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Relative $(p,q,t)L$-th Type and Relative $(p,q,t)L$-th Weak Type Oriented Growth Properties of Composite Entire Functions

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Abstract

In the paper some new results depending on the comparative growth properties of composite entire functions using relative $(p,q,t)L$-th order, relative $(p,q,t)L$-th type and relative $(p,q,t)L$-th weak type of entire function with respect to another entire function are established.

Keywords: *Entire function; relative $(p,q,t)L$-th order; relative $(p,q,t)L$-th type; relative $(p,q,t)L$-th weak type; growth; slowly changing function.*
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1. Introduction, Definitions and Notations

Let $\mathbb{C}$ be the set of all finite complex numbers. For any entire function $f=\sum_{n=0}^{\infty} a_n z^n$ defined on $\mathbb{C}$, the functions $M(r,f)$ known as maximum modulus function of f is defined as $M(r,f)=\max_{|z|=r}|f(z)|$. When f is non-constant, then $M_f(r)$ is strictly increasing and continuous and its inverse $M_f{}^{-1}:(|f(0)|,\infty)\to(0,\infty)$ exists and is such that $\lim_{s\to\infty} M_f{}^{-1}(s)=\infty$.

However let us consider that $x\in[0,\infty)$ and $k\in\mathbb{N}$ where $\mathbb{N}$ is the set of all positive integers. We define $\exp^{[k]}x=\exp(\exp^{[k-1]}x)$ and $\log^{[k]}x=\log(\log^{[k-1]}x)$. We also denote $\log^{[0]}x=x$, $\log^{[-1]}x=\exp x$, $\exp^{[0]}x=x$ and $\exp^{[-1]}x=\log x$. Further we assume that throughout the present paper l,p,q,m and n always denote positive integers and $t\in\mathbb{N}\cup\{-1,0\}$. Now considering this, we just recall that Shen et al. [7] defined the (m,n)-φ order and (m,n)-φ lower order of entire functions f which are as follows:

Definition 1.1. *[7] Let $\varphi:[0,+\infty)\to(0,+\infty)$ be a non-decreasing unbounded function and $m\geq n$. The (m,n)-φ order $\rho^{(m,n)}(f,\varphi)$ and (m,n)-φ lower order $\lambda^{(m,n)}(f,\varphi)$ of entire functions f are defined as:*

$$\rho^{(m,n)}(f,\varphi)=\overline{\lim_{r\to\infty}}\frac{\log^{[m]}M_f(r)}{\log^{[n]}\varphi(r)} \text{ and } \lambda^{(m,n)}(f,\varphi)=\underline{\lim_{r\to\infty}}\frac{\log^{[m]}M_f(r)}{\log^{[n]}\varphi(r)}.$$

If we take $m=p$, $n=1$ and $\varphi(r)=\log^{[q-1]}r$, then the above definition reduce to the following definition:

Definition 1.2. *The (p,q)-th order and (p,q)-th lower order of an entire function f are defined as:*

$$\rho^{(p,q)}(f)=\overline{\lim_{r\to\infty}}\frac{\log^{[p]}M_f(r)}{\log^{[q]}r} \text{ and } \lambda^{(p,q)}(f)=\underline{\lim_{r\to\infty}}\frac{\log^{[p]}M_f(r)}{\log^{[q]}r}.$$

Definition 1.2 avoids the restriction $p\geq q$ of the original definition of (p,q)-th order (respectively (p,q)-th lower order) of entire functions introduced by Juneja et al. [5].

However the above definition is very useful for measuring the growth of entire functions. If $p=l$ and $q=1$ then we write $\rho^{(l,1)}(f)=\rho^{(l)}(f)$ and $\lambda^{(l,1)}(f)=\lambda^{(l)}(f)$ where $\rho^{(l)}(f)$ and $\lambda^{(l)}(f)$ are respectively known as generalized order and generalized lower order of entire function f. For details about generalized order one may see [8]. Also for $p=2$ and $q=1$, we respectively denote $\rho^{(2,1)}(f)$ and $\lambda^{(2,1)}(f)$ by $\rho(f)$ and $\lambda(f)$ which are classical growth indicators such as order and lower order of entire function f.

In this connection we just recall the following definition of index-pair where we will give a minor modification to the original definition (see e.g. [5]) :

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Definition 1.3. *An entire function f is said to have index-pair (p,q) if $b<\rho^{(p,q)}(f)<\infty$ and $\rho^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b=1$ if $p=q$ and $b=0$ for otherwise. Moreover if $0<\rho^{(p,q)}(f)<\infty$, then*

$$\begin{cases} \rho^{(p-n,q)}(f)=\infty & \text{for } n<p, \\ \rho^{(p,q-n)}(f)=0 & \text{for } n<q, \\ \rho^{(p+n,q+n)}(f)=1 & \text{for } n=1,2,\cdots \end{cases}.$$

Similarly for $0<\lambda^{(p,q)}(f)<\infty$, one can easily verify that

$$\begin{cases} \lambda^{(p-n,q)}(f)=\infty & \text{for } n<p, \\ \lambda^{(p,q-n)}(f)=0 & \text{for } n<q, \\ \lambda^{(p+n,q+n)}(f)=1 & \text{for } n=1,2,\cdots \end{cases}.$$

However, the function f is said to be of regular (p,q) growth when (p,q)-th order and (p,q)-th lower order of f are the same. Functions which are not of regular (p,q) growth are said to be of irregular (p,q) growth.

For entire functions, Somasundaram and Thamizharasi [6] introduced the notions of the growth indicators L-order and L-lower order where $L\equiv L(r)$ is a positive continuous function increasing slowly i.e.,$L(ar)\sim L(r)$ as $r\to\infty$ for every positive constant 'a' i.e., $\lim\limits_{r\to\infty}\frac{L(ar)}{L(r)}=1$ where $L\equiv L(r)$ is a positive continuous function increasing slowly. The more generalized concept of L-order and L-lower order for entire function are L^*-order and L^*-lower order. Their definitions are as follows:

Definition 1.4. *[6] The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of an entire function f are defined as*

$$\rho_f^{L^*}=\overline{\lim_{r\to\infty}}\frac{\log^{[2]}M_f(r)}{\log[re^{L(r)}]} \text{ and } \lambda_f^{L^*}=\underline{\lim_{r\to\infty}}\frac{\log^{[2]}M_f(r)}{\log[re^{L(r)}]}.$$

If we take $m=p$, $n=1$ and $\varphi(r)=\log^{[q-1]}r\cdot\exp^{[t+1]}L(r)$, then Definition 1.1 turn into the definitions of $(p,q,t)L$-th order and $(p,q,t)L$-th lower order of an entire function f which are as follows:

$$\rho_f^L(p,q,t)=\overline{\lim_{r\to\infty}}\frac{\log^{[p]}M_f(r)}{\log^{[q]}r+\exp^{[t]}L(r)} \text{ and } \lambda_f^L(p,q,t)=\underline{\lim_{r\to\infty}}\frac{\log^{[p]}M_f(r)}{\log^{[q]}r+\exp^{[t]}L(r)}.$$

In order to compare the relative growth of two entire functions having same non zero finite $(p,q,t)L$-th order, one may introduce the definitions of $(p,q,t)L$-th type (respectively $(p,q,t)L$-th lower type) of entire functions having finite positive finite $(p,q,t)L$-th order in the following manner:

Definition 1.5. *[2] Let f be an entire function with non-zero finite $(p,q,t)L$-th order $\rho_f^L(p,q,t)$. The $(p,q,t)L$-th type denoted by $\sigma_f^L(p,q,t)$ and $(p,q,t)L$-th lower type denoted by $\overline{\sigma}_f^L(p,q,t)$ are respectively defined as follows:*

$$\sigma_f^L(p,q,t)=\overline{\lim_{r\to\infty}}\frac{\log^{[p-1]}M_f(r)}{[\log^{[q-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_f^L(p,q,t)}}$$

and

$$\overline{\sigma}_f^L(p,q,t)=\underline{\lim_{r\to\infty}}\frac{\log^{[p-1]}M_f(r)}{[\log^{[q-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_f^L(p,q,t)}}$$

Analogously in order to determine the relative growth of two entire functions having same non zero finite $(p,q,t)L$-th lower order one may introduce the definition of $(p,q,t)L$-th weak type of entire functions having finite positive $(p,q,t)L$-th lower order in the following way:

Definition 1.6. *[2] The $(p,q,t)L$-th weak type denoted by $\tau_f^L(p,q,t)$ of an entire function f is defined as follows:*

$$\tau_f^L(p,q,t)=\underline{\lim_{r\to+\infty}}\frac{\log^{[p-1]}M_f(r)}{[\log^{[q-1]}r\cdot\exp^{[t+1]}L(r)]^{\lambda_f^L(p,q,t)}},\ 0<\lambda_f^L(p,q,t)<\infty.$$

Also one may define the growth indicator $\overline{\tau}_f^L(p,q,t)$ of an entire function f in the following manner :

$$\overline{\tau}_f^L(p,q,t)=\overline{\lim_{r\to\infty}}\frac{\log^{[p-1]}M_f(r)}{[\log^{[q-1]}r\cdot\exp^{[t+1]}L(r)]^{\lambda_f^L(p,q,t)}},\ 0<\lambda_f^L(p,q,t)<\infty.$$

Mainly the growth investigation of entire functions has usually been done through their maximum moduli in comparison with those of exponential function. But if one is paying attention to evaluate the growth rates of any entire with respect to a new entire function , the notions of relative growth indicators [1] will come. In order to make some progress in the study of relative order, recently Biswas [3] introduce the notion of relative $(p,q,t)L$-th order and relative $(p,q,t)L$-th lower order of an entire function f with respect to another entire function g in the following way:

Definition 1.7. *[3] Let f and g be any two entire functions. Then relative $(p,q,t)L$-th order denoted as $\rho_g^{(p,q,t)L}(f)$ and relative $(p,q,t)L$-th lower order denoted as $\lambda_g^{(p,q,t)L}(f)$ of an entire function f with respect to another entire function g are define by*

$$\rho_g^{(p,q,t)L}(f)=\overline{\lim_{r\to\infty}}\frac{\log^{[p]}M_g^{-1}(M_f(r))}{\log^{[q]}r+\exp^{[t]}L(r)} \text{ and } \lambda_g^{(p,q,t)L}(f)=\underline{\lim_{r\to\infty}}\frac{\log^{[p]}M_g^{-1}(M_f(r))}{\log^{[q]}r+\exp^{[t]}L(r)}.$$

Now to compare the relative growth of two entire functions having same non zero finite relative $(p,q,t)L$-th order with respect to another entire function, one can introduce the notion of relative $(p,q,t)L$-th type (respectively relative $(p,q,t)L$-th lower type) of an entire function with respect to another entire function which is as follows :

Definition 1.8. *[3] If $0<\rho_g^{(p,q,t)L}(f)<\infty$, then relative $(p,q,t)L$-th type $\sigma_g^{(p,q,t)L}(f)$ and relative $(p,q,t)L$-th lower type $\overline{\sigma}_g^{(p,q,t)L}(f)$ of an entire function f with respect to another entire function g are defined as*

$$\sigma_g^{(p,q,t)L}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_g^{-1}(M_f(r))}{[\log^{[q-1]} r\cdot \exp^{[t+1]} L(r)]^{\rho_g^{(p,q,t)L}(f)}} \quad \text{and}$$

$$\overline{\sigma}_g^{(p,q,t)L}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_g^{-1}(M_f(r))}{[\log^{[q-1]} r\cdot \exp^{[t+1]} L(r)]^{\rho_g^{(p,q,t)L}(f)}}.$$

Similarly, one can define relative $(p,q,t)L$-th weak type to determine the relative growth of two entire functions having same nonzero finite relative $(p,q,t)L$-th lower order with respect to another entire function in the following manner:

Definition 1.9. *[3] If $0<\lambda_g^{(p,q,t)L}(f)<\infty$, then relative $(p,q,t)L$-th weak type $\tau_g^{(p,q,t)L}(f)$ of an entire function f with respect to another entire function g is defined as:*

$$\tau_g^{(p,q,t)L}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_g^{-1}(M_f(r))}{[\log^{[q-1]} r\cdot \exp^{[t+1]} L(r)]^{\lambda_g^{(p,q,t)L}(f)}}.$$

Further one may define the growth indicator $\overline{\tau}_g^{(p,q,t)L}(f)$ of an entire function f with respect to an entire function g in the following way :

$$\overline{\tau}_g^{(p,q,t)L}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_g^{-1}(M_f(r))}{[\log^{[q-1]} r\cdot \exp^{[t+1]} L(r)]^{\lambda_g^{(p,q,t)L}(f)}},\ 0<\ \lambda_g^{(p,q,t)L}(f)\ <\infty.$$

In the paper we study some maximum modulus oriented growth properties of composite entire functions on the basis of their relative $(p,q,t)L$-th order, relative $(p,q,t)L$-th type and relative $(p,q,t)L$-th weak type of entire function with respect to another entire function improving some earlier results where $p,q\in\mathbb{N}$ and $t\in\mathbb{N}\cup\{-1,0\}$. We do not explain the standard definitions and notations in the theory of entire functions as those are available in [9].

2. Lemmas.

In this section we present two lemmas which will be needed in the sequel.

Lemma 2.1. *[4] If f and g are any two entire functions then for all sufficiently large values of r,*

$M_{f\circ g}(r)\le M_f(M_g(r))$.

Lemma 2.2. *[4] If f and g are any two entire functions then for all sufficiently large values of r,*

$M_{f\circ g}(r)\ge M_f\left(\frac{1}{16}M_g\left(\frac{r}{2}\right)\right)$.

3. Theorems.

In this section we present the main results of the paper.

Theorem 3.1. *Let f, g and h be any three entire functions such that $0<\lambda_h^{(p,q,t)L}(f)<\infty$ or $0<\rho_h^{(p,q,t)L}(f)<\infty$ and $\sigma_g^L(m,n,t)<\infty$ where $m-1=q$. If $\exp^{[t]}L(M_g(r))=o([\log^{[n-1]} r\cdot\exp^{[t+1]}L(r)]^{\alpha})$ as $r\to\infty$ and for some positive $\alpha<\rho_g^L(m,n,t)$, then*

$$\underline{\lim_{r\to\infty}} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p]} M_h^{-1}\left(M_f\left(\exp^{[q]}[\log^{[n-1]} r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)}\right)\right)} \le \sigma_g^L(m,n,t).$$

Proof. Let us consider $0<\lambda_h^{(p,q,t)L}(f)<\infty$.
Since $M_h^{-1}(r)$ is an increasing function of r, it follows from in view of Lemma 2.1, for a sequence of values of r tending to infinity that

$$\log^{[p]} M_h^{-1}(M_{f\circ g}(r))\le \log^{[p]} M_h^{-1}(M_f(M_g(r)))$$

$$i.e.,\ \log^{[p]} M_h^{-1}(M_{f\circ g}(r))\le (\lambda_h^{(p,q,t)L}(f)+\varepsilon)[\log^{[q]} M_g(r)+\exp^{[t]}L(M_g(r))]$$

$$i.e.,\ \log^{[p]} M_h^{-1}(M_{f\circ g}(r))\le (\lambda_h^{(p,q,t)L}(f)+\varepsilon)[\log^{[m-1]} M_g(r)+\exp^{[t]}L(M_g(r))]$$

$$i.e.,\ \log^{[p]} M_h^{-1}(M_{f\circ g}(r))\le (\lambda_h^{(p,q,t)L}(f)+\varepsilon)\cdot \left[(\sigma_g^L(m,n,t)+\varepsilon)[\log^{[n-1]} r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)}+\exp^{[t]}L(M_g(r))\right]$$

$$i.e.,\ \log^{[p]} M_h^{-1}(M_{f\circ g}(r))\le (\lambda_h^{(p,q,t)L}(f)+\varepsilon)\cdot$$

$$\left[(\sigma_g^L(m,n,t)+\varepsilon)[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}+\exp^{[t]} L(M_g(r))\right]. \tag{3.1}$$

Further, we obtain for all sufficiently large values of r that

$$\log^{[p]} M_h^{-1}\left(M_f\left(\exp^{[q]}[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}\right)\right)\geq$$

$$(\lambda_h^{(p,q,t)L}(f)-\varepsilon)[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}$$

$$+(\lambda_h^{(p,q,t)L}(f)-\varepsilon)\exp^{[t]}\left[L\left(\exp^{[q]}[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}\right)\right]$$

$$\log^{[p]} M_h^{-1}\left(M_f\left(\exp^{[q]}[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}\right)\right)\geq(\lambda_h^{(p,q,t)L}(f)-\varepsilon)[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}.$$

Now from (3.1) and above it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p]} M_h^{-1}\left(M_f\left(\exp^{[q]}[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}\right)\right)}$$

$$\leq\frac{(\lambda_h^{(p,q,t)L}(f)+\varepsilon)\left[(\sigma_g^L(m,n,t)+\varepsilon)[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}+\exp^{[t]} L(M_g(r))\right]}{(\lambda_h^{(p,q,t)L}(f)-\varepsilon)[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}}$$

$$i.e.,\ \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p]} M_h^{-1}\left(M_f\left(\exp^{[q]}[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}\right)\right)}\leq\frac{(\lambda_h^{(p,q,t)L}(f)+\varepsilon)\cdot\left[(\sigma_g^L(m,n,t)+\varepsilon)+\frac{\exp^{[t]} L(M_g(r))}{[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}}\right]}{(\lambda_h^{(p,q,t)L}(f)-\varepsilon)}. \tag{3.2}$$

As $\alpha<\rho_g^L(m,n,t)$ and $\exp^{[t]} L(M_g(r))=o([\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\alpha})$ as $r\to\infty$, we obtain that

$$\lim_{r\to\infty}\frac{\exp^{[t]} L(M_g(r))}{[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}}=0. \tag{3.3}$$

Since $\varepsilon(>0)$ is arbitrary, it follows from (3.2) and (3.3) that

$$\varliminf_{r\to\infty}\frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p]} M_h^{-1}\left(M_f\left(\exp^{[q]}[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}\right)\right)}\leq\sigma_g^L(m,n,t).$$

Similarly if we consider $0<\rho_h^{(p,q,t)L}(f)<\infty$, then using the same technique one can easily verify that

$$\varliminf_{r\to\infty}\frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p]} M_h^{-1}\left(M_f\left(\exp^{[q]}[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}\right)\right)}\leq\sigma_g^L(m,n,t).$$

Thus the theorem is established. □

In the line of Theorem 3.1, the following theorem can be carried out and therefore its proof is omitted:

Theorem 3.2. *Let f, g and h be any three entire functions such that $0<\lambda_h^{(p,q,t)L}(f)<\infty$ or $0<\rho_h^{(p,q,t)L}(f)<\infty$ and $\overline{\tau}_g^L(m,n,t)<\infty$ where $m-1=q$. If $\exp^{[t]} L(M_g(r))=o([\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\alpha})$ as $r\to\infty$ and for some positive $\alpha<\lambda_g^L(m,n,t)$, then*

$$\varliminf_{r\to\infty}\frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p]} M_h^{-1}\left(M_f\left(\exp^{[q]}[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\lambda_g^L(m,n,t)}\right)\right)}\leq\overline{\tau}_g^L(m,n,t).$$

Theorem 3.3. *Let f, g and h be any three entire functions such that $0<\lambda_h^{(p,q,t)L}(f)\leq\rho_h^{(p,q,t)L}(f)<\infty$ and $\sigma_g^L(m,n,t)<\infty$ where $m-1=q$. If $\exp^{[t]} L(M_g(r))=o([\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\alpha})$ as $r\to\infty$ and for some positive $\alpha<\rho_g^L(m,n,t)$, then*

$$\varlimsup_{r\to\infty}\frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p]} M_h^{-1}\left(M_f\left(\exp^{[q]}[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}\right)\right)}\leq\frac{\rho_h^{(p,q,t)L}(f)\cdot\sigma_g^L(m,n,t)}{\lambda_h^{(p,q,t)L}(f)}.$$

Theorem 3.4. *Let f, g and h be any three entire functions such that $0<\lambda_h^{(p,q,t)L}(f)\leq\rho_h^{(p,q,t)L}(f)<\infty$ and $\overline{\tau}_g^L(m,n,t)<\infty$ where $m-1=q$. If $\exp^{[t]} L(M_g(r))=o([\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\alpha})$ as $r\to\infty$ and for some positive $\alpha<\lambda_g^L(m,n,t)$, then*

$$\varlimsup_{r\to\infty}\frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p]} M_h^{-1}\left(M_f\left(\exp^{[q]}[\log^{[n-1]} r\cdot\exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)}\right)\right)}\leq\frac{\rho_h^{(p,q,t)L}(f)\cdot\overline{\tau}_g^L(m,n,t)}{\lambda_h^{(p,q,t)L}(f)}.$$

We omit the proofs of Theorem 3.3 and Theorem 3.4 as those can easily be established in the line of Theorem 3.1 and Theorem 3.2 respectively.

Using the notion of $(m,n,t)L$-th lower type and $(m,n,t)L$-th weak type we may state the following two theorems without their proofs because those can be proved in the line of Theorem 3.3:

Theorem 3.5. *Let f, g and h be any three entire functions such that $0<\lambda_h^{(p,q,t)L}(f)\leq\rho_h^{(p,q,t)L}(f)<\infty$ and $\overline{\sigma}_g^L(m,n,t)<\infty$ where $m-1=q$. If $\exp^{[t]}L(M_g(r))=o([\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\alpha})$ as $r\to\infty$ and for some positive $\alpha<\rho_g^L(m,n,t)$, then*

$$\underline{\lim}_{r\to\infty}\frac{\log^{[p]}M_h^{-1}(M_{f\circ g}(r))}{\log^{[p]}M_h^{-1}\left(M_f\left(\exp^{[q]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)}\right)\right)}\leq\frac{\rho_h^{(p,q,t)L}(f)\cdot\overline{\sigma}_g^L(m,n,t)}{\lambda_h^{(p,q,t)L}(f)}.$$

Theorem 3.6. *Let f, g and h be any three entire functions such that $0<\lambda_h^{(p,q,t)L}(f)\leq\rho_h^{(p,q,t)L}(f)<\infty$ and $\tau_g^L(m,n,t)<\infty$ where $m-1=q$. If $\exp^{[t]}L(M_g(r))=o([\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\alpha})$ as $r\to\infty$ and for some positive $\alpha<\lambda_g^L(m,n,t)$, then*

$$\underline{\lim}_{r\to\infty}\frac{\log^{[p]}M_h^{-1}(M_{f\circ g}(r))}{\log^{[p]}M_h^{-1}\left(M_f\left(\exp^{[q]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\lambda_g^L(m,n,t)}\right)\right)}\leq\frac{\rho_h^{(p,q,t)L}(f)\cdot\tau_g^L(m,n,t)}{\lambda_h^{(p,q,t)L}(f)}.$$

Now we state the following three theorems without their proofs as those can be carried out in the line of Theorem 3.1 and Theorem 3.3 repectively.

Theorem 3.7. *Let f, g, h and k be any four entire functions such that $\lambda_k^{(l,n,t)L}(g)>0$, $\lambda_h^{(p,q,t)L}(f)<\infty$ and $\sigma_g^L(m,n,t)<\infty$ where $m-1=q$. If $\exp^{[t]}L(M_g(r))=o([\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\alpha})$ as $r\to\infty$ and for some positive $\alpha<\rho_g^L(m,n,t)$, then*

$$\underline{\lim}_{r\to\infty}\frac{\log^{[p]}M_h^{-1}(M_{f\circ g}(r))}{\log^{[l]}M_k^{-1}\left(M_g\left(\exp^{[n]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)}\right)\right)}\leq\frac{\lambda_h^{(p,q,t)L}(f)\cdot\sigma_g^L(m,n,t)}{\lambda_k^{(l,n,t)L}(g)}.$$

Theorem 3.8. *Let f, g, h and k be any four entire functions such that $\rho_k^{(l,n,t)L}(g)>0$, $\rho_h^{(p,q,t)L}(f)<\infty$ and $\sigma_g^L(m,n,t)<\infty$ where $m-1=q$. If $\exp^{[t]}L(M_g(r))=o([\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\alpha})$ as $r\to\infty$ and for some positive $\alpha<\rho_g^L(m,n,t)$, then*

$$\underline{\lim}_{r\to\infty}\frac{\log^{[p]}M_h^{-1}(M_{f\circ g}(r))}{\log^{[l]}M_k^{-1}\left(M_g\left(\exp^{[n]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)}\right)\right)}\leq\frac{\rho_h^{(p,q,t)L}(f)\cdot\sigma_g^L(m,n,t)}{\rho_k^{(l,n,t)L}(g)}.$$

Theorem 3.9. *Let f, g, h and k be any four entire functions such that $\lambda_k^{(l,n,t)L}(g)>0$, $\rho_h^{(p,q,t)L}(f)<\infty$ and $\sigma_g^L(m,n,t)<\infty$ where $m-1=q$. If $\exp^{[t]}L(M_g(r))=o([\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\alpha})$ as $r\to\infty$ and for some positive $\alpha<\rho_g^L(m,n,t)$, then*

$$\overline{\lim}_{r\to\infty}\frac{\log^{[p]}M_h^{-1}(M_{f\circ g}(r))}{\log^{[l]}M_k^{-1}\left(M_g\left(\exp^{[n]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)}\right)\right)}\leq\frac{\rho_h^{(p,q,t)L}(f)\cdot\sigma_g^L(m,n,t)}{\lambda_k^{(l,n,t)L}(g)}.$$

Theorem 3.10. *Let f, g, h and k be any four entire functions such that $\lambda_k^{(l,n,t)L}(g)>0$, $\rho_h^{(p,q,t)L}(f)<\infty$ and $\overline{\sigma}_g^L(m,n,t)<\infty$ where $m-1=q$. If $\exp^{[t]}L(M_g(r))=o([\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\alpha})$ as $r\to\infty$ and for some positive $\alpha<\rho_g^L(m,n,t)$, then*

$$\underline{\lim}_{r\to\infty}\frac{\log^{[p]}M_h^{-1}(M_{f\circ g}(r))}{\log^{[l]}M_k^{-1}\left(M_g\left(\exp^{[n]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)}\right)\right)}\leq\frac{\rho_h^{(p,q,t)L}(f)\cdot\overline{\sigma}_g^L(m,n,t)}{\lambda_k^{(l,n,t)L}(g)}.$$

We omit the proof of Theorem 3.10 as it can easily be established in the line of Theorem 3.5.

Further we state the following theorem which is based on (m,n,t)-th L-weak type:

Theorem 3.11. *Let f, g, h and k be any four entire functions such that $\lambda_k^{(l,n,t)L}(g)>0$, $\rho_h^{(p,q,t)L}(f)<\infty$ and $\tau_g^L(m,n,t)<\infty$ where $m-1=q$. If $\exp^{[t]}L(M_g(r))=o([\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\alpha})$ as $r\to\infty$ and for some positive $\alpha<\lambda_g^L(m,n,t)$, then*

$$\underline{\lim}_{r\to\infty}\frac{\log^{[p]}M_h^{-1}(M_{f\circ g}(r))}{\log^{[l]}M_k^{-1}\left(M_g\left(\exp^{[n]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\lambda_g^L(m,n,t)}\right)\right)}\leq\frac{\rho_h^{(p,q,t)L}(f)\cdot\tau_g^L(m,n,t)}{\lambda_k^{(l,n,t)L}(g)}.$$

Proofs of the above theorem can be carried out in the line of Theorem 3.10 and therefore its proof is omitted.

Using the concept of the growth indicator $\overline{\tau}_g^L(m,n,t)$ of an entire function g, we may state the subsequent three theorems without their proofs since those can be carried out in the line of Theorem 3.11.

Theorem 3.12. *Let f, g, h and k be any four entire functions such that $\lambda_k^{(l,n,t)L}(g)>0$, $\lambda_h^{(p,q,t)L}(f)<\infty$ and $\overline{\tau}_g^L(m,n,t)<\infty$ where $m-1=q$. If $\exp^{[t]}L(M_g(r))=o([\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\alpha})$ as $r\to\infty$ and for some positive $\alpha<\lambda_g^L(m,n,t)$, then*

$$\underline{\lim}_{r\to\infty}\frac{\log^{[p]}M_h^{-1}(M_{f\circ g}(r))}{\log^{[l]}M_k^{-1}\left(M_g\left(\exp^{[n]}[\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\lambda_g^L(m,n,t)}\right)\right)}\leq\frac{\lambda_h^{(p,q,t)L}(f)\cdot\overline{\tau}_g^L(m,n,t)}{\lambda_k^{(l,n,t)L}(g)}.$$

Theorem 3.13. *Let f, g, h and k be any four entire functions such that $\rho_k^{(l,n,t)L}(g) > 0$, $\rho_h^{(p,q,t)L}(f) < \infty$ and $\overline{\tau}_g^L(m,n,t) < \infty$ where $m-1=q$. If $\exp^{[t]} L(M_g(r)) = o([\log^{[n-1]} r \cdot \exp^{[t+1]} L(r)]^{\alpha})$ as $r \to \infty$ and for some positive $\alpha < \lambda_g^L(m,n,t)$, then*

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[l]} M_k^{-1}\left(M_g\left(\exp^{[n]}[\log^{[n-1]} r \cdot \exp^{[t+1]} L(r)]^{\lambda_g^L(m,n,t)}\right)\right)} \le \frac{\rho_h^{(p,q,t)L}(f) \cdot \overline{\tau}_g^L(m,n,t)}{\rho_k^{(l,n,t)L}(g)}.$$

Theorem 3.14. *Let f, g, h and k be any four entire functions such that $\lambda_k^{(l,n,t)L}(g) > 0$, $\rho_h^{(p,q,t)L}(f) < \infty$ and $\overline{\tau}_g^L(m,n,t) < \infty$ where $m-1=q$. If $\exp^{[t]} L(M_g(r)) = o([\log^{[n-1]} r \cdot \exp^{[t+1]} L(r)]^{\alpha})$ as $r \to \infty$ and for some positive $\alpha < \lambda_g^L(m,n,t)$, then*

$$\overline{\lim}_{r\to\infty} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[l]} M_k^{-1}\left(M_g\left(\exp^{[n]}[\log^{[n-1]} r \cdot \exp^{[t+1]} L(r)]^{\lambda_g^L(m,n,t)}\right)\right)} \le \frac{\rho_h^{(p,q,t)L}(f) \cdot \overline{\tau}_g^L(m,n,t)}{\lambda_k^{(l,n,t)L}(g)}.$$

Theorem 3.15. *Let f, g and h be any three entire functions such that $\rho_h^{(p,q,t)L}(f) = \rho_g^L(m,n,t)$, $0 < \sigma_g^L(m,n,t) < \infty$ and $\overline{\sigma}_h^{(p,q,t)L}(f) > 0$ where $m-1=n=q$. Then*

$$\overline{\lim}_{r\to\infty} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p-1]} M_h^{-1}(M_f(r)) + \exp^{[t]} L(M_g(r))} \le \begin{cases} \frac{\rho_h^{(p,q,t)L}(f)\sigma_g^L(m,n,t)}{\overline{\sigma}_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} M_h^{-1}(M_f(r))\}, \\ \rho_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} M_h^{-1}(M_f(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}$$

Proof. Since $M_h^{-1}(r)$ is an increasing function of r, it follows from in view of Lemma 2.1 for all sufficiently large values of r that

$$\log^{[p]} M_h^{-1}(M_{f\circ g}(r)) \le \log^{[p]} M_h^{-1}(M_f(M_g(r)))$$

$$i.e.,\ \log^{[p]} M_h^{-1}(M_{f\circ g}(r)) \le (\rho_h^{(p,q,t)L}(f)+\varepsilon)[\log^{[q]} M_g(r) + \exp^{[t]} L(M_g(r))]$$

$$i.e.,\ \log^{[p]} M_h^{-1}(M_{f\circ g}(r)) \le (\rho_h^{(p,q,t)L}(f)+\varepsilon)[\log^{[m-1]} M_g(r) + \exp^{[t]} L(M_g(r))]$$

$$i.e.,\ \log^{[p]} M_h^{-1}(M_{f\circ g}(r)) \le (\rho_h^{(p,q,t)L}(f)+\varepsilon) \cdot \left[(\sigma_g^L(m,n,t)+\varepsilon)[\log^{[n-1]} r \cdot \exp^{[t+1]} L(r)]^{\rho_g^L(m,n,t)} + \exp^{[t]} L(M_g(r))\right].$$

Since $\rho_h^{(p,q,t)L}(f) = \rho_g^L(m,n,t)$, we obtain from above for all sufficiently large values of r that

$$i.e.,\ \log^{[p]} M_h^{-1}(M_{f\circ g}(r)) \le (\rho_h^{(p,q,t)L}(f)+\varepsilon) \cdot$$

$$\left[(\sigma_g^L(m,n,t)+\varepsilon)[\log^{[n-1]} r \cdot \exp^{[t+1]} L(r)]^{\rho_h^{(p,q,t)L}(f)} + \exp^{[t]} L(M_g(r))\right]. \tag{3.4}$$

Again we get for all sufficiently large values of r that

$$\log^{[p-1]} M_h^{-1}(M_f(r)) \ge (\overline{\sigma}_h^{(p,q,t)L}(f)-\varepsilon)[\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\rho_g^{(p,q,t)L}(f)}$$

$$i.e.,\ [\log^{[q-1]} r \cdot \exp^{[t+1]} L(r)]^{\rho_g^{(p,q,t)L}(f)} \le \frac{\log^{[p-1]} M_h^{-1} M_f(r)}{(\overline{\sigma}_h^{(p,q,t)L}(f)-\varepsilon)}$$

$$i.e.,\ [\log^{[n-1]} r \cdot \exp^{[t+1]} L(r)]^{\rho_g^{(p,q,t)L}(f)} \le \frac{\log^{[p-1]} M_h^{-1} M_f(r)}{(\overline{\sigma}_h^{(p,q,t)L}(f)-\varepsilon)}. \tag{3.5}$$

Now from (3.4) and (3.5) it follows for all sufficiently large values of r that

$$\log^{[p]} M_h^{-1}(M_{f\circ g}(r)) \le (\sigma_g^L(m,n,t)+\varepsilon) \cdot \exp^{[t]} L(M_g(r)) + (\rho_h^{(p,q,t)L}(f)+\varepsilon)(\sigma_g^L(m,n,t)+\varepsilon) \cdot \frac{\log^{[p-1]} M_h^{-1} M_f(r)}{(\overline{\sigma}_h^{(p,q,t)L}(f)-\varepsilon)}$$

$$ie.,\ \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p-1]} M_h^{-1}(M_f(r)) + \exp^{[t]} L(M_g(r))} \le \frac{(\rho_h^{(p,q,t)L}(f)+\varepsilon)}{1+\frac{\log^{[p-1]} M_h^{-1} M_f(r)}{\exp^{[t]} L(M_g(r))}} + \frac{\frac{(\rho_h^{(p,q,t)L}(f)+\varepsilon)(\sigma_g^L(m,n,t)+\varepsilon)}{(\overline{\sigma}_h^{(p,q,t)L}(f)-\varepsilon)}}{1+\frac{\exp^{[t]} L(M_g(r))}{\log^{[p-1]} M_h^{-1} M_f(r)}}. \tag{3.6}$$

If $\exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} M_h^{-1}(M_f(r))\}$ then from (3.6) we get that

$$\overline{\lim}_{r\to\infty} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p-1]} M_h^{-1}(M_f(r)) + \exp^{[t]} L(M_g(r))} \le \frac{(\rho_h^{(p,q,t)L}(f)+\varepsilon)(\sigma_g^L(m,n,t)+\varepsilon)}{(\overline{\sigma}_h^{(p,q,t)L}(f)-\varepsilon)}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p-1]} M_h^{-1}(M_f(r))+\exp^{[t]} L(M_g(r))} \leq \frac{\rho_h^{(p,q,t)L}(f)\sigma_g^L(m,n,t)}{\overline{\sigma}_h^{(p,q,t)L}(f)}$$

Again if $\log^{[p-1]} M_h^{-1}(M_f(r)) = o\{\exp^{[t]} L(M_g(r))\}$ then from (3.6) it follows that

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p-1]} M_h^{-1}(M_f(r))+\exp^{[t]} L(M_g(r))} \leq (\rho_h^{(p,q,t)L}(f)+\varepsilon).$$

As $\varepsilon(>0)$ is arbitrary, we obtain from above that

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p-1]} M_h^{-1}(M_f(r))+\exp^{[t]} L(M_g(r))} \leq \rho_h^{(p,q,t)L}(f).$$

Thus the theorem is established. □

Theorem 3.16. *Let f, g and h be any three entire functions such that $\lambda_h^{(p,q,t)L}(f) < 0$, $\rho_h^{(p,q,t)L}(f) = \rho_g^L(m,n,t)$, $0 < \sigma_g^L(m,n,t) < \infty$ and $\overline{\sigma}_h^{(p,q,t)L}(f) > 0$ where $m-1 = n = q$. Then*

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p-1]} M_h^{-1}(M_f(r))+\exp^{[t]} L(M_g(r))} \leq \begin{cases} \frac{\lambda_h^{(p,q,t)L}(f)\sigma_g^L(m,n,t)}{\overline{\sigma}_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} M_h^{-1}(M_f(r))\}, \\ \lambda_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} M_h^{-1}(M_f(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}$$

Theorem 3.17. *Let f, g and h be any three entire functions such that $\rho_h^{(p,q,t)L}(f) = \rho_g^L(m,n,t)$, $0 < \sigma_g^L(m,n,t) < \infty$ and $\sigma_h^{(p,q,t)L}(f) > 0$ where $m-1 = n = q$. Then*

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p-1]} M_h^{-1}(M_f(r))+\exp^{[t]} L(M_g(r))} \leq \begin{cases} \frac{\rho_h^{(p,q,t)L}(f)\sigma_g^L(m,n,t)}{\sigma_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} M_h^{-1}(M_f(r))\}, \\ \rho_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} M_h^{-1}(M_f(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}$$

Theorem 3.18. *Let f, g and h be any three entire functions such that $\rho_h^{(p,q,t)L}(f) = \rho_g^L(m,n,t)$, $0 < \overline{\sigma}_g^L(m,n,t) < \infty$ and $\overline{\sigma}_h^{(p,q,t)L}(f) > 0$ where $m-1 = n = q$. Then*

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p-1]} M_h^{-1}(M_f(r))+\exp^{[t]} L(M_g(r))} \leq \begin{cases} \frac{\rho_h^{(p,q,t)L}(f)\overline{\sigma}_g^L(m,n,t)}{\overline{\sigma}_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} M_h^{-1}(M_f(r))\}, \\ \rho_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} M_h^{-1}(M_f(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}$$

We omit the proof of the above three theorems as those can be carried out in the line of Theorem 3.15.

Similarly using the concept of the growth indicator $\tau_h^{(p,q,t)L}(f)$ and $\overline{\tau}_g^L(m,n,t)$ we may state the subsequent four theorems without their proofs since those can be carried out in the line of Theorem 3.15, Theorem 3.16, Theorem 3.17 and Theorem 3.18 respectively.

Theorem 3.19. *Let f, g and h be any three entire functions such that $\rho_h^{(p,q,t)L}(f) < \infty$, $\lambda_h^{(p,q,t)L}(f) = \lambda_g^L(m,n,t)$, $0 < \overline{\tau}_g^L(m,n,t) < \infty$ and $\tau_h^{(p,q,t)L}(f) > 0$ where $m-1 = n = q$. Then*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p-1]} M_h^{-1}(M_f(r))+\exp^{[t]} L(M_g(r))} \leq \begin{cases} \frac{\rho_h^{(p,q,t)L}(f)\overline{\tau}_g^L(m,n,t)}{\tau_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} M_h^{-1}(M_f(r))\}, \\ \rho_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} M_h^{-1}(M_f(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}$$

Theorem 3.20. *Let f, g and h be any three entire functions such that $\lambda_h^{(p,q,t)L}(f) = \lambda_g^L(m,n,t)$, $0 < \overline{\tau}_g^L(m,n,t) < \infty$ and $\tau_h^{(p,q,t)L}(f) > 0$ where $m-1 = n = q$. Then*

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p-1]} M_h^{-1}(M_f(r))+\exp^{[t]} L(M_g(r))} \leq \begin{cases} \frac{\lambda_h^{(p,q,t)L}(f)\overline{\tau}_g^L(m,n,t)}{\tau_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} M_h^{-1}(M_f(r))\}, \\ \lambda_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} M_h^{-1}(M_f(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}$$

Theorem 3.21. *Let f, g and h be any three entire functions such that $\rho_h^{(p,q,t)L}(f) < \infty$, $\lambda_h^{(p,q,t)L}(f) = \lambda_g^L(m,n,t)$, $0 < \overline{\tau}_g^L(m,n,t) < \infty$ and $\overline{\tau}_h^{(p,q,t)L}(f) > 0$ where $m-1 = n = q$. Then*

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p-1]} M_h^{-1}(M_f(r))+\exp^{[t]} L(M_g(r))} \leq \begin{cases} \frac{\rho_h^{(p,q,t)L}(f)\overline{\tau}_g^L(m,n,t)}{\overline{\tau}_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} M_h^{-1}(M_f(r))\}, \\ \rho_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} M_h^{-1}(M_f(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}.$$

Theorem 3.22. *Let f, g and h be any three entire functions such that $\rho_h^{(p,q,t)L}(f) < \infty$, $\lambda_h^{(p,q,t)L}(f) = \lambda_g^L(m,n,t)$, $0 < \tau_g^L(m,n,t) < \infty$ and $\tau_h^{(p,q,t)L}(f) > 0$ where $m-1 = n = q$. Then*

$$\lim_{r\to\infty} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p-1]} M_h^{-1}(M_f(r)) + \exp^{[t]} L(M_g(r))} \leq \begin{cases} \frac{\rho_h^{(p,q,t)L}(f)\tau_g^L(m,n,t)}{\tau_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} M_h^{-1}(M_f(r))\}, \\ \rho_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} M_h^{-1}(M_f(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}$$

Analogously we state the following four theorems under some different conditions which can also be carried out using the same technique of Theorem 3.15 and therefore their proofs are omitted.

Theorem 3.23. *Let f, g and h be any three entire functions such that $\rho_h^{(p,q,t)L}(f) < \infty$, $\lambda_h^{(p,q,t)L}(f) = \rho_g^L(m,n,t)$, $0 < \sigma_g^L(m,n,t) < \infty$ and $\tau_h^{(p,q,t)L}(f) > 0$ where $m-1 = n = q$. Then*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p-1]} M_h^{-1}(M_f(r)) + \exp^{[t]} L(M_g(r))} \leq \begin{cases} \frac{\rho_h^{(p,q,t)L}(f)\sigma_g^L(m,n,t)}{\tau_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} M_h^{-1}(M_f(r))\}, \\ \rho_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} M_h^{-1}(M_f(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}$$

Theorem 3.24. *Let f, g and h be any three entire functions such that $\lambda_h^{(p,q,t)L}(f) = \rho_g^L(m,n,t)$, $0 < \sigma_g^L(m,n,t) < \infty$ and $\tau_h^{(p,q,t)L}(f) > 0$ where $m-1 = n = q$. Then*

$$\underline{\lim_{r\to\infty}} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p-1]} M_h^{-1}(M_f(r)) + \exp^{[t]} L(M_g(r))} \leq \begin{cases} \frac{\lambda_h^{(p,q,t)L}(f)\sigma_g^L(m,n,t)}{\tau_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} M_h^{-1}(M_f(r))\}, \\ \lambda_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} M_h^{-1}(M_f(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}$$

Theorem 3.25. *Let f, g and h be any three entire functions such that $\rho_h^{(p,q,t)L}(f) = \lambda_g^L(m,n,t)$, $0 < \overline{\tau}_g^L(m,n,t) < \infty$ and $\overline{\sigma}_h^{(p,q,t)L}(f) > 0$ where $m-1 = n = q$. Then*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p-1]} M_h^{-1}(M_f(r)) + \exp^{[t]} L(M_g(r))} \leq \begin{cases} \frac{\rho_h^{(p,q,t)L}(f)\overline{\tau}_g^L(m,n,t)}{\overline{\sigma}_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} M_h^{-1}(M_f(r))\}, \\ \rho_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} M_h^{-1}(M_f(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}$$

Theorem 3.26. *Let f, g and h be any three entire functions such that $\rho_h^{(p,q,t)L}(f) = \lambda_g^L(m,n,t)$, $0 < \overline{\tau}_g^L(m,n,t) < \infty$ and $\overline{\sigma}_h^{(p,q,t)L}(f) > 0$ where $m-1 = n = q$. Then*

$$\underline{\lim_{r\to\infty}} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[p-1]} M_h^{-1}(M_f(r)) + \exp^{[t]} L(M_g(r))} \leq \begin{cases} \frac{\lambda_h^{(p,q,t)L}(f)\overline{\tau}_g^L(m,n,t)}{\overline{\sigma}_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} M_h^{-1}(M_f(r))\}, \\ \lambda_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} M_h^{-1}(M_f(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}$$

Theorem 3.27. *Let f, g and h be any three **entire functions su**ch that (i) $\rho_h^{(p,q,t)L}(f) = \rho_g^L(m,n,t)$, (ii) $0 < \overline{\sigma}_g^L(m,n,t) < \infty$ and (iii) $\sigma_h^{(p,q,t)L}(f) < \infty$ where $m-1 = n = q$ and $\exp^{[t]} L(ar) \sim \exp^{[t]} L(r)$. Then*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(2r))}{\log^{[p-1]} M_h^{-1}(M_f(r)) + \exp^{[t]} L(M_g(r))} \geq \begin{cases} \frac{\rho_h^{(p,q,t)L}(f)\overline{\sigma}_g^L(m,n,t)}{\sigma_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} M_h^{-1}(M_f(r))\}, \\ \rho_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} M_h^{-1}(M_f(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}$$

Proof. Since $M_h^{-1}(r)$ is an increasing function of r, in view of Lemma 2.2 we get for a sequence of values of r tending to infinity that

$$\log^{[p]} M_h^{-1}(M_{f\circ g}(2r)) \geq \log^{[p]} M_h^{-1} M_f\left(\frac{1}{16} M_g(r)\right)$$

$$\begin{aligned} i.e.,\ \log^{[p]} M_h^{-1}(M_{f\circ g}(2r)) &\geq (\rho_h^{(p,q,t)L}(f) - \varepsilon) \cdot \\ &\quad \left[\log^{[q]}\left(\frac{1}{16} M_g(r)\right) + \exp^{[t]} L\left(\frac{1}{16} M_g(r)\right)\right] \end{aligned}$$

$$\begin{aligned} i.e.,\ \log^{[p]} M_h^{-1}(M_{f\circ g}(2r)) &\geq (\rho_h^{(p,q,t)L}(f) - \varepsilon) \cdot \\ &\quad [\log^{[m-1]} M_g(r) + \exp^{[t]} L(M_g(r)) + O(1)] \end{aligned}$$

$$i.e.,\ \log^{[p]} M_h^{-1}(M_{f\circ g}(2r)) \geq (\rho_h^{(p,q,t)L}(f) - \varepsilon) \cdot$$

$$\left[(\overline{\sigma}_g^L(m,n,t)-\varepsilon)[\log^{[n-1]}(r)\cdot\exp^{[t+1]}L(r)]^{\rho_g^L(m,n,t)}+\exp^{[t]}L(M_g(r))+O(1)\right].$$

Now in view of condition (i) we have from above for a sequence of values of r tending to infinity that

$$i.e.,\ \log^{[p]}M(_h^{-1}M_{f\circ g}(2r))\geq(\rho_h^{(p,q,t)L}(f)-\varepsilon)\cdot$$

$$\left[(\overline{\sigma}_g^L(m,n,t)-\varepsilon)[\log^{[n-1]}(r)\cdot\exp^{[t+1]}L(r)]^{\rho_h^{(p,q,t)L}(f)}+\exp^{[t]}L(M_g(r))+O(1)\right]. \tag{3.7}$$

Again we get for all sufficiently large values of r that

$$\log^{[p-1]}M_h^{-1}(M_f(r))\leq(\sigma_h^{(p,q,t)L}(f)+\varepsilon)[\log^{[q-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_h^{(p,q,t)L}(f)}$$

$$i.e.,\ [\log^{[q-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_h^{(p,q,t)L}(f)}\geq\frac{\log^{[p-1]}M_h^{-1}(M_f(r))}{(\sigma_h^{(p,q,t)L}(f)+\varepsilon)}$$

$$i.e.,\ [\log^{[n-1]}r\cdot\exp^{[t+1]}L(r)]^{\rho_h^{(p,q,t)L}(f)}\geq\frac{\log^{[p-1]}M_h^{-1}(M_f(r))}{(\sigma_h^{(p,q,t)L}(f)+\varepsilon)}. \tag{3.8}$$

Now it follows from (3.7) and (3.8) for a sequence of values of r tending to infinity that

$$\log^{[p]}M_h^{-1}(M_{f\circ g}(2r))\geq(\rho_h^{(p,q,t)L}(f)-\varepsilon)(\overline{\sigma}_g^L(m,n,t)-\varepsilon)\cdot\frac{\log^{[p-1]}M_h^{-1}(M_f(r))}{(\sigma_h^{(p,q,t)L}(f)+\varepsilon)}$$
$$+(\rho_h^{(p,q,t)L}(f)-\varepsilon)[\exp^{[t]}L(M_g(r))+O(1)]$$

$$ie.,\ \frac{\log^{[p]}M_h^{-1}(M_{f\circ g}(2r))}{\log^{[p-1]}M_h^{-1}(M_f(r))+\exp^{[t]}L(M_g(r))}\geq\frac{(\rho_h^{(p,q,t)L}(f)-\varepsilon)}{1+\frac{\log^{[p-1]}M_h^{-1}(M_f(r))}{\exp^{[t]}L(M_g(r))}}$$

$$+\frac{\frac{(\rho_h^{(p,q,t)L}(f)-\varepsilon)(\overline{\sigma}_g^L(m,n,t)-\varepsilon)}{(\sigma_h^{(p,q,t)L}(f)+\varepsilon)}}{1+\frac{\exp^{[t]}L(M_g(r))}{\log^{[p-1]}M_h^{-1}(M_f(r))}}+\frac{O(1)}{\log^{[p-1]}M_h^{-1}(M_f(r))+\exp^{[t]}L(M_g(r))}. \tag{3.9}$$

If $\exp^{[t]}L(M_g(r))=o\{\log^{[p-1]}M_h^{-1}(M_f(r))\}$ then from (3.9) we get that

$$\varlimsup_{r\to\infty}\frac{\log^{[p]}M_h^{-1}(M_{f\circ g}(2r))}{\log^{[p-1]}M_h^{-1}(M_f(r))+\exp^{[t]}L(M_g(r))}\geq\frac{(\rho_h^{(p,q,t)L}(f)-\varepsilon)(\overline{\sigma}_g^L(m,n,t)-\varepsilon)}{(\sigma_h^{(p,q,t)L}(f)+\varepsilon)}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\varlimsup_{r\to\infty}\frac{\log^{[p]}M_h^{-1}(M_{f\circ g}(2r))}{\log^{[p-1]}M_h^{-1}(M_f(r))+\exp^{[t]}L(M_g(r))}\geq\frac{\rho_h^{(p,q,t)L}(f)\cdot\overline{\sigma}_g^L(m,n,t)}{\sigma_h^{(p,q,t)L}(f)}$$

Again if $\log^{[p-1]}M_h^{-1}(M_f(r))=o\{\exp^{[t]}L(M_g(r))\}$ then from (3.9) it follows that

$$\varlimsup_{r\to\infty}\frac{\log^{[p]}M_h^{-1}(M_{f\circ g}(2r))}{\log^{[p-1]}M_h^{-1}(M_f(r))+\exp^{[t]}L(M_g(r))}\geq(\rho_h^{(p,q,t)L}(f)-\varepsilon).$$

As $\varepsilon(>0)$ is arbitrary, we obtain from above that

$$\varlimsup_{r\to\infty}\frac{\log^{[p]}M_h^{-1}(M_{f\circ g}(2r))}{\log^{[p-1]}M_h^{-1}(M_f(r))+\exp^{[t]}L(M_g(r))}\geq\rho_h^{(p,q,t)L}(f).$$

Thus the theorem follows. □

Theorem 3.28. *Let f, g and h be any three entire functions such that (i) $\lambda_h^{(p,q,t)L}(f)>0$, (ii) $\rho_h^{(p,q,t)L}(f)=\rho_g^L(m,n,t)$, (iii) $\sigma_g^L(m,n,t)>0$ and (iv) $\sigma_h^{(p,q,t)L}(f)<\infty$ where $m-1=n=q$ and $\exp^{[t]}L(ar)\sim\exp^{[t]}L(r)$. Then*

$$\varlimsup_{r\to\infty}\frac{\log^{[p]}M_h^{-1}(M_{f\circ g}(2r))}{\log^{[p-1]}M_h^{-1}(M_f(r))+\exp^{[t]}L(M_g(r))}\geq\begin{cases}\frac{\lambda_h^{(p,q,t)L}(f)\sigma_g^L(m,n,t)}{\sigma_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]}L(M_g(r))=o\{\log^{[p-1]}M_h^{-1}(M_f(r))\},\\ \lambda_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]}M_h^{-1}(M_f(r))=o\{\exp^{[t]}L(M_g(r))\}.\end{cases}$$

Theorem 3.29. *Let f, g and h be any three entire functions such that (i) $\lambda_h^{(p,q,t)L}(f) > 0$, (ii) $\rho_h^{(p,q,t)L}(f) = \rho_g^L(m,n,t)$, (iii) $\overline{\sigma}_g^L(m,n,t) > 0$ and (iv) $\overline{\sigma}_h^{(p,q,t)L}(f) < \infty$ where $m-1 = n = q$ and $\exp^{[t]} L(ar) \sim \exp^{[t]} L(r)$. Then*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(2r))}{\log^{[p-1]} M_h^{-1}(M_f(r)) + \exp^{[t]} L(M_g(r))} \geq \begin{cases} \frac{\lambda_h^{(p,q,t)L}(f)\overline{\sigma}_g^L(m,n,t)}{\overline{\sigma}_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} M_h^{-1}(M_f(r))\}, \\ \lambda_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} M_h^{-1}(M_f(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}$$

Theorem 3.30. *Let f, g and h be any three entire functions such that (i) $\lambda_h^{(p,q,t)L}(f) > 0$, (ii) $\rho_h^{(p,q,t)L}(f) = \rho_g^L(m,n,t)$, (iii) $\overline{\sigma}_g^L(m,n,t) > 0$ and (iv) $\sigma_h^{(p,q,t)L}(f) < \infty$ where $m-1 = n = q$ and $\exp^{[t]} L(ar) \sim \exp^{[t]} L(r)$. Then*

$$\underline{\lim_{r\to\infty}} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(2r))}{\log^{[p-1]} M_h^{-1}(M_f(r)) + \exp^{[t]} L(M_g(r))} \geq \begin{cases} \frac{\lambda_h^{(p,q,t)L}(f)\overline{\sigma}_g^L(m,n,t)}{\sigma_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} M_h^{-1}(M_f(r))\}, \\ \lambda_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} M_h^{-1}(M_f(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}$$

We omit the proofs of the above three theorems as those can be carried out in the line of Theorem 3.27.

Similarly using the concept of the growth indicator $\tau_h^{(p,q,t)L}(f)$ and $\tau_g^L(m,n,t)$ we may state the subsequent four theorems without their proofs since those can be carried out in the line of Theorem 3.27, Theorem 3.28, Theorem 3.29 and Theorem 3.30 respectively.

Theorem 3.31. *Let f, g and h be any three entire functions such that (i) $\rho_h^{(p,q,t)L}(f) > 0$, (ii) $\lambda_h^{(p,q,t)L}(f) = \lambda_g^L(m,n,t)$, (iii) $0 < \tau_g^L(m,n,t) < \infty$ and (iv) $\overline{\tau}_h^{(p,q,t)L}(f) < \infty$ where $m-1 = n = q$ and $\exp^{[t]} L(ar) \sim \exp^{[t]} L(r)$. Then*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(2r))}{\log^{[p-1]} M_h^{-1}(M_f(r)) + \exp^{[t]} L(M_g(r))} \geq \begin{cases} \frac{\rho_h^{(p,q,t)L}(f)\tau_g^L(m,n,t)}{\overline{\tau}_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} M_h^{-1}(M_f(r))\}, \\ \rho_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} M_h^{-1}(M_f(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}$$

Theorem 3.32. *Let f, g and h be any three entire functions such that (i) $\lambda_h^{(p,q,t)L}(f) = \lambda_g^L(m,n,t)$, (ii) $0 < \overline{\tau}_g^L(m,n,t) < \infty$ and (iii) $\overline{\tau}_h^{(p,q,t)L}(f) < \infty$ where $m-1 = n = q$ and $\exp^{[t]} L(ar) \sim \exp^{[t]} L(r)$. Then*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(2r))}{\log^{[p-1]} M_h^{-1}(M_f(r)) + \exp^{[t]} L(M_g(r))} \geq \begin{cases} \frac{\lambda_h^{(p,q,t)L}(f)\overline{\tau}_g^L(m,n,t)}{\overline{\tau}_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} M_h^{-1}(M_f(r))\}, \\ \lambda_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} M_h^{-1}(M_f(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}$$

Theorem 3.33. *Let f, g and h be any three entire functions such that (i) $\lambda_h^{(p,q,t)L}(f) = \lambda_g^L(m,n,t)$, (ii) $0 < \tau_g^L(m,n,t) < \infty$ and (iii) $\tau_h^{(p,q,t)L}(f) < \infty$ where $m-1 = n = q$ and $\exp^{[t]} L(ar) \sim \exp^{[t]} L(r)$. Then*

$$\underline{\lim_{r\to\infty}} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(2r))}{\log^{[p-1]} M_h^{-1}(M_f(r)) + \exp^{[t]} L(M_g(r))} \geq \begin{cases} \frac{\lambda_h^{(p,q,t)L}(f)\tau_g^L(m,n,t)}{\tau_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} M_h^{-1}(M_f(r))\}, \\ \lambda_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} M_h^{-1}(M_f(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}$$

Theorem 3.34. *Let f, g and h be any three entire functions such that (i) $\lambda_h^{(p,q,t)L}(f) = \lambda_g^L(m,n,t)$, (ii) $0 < \tau_g^L(m,n,t) < \infty$ and (iii) $\overline{\tau}_h^{(p,q,t)L}(f) < \infty$ where $m-1 = n = q$ and $\exp^{[t]} L(ar) \sim \exp^{[t]} L(r)$. Then*

$$\underline{\lim_{r\to\infty}} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(2r))}{\log^{[p-1]} M_h^{-1}(M_f(r)) + \exp^{[t]} L(M_g(r))} \geq \begin{cases} \frac{\lambda_h^{(p,q,t)L}(f)\tau_g^L(m,n,t)}{\overline{\tau}_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} M_h^{-1}(M_f(r))\}, \\ \lambda_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} M_h^{-1}(M_f(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}$$

Analogously we state the following four theorems under some different conditions which can also be carried out using the same technique of Theorem 3.27 and therefore their proofs are omitted.

Theorem 3.35. *Let f, g and h be any three entire functions such that (i) $\rho_h^{(p,q,t)L}(f) > 0$, (ii) $\lambda_h^{(p,q,t)L}(f) = \rho_g^L(m,n,t)$, (iii) $0 < \overline{\sigma}_g^L(m,n,t) < \infty$ and (iv) $\overline{\tau}_h^{(p,q,t)L}(f) < \infty$ where $m-1 = n = q$ and $\exp^{[t]} L(ar) \sim \exp^{[t]} L(r)$. Then*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(2r))}{\log^{[p-1]} M_h^{-1}(M_f(r)) + \exp^{[t]} L(M_g(r))} \geq \begin{cases} \frac{\rho_h^{(p,q,t)L}(f)\overline{\sigma}_g^L(m,n,t)}{\overline{\tau}_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} M_h^{-1}(M_f(r))\}, \\ \rho_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} M_h^{-1}(M_f(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}$$

Theorem 3.36. *Let f, g and h be any three entire functions such that (i) $\lambda_h^{(p,q,t)L}(f) = \rho_g^L(m,n,t)$, (ii) $0 < \overline{\sigma}_g^L(m,n,t) < \infty$ and (iii) $\overline{\tau}_h^{(p,q,t)L}(f) < \infty$ where $m-1 = n = q$ and $\exp^{[t]} L(ar) \sim \exp^{[t]} L(r)$. Then*

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(2r))}{\log^{[p-1]} M_h^{-1}(M_f(r)) + \exp^{[t]} L(M_g(r))} \geq \begin{cases} \frac{\lambda_h^{(p,q,t)L}(f)\overline{\sigma}_g^L(m,n,t)}{\overline{\tau}_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} M_h^{-1}(M_f(r))\}, \\ \lambda_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} M_h^{-1}(M_f(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}$$

Theorem 3.37. *Let f, g and h be any three entire functions such that (i) $\rho_h^{(p,q,t)L}(f) = \lambda_g^L(m,n,t)$, (ii) $0 < \tau_g^L(m,n,t) < \infty$ and (iii) $\sigma_h^{(p,q,t)L}(f) < \infty$ where $m-1 = n = q$ and $\exp^{[t]} L(ar) \sim \exp^{[t]} L(r)$. Then*

$$\overline{\lim}_{r\to\infty} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(2r))}{\log^{[p-1]} M_h^{-1}(M_f(r)) + \exp^{[t]} L(M_g(r))} \geq \begin{cases} \frac{\rho_h^{(p,q,t)L}(f)\tau_g^L(m,n,t)}{\sigma_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} M_h^{-1}(M_f(r))\}, \\ \rho_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} M_h^{-1}(M_f(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}$$

Theorem 3.38. *Let f, g and h be any three entire functions such that (i) $\lambda_h^{(p,q,t)L}(f) > 0$, (ii) $\rho_h^{(p,q,t)L}(f) = \lambda_g^L(m,n,t)$, (iii) $0 < \tau_g^L(m,n,t) < \infty$ and (iv) $\sigma_h^{(p,q,t)L}(f) < \infty$ where $m-1 = n = q$ and $\exp^{[t]} L(ar) \sim \exp^{[t]} L(r)$. Then*

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} M_h^{-1}(M_{f\circ g}(2r))}{\log^{[p-1]} M_h^{-1}(M_f(r)) + \exp^{[t]} L(M_g(r))} \geq \begin{cases} \frac{\lambda_h^{(p,q,t)L}(f)\tau_g^L(m,n,t)}{\sigma_h^{(p,q,t)L}(f)} & \text{if } \exp^{[t]} L(M_g(r)) = o\{\log^{[p-1]} M_h^{-1}(M_f(r))\}, \\ \lambda_h^{(p,q,t)L}(f) & \text{if } \log^{[p-1]} M_h^{-1}(M_f(r)) = o\{\exp^{[t]} L(M_g(r))\}. \end{cases}$$

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