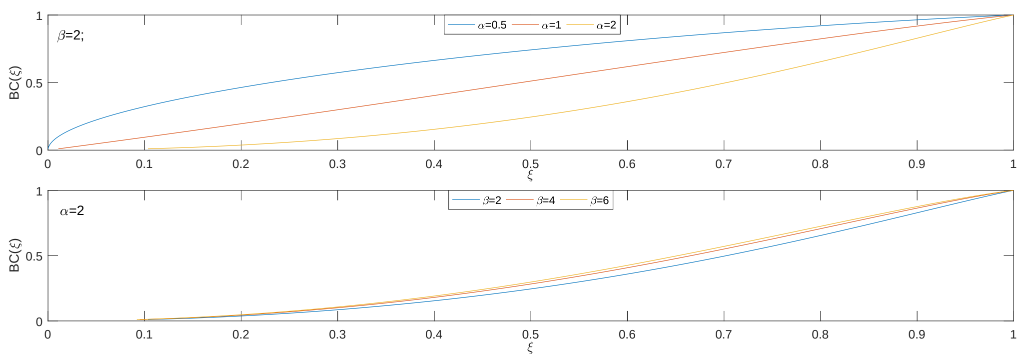

FIGURE 4. The Bonferroni curves of $DUS-K(\alpha,\beta)$ distribution

PROPOSITION 1. *Let Y and X be independent stress and strength random variables that follow Dus-K distribution with parameters (α,β_1) and (α,β_2), respectively. Then, the stress–strength reliability R is*

$$R = \frac{1}{(e-1)^2}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\frac{(-1)^{j+i}}{n_1!n_2!} \times \binom{n_1}{i}\binom{n_2}{j}\frac{\beta_1}{i\beta_1+j\beta_2+\beta_1}-\frac{1}{e-1},$$

where β_1 and β_2 are positive integers.

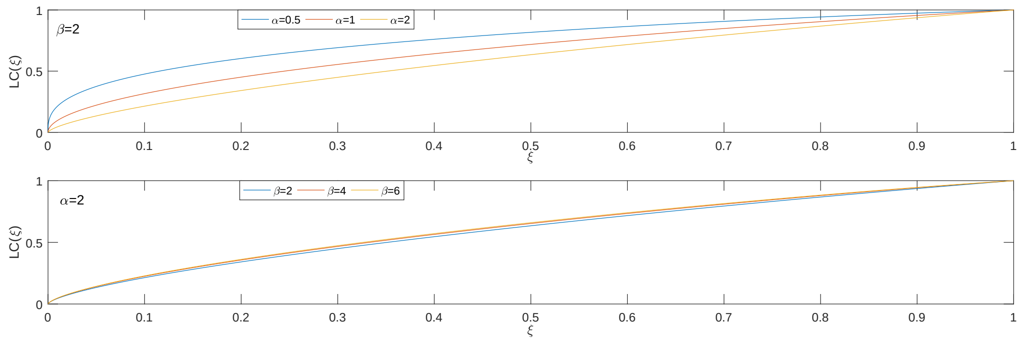

FIGURE 5. The Lorenz curves of $DUS-K(\alpha,\beta)$ distribution

PROOF. The stress-strength reliability can be written as

$$
\begin{aligned}
R &= \int_0^1 P\left(Y<X \mid X=x\right) f_X\left(x\right) dx \\
&= \int_0^1 F_Y\left(x\right) f_X\left(x\right) dx \\
&= \frac{1}{\left(e-1\right)^2} \int_0^1 \left(\exp\left(1-\left(1-x^{\alpha}\right)^{\beta_1}\right)-1\right) \\
&\quad \times \alpha\beta_2 x^{\alpha-1}\left(1-x^{\alpha}\right)^{\beta_2-1} \exp\left(1-\left(1-x^{\alpha}\right)^{\beta_2}\right) dx.
\end{aligned} \tag{2.17}
$$

Hence the proof follows by using Taylor and binomial expansions on terms in (2.17).
It is said that X is smaller than Y according to likelihood ratio ordering if

$$\frac{f_X\left(x\right)}{f_Y\left(x\right)} \text{ is nondecreasing in } x,$$

where $f_X\left(\cdot\right)$ and $f_Y\left(\cdot\right)$ are the pdfs of X and Y random variables, respectively. We write $X \leq_{lr} Y$ to represent that the random variable X is smaller than Y in the likelihood ratio ordering. The following proposition gives likelihood ratio order properties for the random variables with Dus-K distribution.

PROPOSITION 2. *Let* $X \sim$*DUS-K*(α,β_1) *and* $Y \sim$*DUS-K*(α,β_2)*. If* $\beta_1>\beta_2$ *then* $X \leq_{lr} Y$*.*

PROOF. For any $x \in (0,1)$ the ratio of the densities of X and Y is given by

$$g\left(x\right)=\frac{\beta_1\left(1-x^{\alpha}\right)^{\beta_1-1}\exp\left(1-\left(1-x^{\alpha}\right)^{\beta_1}\right)}{\beta_2\left(1-x^{\alpha}\right)^{\beta_2-1}\exp\left(1-\left(1-x^{\alpha}\right)^{\beta_2}\right)}.$$

$X \leq_{lr} Y$ is equivalent to $g\left(x\right)$ is decreasing in x. Let us consider

$$\frac{d\log\left(g\left(x\right)\right)}{dx}=r\left(x\right)h\left(x\right),$$

where

$$r(x) = \frac{\alpha x^{\alpha}}{x(1-x^{\alpha})}$$

and

$$h(x) = \beta_1 \left[(1-x^{\alpha})^{\beta_1} - 1 \right] - \beta_2 \left[(1-x^{\alpha})^{\beta_2} - 1 \right].$$

It is pointed out that we can easily write $r(x) > 0$ for all $\alpha > 0$ and $x \in (0,1)$. It is also clearly that $(1-x^{\alpha})^{\beta}$ is decreasing function in β for $x \in (0,1)$. Then we can write

$$\begin{aligned}
(\beta_1 > \beta_2) &\Longrightarrow (1-x^{\alpha})^{\beta_2} > (1-x^{\alpha})^{\beta_1} \\
&\Longrightarrow (1-x^{\alpha})^{\beta_2} - 1 > (1-x^{\alpha})^{\beta_1} - 1 \\
&\Longrightarrow \beta_2 \left[(1-x^{\alpha})^{\beta_2} - 1 \right] > \beta_1 \left[(1-x^{\alpha})^{\beta_1} - 1 \right] \\
&\Longrightarrow \beta_1 \left[(1-x^{\alpha})^{\beta_1} - 1 \right] - \beta_2 \left[(1-x^{\alpha})^{\beta_2} - 1 \right] < 0 \\
&\Longrightarrow h(x) < 0
\end{aligned}$$

Therefore, we have

$$\frac{d \log(g(x))}{dx} = \underbrace{r(x)}_{>0} \underbrace{h(x)}_{<0} < 0$$

for $\beta_1 > \beta_2$. The last inequality shows that $g(x)$ is decreasing in x and it implies $X \leq_{lr} Y$ for $\beta_1 > \beta_2$.

Corollary 1. *It follows from [11] that X is also smaller than Y in the hazard rate, mean residual life and stochastic orders under the conditions given in Proposition 2.*

3. Maximum likelihood estimation

Let $X_1, X_2, \ldots, X_n$ be the i.i.d sample from $DUS-K(a,\beta)$, then the likelihood and log-likelihood functions can be written as

$$L(\alpha,\beta) = \prod_{i=1}^{n} \left(\frac{1}{e-1} \alpha \beta x_i^{\alpha-1} (1-x_i^{\alpha})^{\beta-1} e^{\left(1-(1-x_i^{\alpha})^{\beta}\right)} \right) \tag{3.1}$$

and

$$\begin{aligned}
\ell(\alpha,\beta) = & -n\log(e-1) + n\log(\alpha) + n\log(\beta) + (\alpha-1)\sum_{i=1}^{n} \log(x_i) \\
& + (\beta-1)\sum_{i=1}^{n} \log(1-x_i^{\alpha}) + \sum_{i=1}^{n} \left(1-(1-x_i^{\alpha})^{\beta}\right),
\end{aligned} \tag{3.2}$$

respectively. The corresponding likelihood equations are found to be

$$\begin{aligned}
\frac{\partial \ell(\alpha,\beta)}{\partial \alpha} = & \sum_{i=1}^{n} \log(x_i) + (\beta-1)\sum_{i=1}^{n} \left(-\frac{x_i^{\alpha}\log(x_i)}{1-x_i^{\alpha}} \right) \\
& + \sum_{i=1}^{n} \frac{(1-x_i^{\alpha})^{\beta} \beta x_i^{\alpha} \log(x_i)}{1-x_i^{\alpha}} + \frac{n}{\alpha} = 0,
\end{aligned} \tag{3.3}$$

$$\frac{\partial \ell(\alpha,\beta)}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \log(1-x_i^{\alpha}) + \sum_{i=1}^{n} \left(-(1-x_i^{\alpha})^{\beta} \log((1-x_i^{\alpha})) \right) = 0. \tag{3.4}$$

The Eqs. (3.3) and (3.4) cannot be solved explicitly. It can be solved by some iterative methods. In the next section, the fminsearch (MATLAB function) command is used for this purpose. fminsearch function uses the simplex search method of [8].

4. Simulation study

In this section, a simulation study is conducted to observe the properties of MLE discussed in the Section 3. In Table 2, for different choices of (n, α, β), we present the biases and mean squares errors (MSEs) of the estimates with 5000 replications. From Tables 1, it is observed that the MLEs are biased but asymptotically unbiased. Also, when the sample size n increases, the bias and MSEs of the MLEs decrease to zero as desired.

TABLE 2. Bias and MSEs of MLE estimators for selected parameters

			Bias		MSE	
α	β	n	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\alpha}$	$\widehat{\beta}$
2	2	50	0.0870	0.1212	0.0075	0.0147
		100	0.0466	0.0644	0.0021	0.0041
		200	0.0256	0.0330	0.0006	0.0010
		300	0.0162	0.0210	0.0002	0.0004
		500	0.0098	0.0131	0.0001	0.0001
3	0.5	50	0.2787	0.0185	0.0787	0.0003
		100	0.1429	0.0101	0.0204	0.0001
		200	0.0749	0.0050	0.0056	0.0000
		300	0.0467	0.0031	0.0021	0.0000
		500	0.0283	0.0020	0.0008	0.0000
0.7	1.5	50	0.0348	0.0885	0.0012	0.0078
		100	0.0179	0.0424	0.0003	0.0018
		200	0.0099	0.0221	0.0001	0.0004
		300	0.0062	0.0140	0.0000	0.0001
		500	0.0038	0.0087	0.0000	0.0000

5. Real data application

In this section, we provide an application with real data to illustrate the flexibility of the $DUS-K(\alpha, \beta)$ model. For illustrative purposes, we consider a real data set and compare with some statistical distributions. The data set represents the total milk production in the first birth of 107 cows from the SINDI race. This data can be found in [3]. We consider the Kumaraswamy (Kw) ([7]), exponentiated Kumararaswamy (EKw) ([9]), Weibull (W) ([13]) and beta (B) distributions to compare the fitting ability of the $DUS-K(\alpha, \beta)$ distribution.

The pdf of the distributions used in comparison study are given as follows:

Kw distribution:

$$f_{Kw}(x; \alpha, \beta) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}, \ \alpha, \beta > 0$$

EKw distribution:

$$f_{EKw}(x; \theta, \alpha, \beta) = \alpha\beta\theta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}\left(1-(1-x^{\alpha})^{\beta}\right)^{\theta-1}, \ \theta, \alpha, \beta > 0$$

W distribution

$$f_W(x; \alpha, \theta) = \frac{\alpha}{\theta}\left(\frac{x}{\theta}\right)^{\alpha-1}\exp\left(-\left(\frac{x}{\theta}\right)\right)^{\alpha}, \ \alpha, \theta > 0$$

B distribution

$$f_B(x;a,b) = \frac{1}{B(a,b)} x^{a-1} (1-x)^{b-1}, \ a,b > 0$$

The unknown parameters are estimated for each distribution by the maximum likelihood method. The goodness-of-fit statistics including the values of the Akaike information criterion (AIC), Bayesian information criterion (BIC) where the lower values of AIC,BIC and -2ℓ values are presented in Table 3. From the Table 3, we observed that the $DUS-K$ model is the best model to fit the milk data.

Table 3. Results of AIC, BIC and log-likelihood for $DUS-K$ and other distributions for the data set summaries

$Model$	$Parameters$	AIC	BIC	-2ℓ
Dus-K	$\hat{\alpha} = 1.9198, \hat{\beta} = 3.6421$	-50.360	-45.015	-54.361
Kw	$\hat{\alpha} = 2.1949, \widehat{\beta} = 3.4363$	-46.789	-41.443	-50.789
EKw	$\hat{\theta} = 0.3361, \hat{\alpha} = 5.315, \widehat{\beta} = 7.141$	-27.557	-49.114	-41.443
W	$\hat{\alpha} = 2.6012, \hat{\theta} = 0.5236$	-38.695	-33.349	-42.695
B	$\hat{a} = 2.4125, \widehat{b} = 2.8296$	-43.554	-38.208	-23.777

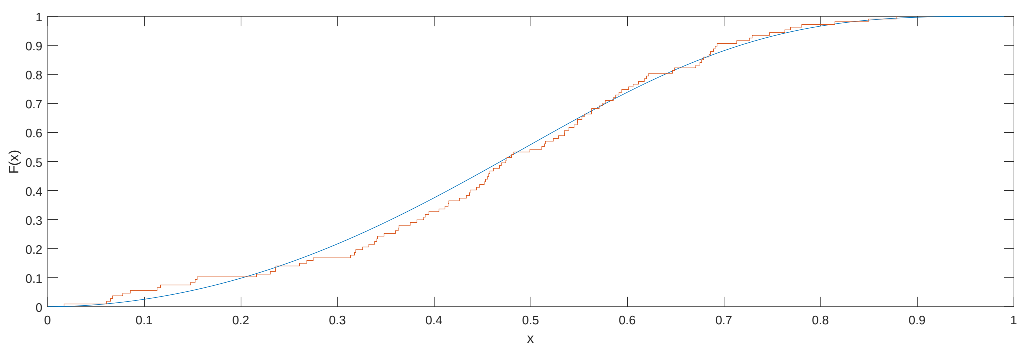

Figure 6. Empirical and fitted distribution function based on milk data

6. Conclusions

In this paper, we introduce a new lifetime distribution by using Dus transformation on Kumaraswamy cdf. Several characteristics have been calculated for the new distribution. Based on our example with the real data, we observed that Dus transformation increases the modelling capability of Kumaraswamy distribution. The proposed distribution has been found to be better than the other well-known distributions in terms of AIC.

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