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Diffusion Equation Including Local Fractional Derivative and Dirichlet Boundary Conditions

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Abstract

In this research, we discuss the construction of analytic solution of homogenous initial boundary value problem including PDEs of fractional order. Since homogenous initial boundary value problem involves local fractional order derivative, it has classical initial and boundary conditions. By means of separation of variables method and the inner product defined on $L^2[0,l]$, the solution is constructed in the form of a Fourier series with respect to the eigenfunctions of a corresponding Sturm-Liouville eigenvalue problem including fractional derivative in local sense used in this study. Illustrative example presents the applicability and influence of separation of variables method on fractional mathematical problems.

Keywords: *Local Fractional Derivative; Dirichlet boundary conditions; Spectral method; Separation of variables.*

2010 Mathematics Subject Classification: *26A33; 65M70.*

1. Introduction

Since mathematical models including fractional derivatives play a vital role fractional derivatives draw a growing attention of many researchers in various branches of sciences. Therefore there are many different fractional derivatives such as Caputo, Riemann-Liouville, Atangana-Baleanu. However these fractional derivatives do not satisfy most important properties of ordinary derivative which leads to many diffuculties to analyze or obtain the solution of fractional mathematical models. As a result many scientists focus on defining new fractional derivatives to cover the setbacks of the defined ones. Moreover the success of mathematical modelling of systems or processes depends on the fractional derivative, it involves, since the correct choice of the fractional derivative allows us to model the real data of systems or processes accurately.

In order to the define new fractional derivatives, various methods exists and these ones are classfied based on their features and formation such as nonlocal fractional derivatives and local fractional derivatives. the proportional derivative is a newly defined fractional derivative which is generally defined as

$$^{P}D_{\alpha}\, f(t) = K_1(\alpha,t)\; f(t) + K_0(\alpha,t)\, f'(t), \tag{1.1}$$

where the functions K_0 and K_1 satisfy certain properties in terms of limit [1] and f is a differentiable function. Notice that this derivative can be regarded as an extension of conformable derivative and is used in control theory.

In this study we focus on obtaining the solution of following fractional diffusion equation including various proportional derivative operator by making use of the separation of variables method:

$$^{P}D_t^{\alpha} u(x,t) = u_{xx}(x,t) + \gamma u(x,t), \tag{1.2}$$

$$u(0,t) = u(l,t) = 0, \tag{1.3}$$

$$u(x,0) = f(x) \tag{1.4}$$

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where $0<\alpha<1, 0\leq x\leq l, 0\leq t\leq T, \gamma\in\mathbb{R}$. Here we use the following forms of the proportional derivatives:

$$ {}^{P}D_{\alpha}f(t)=K_1(\alpha)\ f(t)+K_0(\alpha)f'(t). \tag{1.5}$$

Especially we consider the following ones:

$$ {}_{1}^{P}D_{\alpha}f(t)=(1-\alpha)\ f(t)+\alpha f'(t) \tag{1.6}$$

and

$$ {}_{2}^{P}D_{\alpha}f(t)=\left(1-\alpha^2\right)\ f(t)+\alpha^2 f'(t). \tag{1.7}$$

From a physical aspect, the intrinsic nature of the physical system can be reflected to the mathematical model of the system by using fractional derivatives. Therefore the solution of the fractional mathematical model is in excellent agreement with the predictions and experimental measurement of it. The systems whose behaviour is non-local can be modelled better by fractional mathematical models and the degree of its non-locality can be arranged by the order of fractional derivative. In order to analyze the diffusion in a non-homogenous medium that has memory effects it is better to analyze the solution of the fractional mathematical model for this diffusion. As a result in order to model a process, the correct choices of fractional derivative and its order must be determined.
In this study, local fractional derivative is used to model diffusion problems as in the case of non-local fractional derivative, models including local fractional derivatives gives better results than models including integer order derivatives. In the mathematical modelling of diffusion problem for different matters such as liquid, gas and temperature, the suitable fractional order α is chosen, since the diffusion coefficient depends on the order α of fractional derivative [2]. This mathematical modelling describe the behaviour of matter in a phase. There are many published work on the diffusion of various matters in science especially in fluid mechanics and gas dynamics [3], [4], [5], [6], [7]. From this aspect, analysis of this problem plays an important role in application. Moreover sub-diffusion cases for which $0<\alpha<1$ are under consideration. The solution of the fractional mathematical model of sub-diffusion cases behaves much slower than the solution of the integer-order mathematical model unlike fractional mathematical model for super-diffusion.

2. Main Results

Let us consider the following problem including the proportional derivative in (1.6)

$$ {}^{P}D_{t}^{\alpha}u(x,t)=u_{xx}(x,t)+\gamma u(x,t), \tag{2.1}$$

$$ u(0,t)=u(l,t)=0, \tag{2.2}$$

$$ u(x,0)=f(x) \tag{2.3}$$

where $0<\alpha<1, 0\leq x\leq l, 0\leq t\leq T, \gamma\in\mathbb{R}$.
By means of separation of variables method, The generalized solution of above problem is constructed in analytical form. Thus a solution of problem (2.1)-(2.3) have the following form:

$$ u(x,t;\alpha)=X(x)\ T(t;\alpha) \tag{2.4}$$

where $0\leq x\leq l, 0\leq t\leq T$.
Plugging (2.4) (11) into (2.1) and arranging it, we have

$$ \frac{{}^{P}D_{t}^{\alpha}(T(t;\alpha))}{T(t;\alpha)}-\gamma=\frac{X''(x)}{X(x)}=-\lambda. \tag{2.5}$$

Equation (2.5) produce a fractional equation with respect to time and an ordinary differential equation with respect to space. The first ordinary differential equation is obtained by taking the equation on the right hand side of Eq. (2.5). Hence with boundary conditions (2.2), we have the following problem:

$$ X''(x)+\lambda X(x)=0, \tag{2.6}$$

$$ X(0)=X(l)=0. \tag{2.7}$$

The solution of eigenvalue problem (2.6)-(2.7) is accomplished by making use of the exponential function of the following form:

$$ X(x)=e^{rx}. \tag{2.8}$$

Hence the characteristic equation is computed in the following form:

$$r^2+\lambda=0. \tag{2.9}$$

Case 1. If $\lambda=0$, the characteristic equation have two coincident roots $r_1=r_2$, leading to the general solution of the eigenvalue problem (2.6)-(2.7) having the following form:

$$X(x)=c_1x+c_2. \tag{2.10}$$

The first boundary condition yields

$$X(0)=0=c_2 \tag{2.11}$$

which leads to the following solution

$$X(x)=c_1x. \tag{2.12}$$

Similarly second boundary condition leads to

$$X(l)=c_1l=0 \Longrightarrow c_1=0 \tag{2.13}$$

which implies that $X(x)=0$ which implies that there is not any solution for $\lambda=0$.

Case 2. If $\lambda<0$, the Eq. (2.9) have two distinct real roots $r_{1,2}=\mp\sqrt{-\lambda}$ yielding the general solution of the problem (2.6)-(2.7) in the following form:

$$X(x)=c_1\cosh(\sqrt{-\lambda}x)+c_2\sinh(\sqrt{-\lambda}x). \tag{2.14}$$

By making use of the first boundary condition, we have

$$X(0)=c_1=0 \tag{2.15}$$

which leads to the following solution

$$X(x)=c_2\sinh(\sqrt{-\lambda}x). \tag{2.16}$$

Similarly second boundary condition leads to

$$X(l)=c_2\sinh(\sqrt{-\lambda}l)=0 \Longrightarrow c_2=0 \tag{2.17}$$

which implies that $X(x)=0$ which implies that there is not any solution for $\lambda<0$.

Case 3. If $\lambda>0$, the characteristic equation have two complex roots yielding the general solution of the problem (2.6)-(2.7) in the following form:

$$X(x)=c_1\cos\left(\sqrt{\lambda}x\right)+ic_2\sin(\sqrt{\lambda}x). \tag{2.18}$$

By making use of the first boundary condition we have

$$X(0)=0=c_1. \tag{2.19}$$

Hence the solution becomes

$$X(x)=ic_2\sin(\sqrt{\lambda}x). \tag{2.20}$$

Similarly last boundary condition leads to

$$X(l)=ic_2\sin\left(\sqrt{\lambda}l\right)=0 \tag{2.21}$$

which implies that

$$\sin\left(\sqrt{\lambda}l\right)=0 \tag{2.22}$$

which yields the following eigenvalues

$$\lambda_n = \frac{w_n^2}{l^2}, \lambda_1 < \lambda_2 < \lambda_3 < \ldots, n = 0,1,2,3,\ldots \tag{2.23}$$

where $w_n = n\pi, n = 0,1,2,3,\ldots$ satisfy the equation $sin(w_n) = 0$.
As a result the solution is obtained as follows:

$$X_n(x) = sin\left(\frac{n\pi x}{l}\right), n = 0,1,2,3,\ldots \tag{2.24}$$

The second equation in (2.5) for eigenvalue λ_n yields the fractional differential equation below:

$$\begin{aligned}
&\frac{{}^{P}D_t^{\alpha}(T(t;\alpha))}{T(t;\alpha)} = (\gamma - \lambda),\\
&\frac{K_1(\alpha)\ T_n(t;\alpha) + K_0(\alpha)T'_n(t;\alpha)}{T_n(t;\alpha)} = \left(\gamma - \left(\frac{n\pi}{l}\right)^2\right),\\
&K_0(\alpha)T_n'(t;\alpha) + \left(\left(\left(\frac{n\pi}{l}\right)^2 - \gamma\right) + K_1(\alpha)\right)T_n(t;\alpha) = 0,
\end{aligned} \tag{2.25}$$

which yields the following solution

$$T_n(t;\alpha) = exp\left(-\frac{\left(\left(\frac{n\pi}{l}\right)^2 - \gamma\right) + K_1(\alpha)}{K_0(\alpha)}t\right), n = 0,1,2,3,\ldots \tag{2.26}$$

The solution for every eigenvalue λ_n is constructed as

$$\begin{aligned}
u_n(x,t;\alpha) &= X_n(x)T_n(t;\alpha)\\
&= exp\left(-\frac{\left(\left(\frac{n\pi}{l}\right)^2 - \gamma\right) + K_1(\alpha)}{K_0(\alpha)}t\right) sin\left(\frac{n\pi x}{l}\right), n = 0,1,2,3,\ldots
\end{aligned} \tag{2.27}$$

which leads to the following general solution

$$u(x,t;\alpha) = \sum_{n=0}^{\infty} A_n sin\left(\frac{n\pi x}{l}\right) exp\left(-\frac{\left(\left(\frac{n\pi}{l}\right)^2 - \gamma\right) + K_1(\alpha)}{K_0(\alpha)}t\right). \tag{2.28}$$

Note that it satisfies boundary condition and fractional differential equation.
The coefficients of general solution are established by taking the following initial condition into account:

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} A_n sin\left(\frac{n\pi x}{l}\right). \tag{2.29}$$

The coefficients A_n for $n = 0,1,2,3,\ldots$ determined by the help of inner product defined on $L^2[0,l]$:

$$A_n = \frac{2}{l}\int_0^l f(x) sin\left(\frac{n\pi x}{l}\right). \tag{2.30}$$

3. Illustrative Example

In this section, we first consider the following initial Dirichlet boundary value problem:

$$\begin{aligned}
&u_t(x,t) = u_{xx}(x,t) + \gamma u(x,t),\\
&u(0,t) = 0,\ u(1,t) = 0,\\
&u(x,0) = 3\sin(2\pi x) - 4\sin(3\pi x)
\end{aligned} \tag{3.1}$$

which has the solution in the following form:

$$u(x,t) = 3\sin(2\pi x)e^{(\gamma - 4\pi^2)t} - 4\sin(3\pi x)e^{(\gamma - 9\pi^2)t} \tag{3.2}$$

where $0 \le x \le 1, 0 \le t \le T$.

Example 1. Now let the following problem called fractional heat-like problem be taken into consideration:

$${}_{1}^{P}D_{t}^{\alpha}u(x,t)=u_{xx}(x,t)+\gamma u(x,t), \tag{3.3}$$

$$u(0,t)=0,\ u(1,t)=0, \tag{3.4}$$

$$u(x,0)=3\sin(2\pi x)-4\sin(3\pi x) \tag{3.5}$$

where $0<\alpha<1, 0\leq x\leq 1,\ 0\leq t\leq T$. It is clear from Eq. (2.28) that the solution of above problem can be obtained in the following form:

$$u(x,t;\alpha)=\sum_{n=0}^{\infty}A_{n}sin\left(\frac{n\pi x}{l}\right)exp\left(-\frac{\left(\left(\frac{n\pi}{l}\right)^{2}-\gamma\right)+1-\alpha}{\alpha}t\right). \tag{3.6}$$

Plugging $t=0$ in to the general solution (3.6) and making equal to the initial condition (3.5) we have

$$3\sin(2\pi x)-4\sin(3\pi x)=\sum_{n=0}^{\infty}A_{n}sin\left(\frac{n\pi x}{l}\right). \tag{3.7}$$

The coefficients A_n for $n=0,1,2,3,\ldots$ are determined by the help of the inner product as follows:

$$A_{n}=2\int_{0}^{1}(3\sin(2\pi x)-4\sin(3\pi x))\sin\left(\frac{n\pi x}{l}\right). \tag{3.8}$$

For $n\neq 2$ and $n\neq 3A_{n}=0.n=2$ and $n=3$ we get

$$A_{2}=3, A_{3}=-4. \tag{3.9}$$

As a result

$$u(x,t;\alpha)=3sin(2\pi x)exp\left(-\frac{(4\pi^{2}-\gamma)+1-\alpha}{\alpha}t\right)$$
$$-4sin(3\pi x)exp\left(-\frac{(9\pi^{2}-\gamma)+1-\alpha}{\alpha}t\right). \tag{3.10}$$

It is important to note that plugging $\alpha=1$ in to the solution (3.10) gives the solution (3.2) which confirm the accuracy of the method we apply.

Example 2. Now let the following problem called fractional heat-like problem be taken into consideration:

$${}_{2}^{P}D_{t}^{\alpha}u(x,t)=u_{xx}(x,t), \tag{3.11}$$

$$u(0,t)=0, u(1,t)=0, \tag{3.12}$$

$$u(x,0)=3\sin(2\pi x)-4\sin(3\pi x) \tag{3.13}$$

where $0<\alpha<1, -1\leq x\leq 1,\ 0\leq t\leq T$. It is clear from Eq. (2.28) that the solution of above problem can be obtained in the following form:

$$u(x,t;\alpha)=\sum_{n=0}^{\infty}A_{n}sin\left(\frac{n\pi x}{l}\right)exp\left(-\frac{\left(\left(\frac{n\pi}{l}\right)^{2}-\gamma\right)+1-\alpha^{2}}{\alpha^{2}}t\right). \tag{3.14}$$

As in Example 1, after similar computations the solution can be constructed as follows:

$$u(x,t;\alpha)=3sin(2\pi x)exp\left(-\frac{(4\pi^{2}-\gamma)+1-\alpha^{2}}{\alpha^{2}}t\right)$$
$$-4sin(3\pi x)exp\left(-\frac{(9\pi^{2}-\gamma)+1-\alpha^{2}}{\alpha^{2}}t\right). \tag{3.15}$$

The graphics of solutions, obtained by MATLAB, for Ex.1, Ex. 2 and Problem (3.1) in 2D and 3D are given in Fig.3.1 and Fig.3.1 respectively.

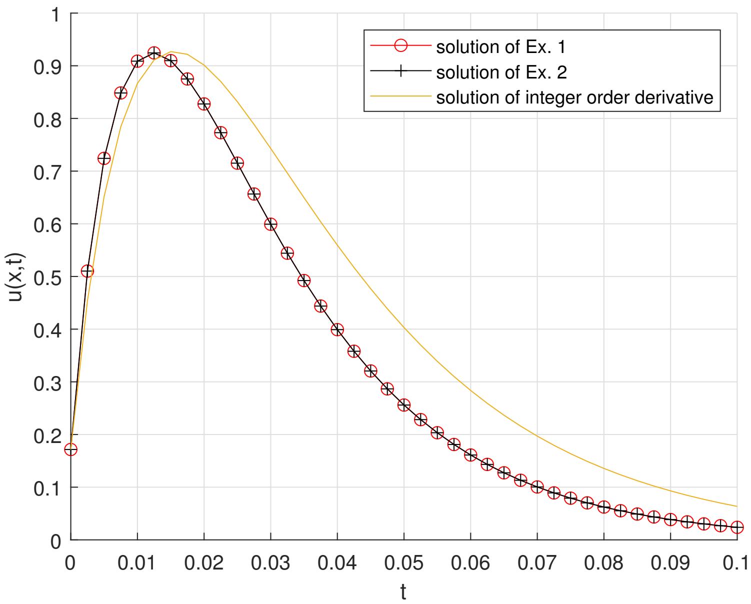

Figure 3.1: The graphics of solutions for Ex. 1 and Ex. 2 in 2D at $x = 0.25$ for $\alpha = 0.8$ and $\gamma = 1$.

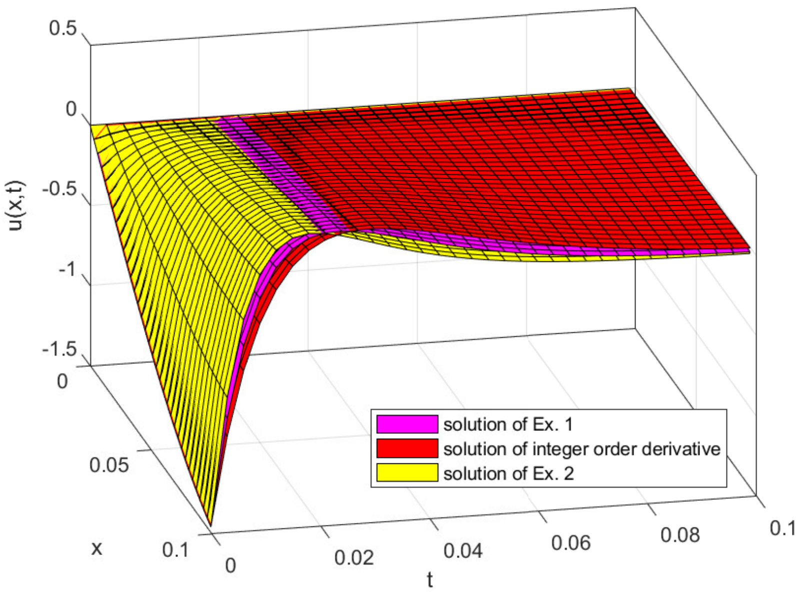

Figure 3.2: The graphics of solutions for Ex. 1 and Ex. 2 in 3D for $\alpha = 0.8$ and $\gamma = 1$.

4. Conclusion

In this study, the analytic solution of time fractional diffusion problem including local fractional derivatives in one dimension is constructed analytically in Fourier series form. Taking the separation of variables into account, the solution is formed in the form of a Fourier series with respect to the eigenfunctions of a corresponding Sturm-Liouville eigenvalue problem including fractional derivative in a proportional sense. Based on the analytic solution, we reach the conclusion that diffusion processes decays exponential with time until initial condition is reached. As α tends to 0, the rate of decaying increases. This implies that in the mathematical model for diffusion of the matter which has small diffusion rate the value of α must be close to 0. This model can account for various diffusion processes of various methods.

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