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From Discrete to Continuous: Garch Volatility Modeling of the Bitcoin

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ABSTRACT

The aim of the study is to determine the most appropriate discrete model for the volatility of Bitcoin returns using the discrete-time GARCH model and its extensions and compare it with the Lévy-driven continuous-time GARCH model. For this purpose, the volatility of Bitcoin returns is modeled using daily data of the Bitcoin / United States Dollar exchange rate. By comparing discrete-time models according to information criteria and likelihood values, the All-GARCH model with Johnson's-SU innovations is found as the most adequate model. The persistence of the volatility and half-life of the volatility of the returns are calculated according to the estimation of the discrete model. This discrete model has been compared with the continuous model in which the Lévy increments are derived from the compound Poisson process using various error measurements. In conclusion, it is found that the continuous-time GARCH model shows a better performance in predicting volatility.

Keywords: Volatility, GARCH, COGARCH, Compound Poisson, Lévy Process, Bitcoin, Stochastic Modeling.

JEL Classification Codes: C22, C52, C58.

INTRODUCTION

Cryptocurrencies are electronic currencies that use cryptography to secure the transactions made with it and to control the process of introducing new currencies. They are considered as a subset of alternative currencies in general and, in a narrower sense, digital currencies. Cryptocurrencies show the peculiarities of their distribution and having a public book, rather than the centralization of the control mechanism. Moreover, they are not controlled by any government, financial institution, or central bank. Therefore, the process of creating new currencies, in other words, the mechanism of money supply through emissions, does not depend on monetary and fiscal policies. New currencies, in publicly known amounts, are produced by the system they are linked to. This production is done collectively by the created system. Cryptocurrencies can be evaluated in the category of virtual currencies that are not regulated and in digital format according to the classification made by the European Central Bank. Accordingly, non-regulation states that the currency is not in a market-traded structure regulated by any official institution. However, the fact that it has a digital format indicates that money does not need to be represented theoretically with any physical material (Plassaras, 2013).

Although cryptocurrencies were initially traded in the market under the leadership of very few cryptocurrencies, they are being used by more and more people day by day due to their accepted positive features. These features can be listed as being anonymous, low transaction fees and being available anywhere with an internet connection. Blockchain technology, a technological innovation brought by cryptocurrencies and especially the predecessor currency Bitcoin, has led to transformation in many areas that affect people's lives.

Satoshi Nakamoto introduced Bitcoin to the world with an article titled "Bitcoin: A Peer-to-Peer Electronic Cash System " written in 2008. The innovation brought by Nakamoto is that it avoids double spending, with a mechanism that uses diffuse processor power that approves the transfer every 10 minutes. Nakamoto proposed a new solution to the problem of information sharing in a potentially fraudulent, insecure, and scattered processor network without using a central authority. Further, he proposed a solution to the double-spending problem using a peer-to-peer network (Nakamoto, 2008).

Nakamoto disappeared in April 2011 and despite this, the system is completely transparent and continues to perform its operations in the same manner within the framework of mathematics principles. The total speed of the Bitcoin network, which started operating in 2009, is higher than the power of the fastest computers in the world. In 2010, it was seen that Bitcoin was used for the first time in history by ordering pizza as a means of payment. Many cryptocurrencies have been derived since the first Bitcoin exchange platform, Mt-Gox, was founded. These currencies have become financial products that are

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traded on many exchange platforms and future contracts on cryptocurrencies are done in the markets. 2013 was the first turning point for Bitcoin, which was valued between $ 10 and $ 50 in 2011-2013. Rising above $ 100 in the first months of 2013, Bitcoin caught a serious break during the year and managed to reach over $ 1000 by the end of the year. This output, which is very important for the early investors of Bitcoin, was welcomed in the Bitcoin communities, while it was talked that the crypto money is now entering an uptrend period, but just two weeks later, Bitcoin, which was traded below $ 1000 again, to exceed $ 1000 for the second time. he had to wait another year or so. Bitcoin price exceeded $ 10,000 in 2017. Then, within a very short time, Bitcoin reached its highest value of $ 20,089 (btcturk.com, 2019). Now it is traded below $ 10 thousand. These price fluctuations show that the volatility in the cryptocurrency market is quite high. Therefore, the importance of modeling volatility has emerged in the cryptocurrency market.

Financial markets may act quite overdramatically, also the prices of the financial products may seem very volatile with fundamental changes. These kinds of facts have been studied over the years and are still extensively studied. Volatility as a phenomenon and a concept is at the center of modern financial markets and academic research. The link between volatility and risk has been somewhat incomprehensible, but volatility in financial markets should not be perceived as a bad thing. In fact, while naturally, existing volatility in the financial markets can be the basic structure for effective price setting, dependence in volatility refers to predictability.

The volatility studies have an extensive area in the field of financial economics. The changes in the volatility of a financial asset affect the asset pricing models used for obtaining equilibrium prices. Therefore, the mean-variance theory is a basis for investment management while derivative pricing methods are based on reliable volatility estimates. Market analysts, corporate treasurers, and portfolio managers closely monitor volatility trends since price changes can have a great impact on investment and risk management decisions.

Kalotychou et al. (2009) elaborated on the four proposals that reveal the relationship between information, volume (liquidity), and volatility. These approaches can be counted as the mixture of distributions hypothesis (MDH), the sequential information hypothesis (SIH), the dispersion of beliefs approach (DBA), and the information trading volume model (ITVM). MDH assumes that volume and volatility are simultaneously and positively related and act in partnership with a stochastic variable that is defined as the information flow. SIH shows that price volatility is predictable based on transaction volume information. The DBA approach states that financial markets involve both informed and uninformed sets of investors and that uninformed investors react to volume/price changes as if these changes include new information. On the other hand, since informed investors have homogeneous beliefs, they make transactions that reflect their prices at fair values. Therefore, uninformed investors are expected to undermine prices and make the prices highly volatile. The ITVM approach is based on the idea that volume has an important place in the information in an environment where investors receive different signals from the pricing level. Although these mentioned approaches have shortcomings, it shows that the effect of volatility on information and liquidity is indisputable.

Mandelbrot (1963) stated that the financial time series has no autocorrelated increments and are not usually stationary, but their squares present autocorrelation. Further, he showed that there exist volatility clusters which are the characteristic of financial returns mirror non-normal returns' distribution. The distribution of financial returns is not normal because of the leptokurtic shape (fat tails). The source of the volatility clusters is the direction and magnitude of the price changes. Volatility clustering occurs towards major/minor price changes after major/minor changes in both directions.

Volatility is a natural consequence of the trade that takes place with the arrival of news and the subsequent reactions of investors. After reaching information to the markets, the successive movements of market actors will force the price to reach the equilibrium point. The updates of expectations and the subsequent positions of market actors will be reflected in the liquidity of a market. Since the information flow is continuous, information, liquidity, and volatility are expected to be related.

The leverage-effect, which is introduced by Black (1976), is caused by fluctuations in prices and is assumed as one of the important stylized facts of financial time series. The leverage-effect helps to describe that unexpected negative shocks have a greater influence on the volatility than positive shocks. Thus, it can be concluded as bad news in stock markets affect volatility more than good news. Furthermore, the leverage effect shows that the volatility of an asset has asymmetric properties.

In financial studies, it is assumed that if the relationship between expected return and expected volatility is positive and future cash flow is not affected by this, the instantaneous index value will decrease and vice-versa. This situation is known as volatility feedback in the literature. This theory is based on the assumptions that there is a positive relationship between expected return and expected volatility, while at the same time volatility is being persistent. Another important topic to examine and discuss for financial time series is the long-term persistence of volatility. It is a measure of how persistent the shock of today's price will have an impact on future unconditional variance. Volatility is persistent if unconditional variance converges to infinity. The duration of volatility persistence can be measured over its half-life.

In recent years, researchers have made remarkable progress in modeling the volatility of financial markets, considering the characteristics of asset returns that were not previously considered. The time intervals between observations of financial data are constant is one of the underlying assumptions of time-series studies. However, the changes in prices and the receiving of new information can occur at irregular time intervals.

From this point of view, Engle opened a new page in the volatility modeling in his study using the UK inflation data in 1982, and Engle won the Nobel Prize in 2003 for this work. Engle (1982) proposed that, while the unconditional variance is fixed, when the conditional variance is time-dependent, this conditional variance is a function of the squares of the residuals obtained from a conditional mean equation. In ARCH model the squares of the returns/log-returns can be used instead of the squares of the residuals. This work by Engle has taken its place in the world of economy and finance under the name of ARCH (Autoregressive Conditional Heteroskedastic). The ARCH model is important not only it considers some of the empirical findings in financial asset returns, but also it finds application in many different areas such as microeconomics, macroeconomics, electronics, computer sciences and brain wave studies. The variance prediction in the ARCH model also includes information from previous periods. The model allows defining error term variance as a function of squares of previous term error terms. The ARCH model, which was insufficient to capture some of the stylized features of the financial time series mentioned above, was developed as a generalized ARCH (GARCH) model by Bollerslev (1986). In GARCH models, the conditional variance in the instant period is not only dependent on the historical values of the error terms, but also on the conditional variances in the past. Therefore, the conditional variance is affected by both past values of residuals and conditional variance values.

Bollerslev (2010) provide an easy-to-use encyclopedic-type reference guide to the long list of ARCH acronyms. Although he has listed well over 100 variants of the original model, the GARCH model extensions are defined utilizing Hentschel's approach in "rugarch" R-package of Ghalanos (2020a; 2020b) in this paper. Furthermore, along with the discrete volatility models, this paper contains the Lévy Driven Continuous GARCH (COGARCH) model, which is introduced to the literature by Klüpellberg et al (2004). COGARCH model is applied via R-based software "yuimagui" developed by Iacus and Yoshida (2018).

In this study, the eight extensions of discrete GARCH-type and Lévy Driven COGARCH models are utilized to model the volatility of Bitcoin returns. Volatility is estimated using discrete and continuous GARCH models and forecasting performances are measured. For this purpose, this paper is organized as follows. The paper first starts with a section with a short literature review on Bitcoin volatility studies. In the third section, the features of the discrete-time GARCH model and the extensions of this model are examined in detail. Afterward, the continuous-time GARCH model is introduced in the fourth section. In the following chapters, the descriptive of the data set and the findings of applied models are given, respectively. The study is ended with a brief conclusion part.

LITERATURE REVIEW

Naturally, studies in this area have accelerated with the rapid increase of the value of Bitcoin and the volume of the cryptocurrency market. Especially the introduction of other cryptocurrencies in the financial markets under the leadership of Bitcoin and the development of regulations for cryptocurrencies have gradually increased the number of studies. From this part of the study, the abbreviation BTC has been used for Bitcoin. Studies on BTC volatility can be briefly listed as follows.

Chen et al. (2016) conducted an econometric analysis of Crypto Money Index (CRIX) returns in their study. The CRIX index is created from the 30 most traded cryptocurrencies in the market. ARIMA-GARCH model was applied for conditional mean and variance on CRIX index returns. At the same time, the volatility relationship between CRIX, Exact CRIX (ECRIX) and Exact Full CRIX (EFCRIX) indices, which are from the CRIX family, was estimated by multivariate GARCH models. They concluded that student-t GARCH(1,1) satisfied the best fit in the univariate case and DCC-GARCH(1,1) was the proper process to show the volatility clustering and time-varying covariances between three CRIX indices.

Dyhrberg (2016) estimated the BTC volatility by using the GARCH model in his study of variables such as exchange rate and gold in addition to the BTC variable and concluded that BTC showed that it could be used as a hedging tool for investors who avoided risk due to bad news expectations.

Bouri et al. (2017a) examined the relationship of BTC with other assets such as gold, oil, general commodity index and US Dollar. GARCH volatility model was used in the study and, unlike Dyhrberg (2016), it was found that BTC has a weak hedge structure. Analyzes show that BTC can serve as an effective diversifier in some cases. The study also concludes that BTC is a safe harbour against weekly excessive fluctuations in Asian stocks.

Bouri et al. (2017b) examined the return-volatility relationship in the Bitcoin market around the price crash of 2013 using symmetrical and asymmetrical GARCH models. They examined the relationship between the GJR-GARCH model, which is an asymmetric model, and the volatility of the BTC series before and after 2013. After their comparison, they found that the GJR-GARCH model explained BTC returns better volatility. At the same time, the researchers found an inverse relationship between the US volatility index and BTC volatility.

Bouri et al. (2017c) try to find out whether Bitcoin can serve as a diversifier, hedge, or safe-haven for commodities in general and for energy commodities in particular. Their study shows that BTC exhibits hedge and safe-haven properties for the general commodity index and for the energy commodity index, for the entire period and the pre-crash period in 2013. They fit a multivariate GARCH model and concluded that there is a weak correlation between BTC and energy commodities.

Chu et al. (2017) modelled the volatility of the seven most popular cryptocurrencies via twelve GARCH models and compared them according to five criteria. Moreover, they made a model comparison of the best fitting models using forecasts and acceptability of value at risk estimates. The normal distributed integrated GARCH (IGARCH) was founded best-fitting model depends on information criteria for the BTC volatility. Katsiampa (2017) compared the volatility forecast of BTC with the appropriate GARCH models. It is concluded that AR-CGARCH model is an adequate model and gives the optimal fit for the volatility of daily closing prices of BTC. Urquhart (2017) compared the heterogeneous autoregression (HAR) model with the GARCH models depending on their forecasting ability for the BTC market. As a result, they illustrated that HAR models are more robust in modelling Bitcoin volatility than traditional GARCH models.

Klein et al. (2018) first analyzed and compared the conditional variance properties of BTC and Gold in their studies that questioned whether BTC is new Gold. As a result, they found a structural difference in conditional variances. They continued their work by applying BEKK-GARCH, which is a multivariate model that predicts the correlation transition between BTC and Gold. They concluded that BTC behaves as the exact opposite and it positively correlates with downward markets. Stavroyiannis and Babalos (2017) examined the dynamic properties and the relation of Bitcoin and the Standard and Poor's index, using a variety of econometric approaches through univariate FIAGARCH and multivariate BEKK-GARCH models and vector autoregressive specifications. Their results indicate that Bitcoin does not exhibit any of the hedge, diversified, or safe-haven properties; rather, its attributes are independent of US market developments.

Peng et al. (2018) estimated the conditional mean and volatility of the three cryptocurrencies including BTC using the Support Vector Regression GARCH (SVR-GARCH) model and compared with the GARCH family models. They combined the traditional GARCH model with a machine learning approach to estimate volatility. As a result of their work, they found SVR-GARCH models showed better performance than GARCH, EGARCH and GJR-GARCH models with Normal, Student's t and Skewed Student's t distributions.

Cermak (2018) stated in his thesis that the biggest obstacle for BTC to be an alternative currency is price volatility. Cermak (2018) analyzed the volatility of the currencies of the countries with the highest BTC transaction volume and BTC volatility by applying the GARCH (1,1) model. The researcher pointed out that BTC already behaves similarly to fiat currencies in China, the U.S. and the European Union but not in Japan. Moreover, as a result of the thesis, Cermak indicated that BTC acts as a safe-haven asset in China and the volatility of BTC has been steadily decreasing throughout its lifetime. The most important interpretation of the study is that BTC has a decreasing trend of volatility for six years and if this trend continues the volatility of BTC reaches the volatility levels of fiat currencies soon and become a functioning alternative to fiat currencies.

Ardia et al (2019) show that there exist regime changes in the log-returns volatility of BTC using the Markov–switching GARCH (MSGARCH) models. Moreover, they compare the MSGARCH to traditional single–regime GARCH via one–day ahead Value–at–Risk (VaR). The most significant part of the study is that they use the Bayesian approach to estimate the parameters and the VaR forecasts. They conclude the study there is strong evidence that MSGARCH models outperform single–regime specifications when predicting the VaR.

Mba and Mwambi (2020) claim that the returns of Bitcoin have a form regime-switching, therefore regime-switching models could be more successful to capture these dynamics and Markov-switching COGARCH-R-vine (MSCOGARCH) model is fitted to select portfolio. They also compare the MSCOGARCH with the single-regime COGARCH-R-vine using the expected shortfall risk. According to the comparison result, MSCOGARCH outperforms the single-regime.

Studies show that while modeling the return volatility of cryptocurrencies is a new field of study, discussions such as increasing rules in the crypto money market and taxation are important for investors' perception. The consequences of these discussions and the increased transaction costs of cryptocurrencies affect the volatility of returns. In particular, the volatility of BTC, the largest currency in the cryptocurrency market, becomes more important. In this study, the volatility of BTC will be estimated using discrete and continuous GARCH models.

GARCH MODEL

One can state the log-return of the financial time series X_t as

$$X_t = \mu_t + a_t \tag{3.1}$$

where μ_t is conditional mean and a_t is residuals. The residuals can be expressed

$$a_t = \sigma_t \varepsilon_t,\ \varepsilon_t \sim f_v(0,1) \tag{3.2}$$

where σ_t and ε_t are volatility process and innovation process respectively and $f_v(0,1)$ represent probability density function that has zero mean and unit variance. In a non-normal distribution case, is a set of additional distributional parameters which are used for the scale and shape of the distribution.

The variance equation of the GARCH (p,q) model (Bolerslev, 1986) can be expressed in two ways which are given in equations (3.3) and (3.4) as

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i a_{t-i}^2 + \sum_{i=1}^{q} \beta_i \sigma_{t-i}^2 \tag{3.3}$$

$$\sigma_t^2 = \alpha_0 + \alpha(B)a_{t-1}^2 + \beta(B)\sigma_{t-1}^2 \quad (3.4)$$

where $\alpha(B)$ and $\beta(B)$ are polynomials of degrees p and q respectively, where B denotes backward shift operator.

GARCH(p,q) model is assumed covariance stationary when $E(a_t) = 0,\ var(a_t) = \alpha_0 / (1 - \alpha(1) - \beta(1))$ and $\text{cov}(a_t, a_s) = 0$ for $t \neq s$ if and only if $\alpha(1) + \beta(1) < 1$. Bolerslev (1986) used the Maximum Likelihood Estimation Method (MLE) for the model's parameter estimation. There are many Bayesian methods that have been developed for parameter estimation of the GARCH model. Although Bayesian methods give better results, MLE method, which is a frequentist approach, is widely used due to the calculation difficulty encountered in some Bayesian methods. The MLE method by maximizing the given log-likelihood function

$$L(\omega) = ln\prod_t f_v\left(a_t, E(a_t|I_{t-1}), \sigma_t\right) \quad (3.5)$$

where $E(a_t|I_{t-1})$ denotes the expected mean of residuals and ω is set of parameters the variance equation $\omega = \left(v, \alpha_0, \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q\right)$.

The Properties of GARCH Models

The GARCH (1,1) model is successful in capturing the characteristics of financial time series. Therefore, the general features of the GARCH model is represented via GARCH (1,1) model.

i. The GARCH (1,1) model, which has a similar structure to the Autoregressive Moving Average (ARMA) model, is as follows

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \quad (3.6)$$

where $\alpha_0 > 0, \alpha_1 > 0, \beta_1 > 0$. If $\alpha_1 + \beta_1 < 1$ GARCH process is weak stationary.

ii. GARCH models can also be fitted without the need for a conditional mean model, accepting observed log-returns as residuals. First, $z_t = a_t^2 - \sigma_t^2$ is defined to show that the GARCH model is an ARMA process. So,

$$a_t^2 - z_t = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1(a_{t-1}^2 - z_{t-1})$$

$$a_t^2 = \alpha_0 + (\alpha_1 + \beta_1)a_{t-1}^2 + z_t - \beta_1 z_{t-1} \quad (3.7)$$

which is an ARMA(1,1) process on squared residuals.

iii. The unconditional variance of a_t is

$$Var(a_t) = \alpha_0 + \alpha_1 E[a_{t-1}^2] + \beta_1 E[\sigma_{t-1}^2]$$
$$= \alpha_0 + (\alpha_1 + \beta_1)E[a_{t-1}^2]$$

and since a_t is a stationary process and $\alpha_1 + \beta_1 < 1$.

$$Var(a_t) = \frac{\alpha_0}{1 - \alpha_1 - \beta_1} \quad (3.8)$$

iv. ARCH(∞) is equivalent to GARCH(1,1).

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$
$$= \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1\left(\alpha_0 + \alpha_1 a_{t-2}^2 + \beta_1 \sigma_{t-2}^2\right)$$
$$\ldots$$
$$= \alpha_0\left(1 + \beta_1 + \beta_1^2 + \beta_1^3 + \ldots\right) + \alpha_1(a_{t-1}^2 + \beta_1 a_{t-2}^2 + \beta_1^2 a_{t-3}^2 + \ldots)$$

$$\sigma_t^2 = \frac{\alpha_0}{1 - \beta_1} + \alpha_1 \sum_{i=0}^{\infty} \beta_1^i a_{t-1-i}^2 \quad (3.9)$$

So, it is possible to say that the conditional variance at time t is equal to the weighted sum of past squared residuals. Further, the weights decrease as going further back in time.

v. The unconditional variance of returns $E[\sigma^2] = \alpha_0/(1 - \alpha_1 - \beta_1)$ is plugged into the Equation (3.6) where $\alpha_0 = (1 - \alpha_1 - \beta_1)E[\sigma^2]$, then it is

$$\sigma_t^2 = (1 - \alpha_1 - \beta_1)E[\sigma^2] + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \quad (3.10)$$

vi. This form of the GARCH(1,1) model makes it easy to observe that the next period's conditional variance is equal to the weighted combination of the unconditional variance of returns, last period's squared residuals and last period's conditional variance with weights $(1 - \alpha_1 - \beta_1), \alpha_1, \beta_1$ respectively.

vii. α_1 and β_1 are considered ARCH term and GARCH term respectively in the GARCH equation given in Equation (3.6). The ARCH term α_1 measures how much the volatility shock that exists today affects volatility tomorrow. It also shows the short-run persistence of the shocks on the return variance.

viii. The model given in Equation (3.6), the coefficient of the lag value of the conditional variance, that is, the GARCH term coefficient β_1 shows the effect of the old shocks on the long-run persistence of volatility. In the literature, there are various studies that show GARCH(1,1) is an adequate model to capture the volatility clustering, one of the most important ones is Akgiray (1989). As the frequency of the observed data decreases, that is, from daily to weekly, from weekly to monthly, the ARCH effect decreases. In the model, the sum of the coefficients of the terms ARCH and GARCH $(\alpha_1 + \beta_1)$ indicates volatility persistence and measures the rate of decay of the volatility feedback effect over time. The sum of $(\alpha_1 + \beta_1)$ close to 1 indicates high persistence, meaning that volatility shocks will be felt even less in the future. Although the decay of shocks occurs over a period of more than a month, the reversion to the mean of long-run variance occurs within a few days. The fall in persistency when monthly data is used weakens the predictability of volatility based on available information. Volatility persistence makes able to predict future economic variables and the changes in the risk-return trade-off over business cycles.

ix. In relation to the volatility persistence measurement, the 'half-life' (denoted by $h2l$) defined as the number of days it takes for half of the expected reversion back towards to the expected variance value, can be calculated as following.

$$h2l = -ln2/ln(\alpha_1 + \beta_1) \tag{3.11}$$

The Estimation of GARCH Model

In this section, parameter estimation of GARCH (p, q) process, in which innovations follow a normal distribution, is done using MLE method. If $f_v(0,1)$ is assumed a normal distribution with zero mean and unit variance then the likelihood function is

$$f(a_t \Big| \omega, a_0, a_1, \ldots, a_{t-1}) = \prod_{t=p+1}^{T} \frac{1}{\sqrt{2\pi\sigma_t^2}} e^{-\frac{a_t^2}{2\sigma_t^2}} \tag{3.12}$$

and the log-likelihood function is

$$L = ln(\underline{x}|\omega) = \sum_{t=p+1}^{T} \{-\frac{1}{2} ln\,(2\pi) - \frac{1}{2} ln(\sigma_t^2) - \frac{1}{2}\frac{a_t^2}{\sigma_t^2}\} \tag{3.13}$$

For the MLE of the GARCH(p,q) process under the normality assumption the conditional likelihood of the t^{th} observation is $l_t = -\frac{1}{2}ln(\sigma_t^2) - \frac{1}{2}\frac{a_t^2}{\sigma_t^2}$ where the constant term is omitted. The partial derivatives of l_t with respect to parameter vector are

$$\frac{\partial l_t}{\partial \omega} = \left(\frac{a_t^2}{2\sigma_t^4} - \frac{1}{2\sigma_t^2}\right)\frac{\partial(\sigma_t^2)}{\partial \omega}$$

$$\frac{\partial^2 l_t}{\partial \omega\, \partial \omega^T} = \left(\frac{a_t^2}{2\sigma_t^4} - \frac{1}{2\sigma_t^2}\right)\frac{\partial^2(\sigma_t^2)}{\partial \omega\, \partial \omega^T} + \left(\frac{1}{2\sigma_t^4} - \frac{a_t^2}{\sigma_t^6}\right)\frac{\partial(\sigma_t^2)}{\partial \omega}\frac{\partial(\sigma_t^2)}{\partial \omega^T}$$

where

$$\frac{\partial(\sigma_t^2)}{\partial \omega} = \left(1, a_{t-1}^2, \ldots, a_{t-p}^2,\ \sigma_{t-1}^2, \ldots, \sigma_{t-q}^2\right)^T + \sum_{i=1}^{q} \beta_i \frac{\partial(\sigma_{t-i}^2)}{\partial \omega}$$

The gradient of the log-likelihood function is

$$\nabla L = \frac{1}{2}\sum_{i=1}^{T}\left(\frac{a_t^2}{2\sigma_t^4} - \frac{1}{2\sigma_t^2}\right)\frac{\partial(\sigma_t^2)}{\partial \omega} \tag{3.14}$$

where $L = \sum_{t=1}^{T} l_t$.

The Fisher information matrix

$$I = \frac{1}{2}\sum_{i=1}^{T} E\left[\left(\frac{a_t^2}{2\sigma_t^4} - \frac{1}{2\sigma_t^2}\right)\frac{\partial^2(\sigma_t^2)}{\partial \omega\, \partial \omega^T} + \left(\frac{1}{2\sigma_t^4} - \frac{a_t^2}{\sigma_t^6}\right)\frac{\partial(\sigma_t^2)}{\partial \omega}\frac{\partial(\sigma_t^2)}{\partial \omega^T}\right]$$

$$I = -\frac{1}{2}\sum_{i=1}^{T} E\left[\frac{1}{\sigma_t^2}\frac{\partial^2(\sigma_t^2)}{\partial \omega\, \partial \omega^T}\right] \tag{3.15}$$

So, the ML estimates can be found using the iteration scheme $\underline{\alpha}_{k+1} = \underline{\alpha}_k - J^{-1}\nabla L$ where. J^{-1} is inverse of Fischer information matrix.

Parameter estimation of Normal-GARCH (1,1) model is easily done by putting $\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2$ in log-likelihood function in equation (3.13) and taking partial derivatives of log-likelihood function with respect to parameter set $\omega = (\alpha_0,\ \alpha_1,\ \beta_1)$ and applying the Equations (3.14) and (3.15).

The Discrete Time GARCH-type Models

In this study, the maximum order of the model is determined as one since simplicity and the model is adequate (Akgiray, 1989) and better (Jafari *et al*., 2007). In addition to standard GARCH, ten of its extension models are considered. Moreover, the distribution of innovations, respectively, the normal distribution (norm), skew-normal distribution (snorm), Student's t distribution (std), skew Student's distribution (sstd), generalized error distribution (ged), skew generalized error distribution (sged), normal inverse Gaussian distribution (nig) and Johnson's SU distribution (jsu) is considered. In the continuation of this section, Ghalanos' (2020b) study "Introduction to the rugarch package – Version 1.3-8" is followed which is a well-written manual for R. The mentioned models are

i. The standard GARCH model of Bollerslev (1986) that is discussed in details in the previous section is denoted by GARCH(1,1) and given in equation (3.6) as following

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

ii. Engle and Bollerslev (1986) proposed a model that is a strictly stationary form of the standard GARCH model is denoted integrated GARCH (iGARCH) where the persistence parameter $\alpha_1 + \beta_1 = 1$. Due to unit persistence, unconditional variance, half-life and other results cannot be calculated. The stationarity of the model has been discussed and demonstrated in many studies in the literature. However, before accepting iGARCH as the preferred model, it is necessary to investigate the possibility of a structural break.

iii. Nelson (1991) proposed the exponential GARCH (EGARCH) since the positive and negative errors have the same effect on the volatility in standard GARCH models. In other words, the effect of negative and positive errors have an asymmetric effect is a weakness of the standard model. As it is mentioned before, although the negative errors have the same magnitude as positive ones, the influence of negative shocks is greater than the positive shocks in real. In light of all of these facts, the EGARCH model is based on the idea of the weighted innovations to allow for asymmetric effects between positive and negative asset returns. Thus the model can be expressed as follows

$$\log\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1} + \gamma_1\left[\left|a_{t-1}\right| - E\left(\left|a_{t-1}\right|\right)\right] \tag{3.16}$$

where $\alpha_0 > 0, \alpha_1 > 0, \beta_1 > 0$ and $\gamma_1 > 0$. α_1 is a measure of the sign effect, and γ_1 is a measure of the size effect. Further, β_1 is a persistence parameter. In the standard model, the conditional variance is a function of past innovations. The difference of the EGARCH model from the standard model is that the conditional variance is written as a function of standardized innovations.

iv. The GJRGARCH model which is developed by Glosten et al. (1993) has an indicator function that asymmetrically models positive and negative shocks on conditional variance. The conditional variance of GJRGARCH(1,1) is

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \xi_1 I_{t-1} a_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \quad (3.17)$$

where ξ_1 shows the leverage term and the indicator function I takes on value of 1 when $a_i < 0$ for $i = 0,1,2,..$ for otherwise 0. In this case, persistence is $\alpha_1 + \beta_1 + \xi_1 \kappa$ where κ is the expected value of the standardized residuals a_t below zero (efectively the probability of being below zero).

$$\boldsymbol{\kappa} = \boldsymbol{E}\left[I_{t-j} a_{t-j}^2\right] = \int_{-\infty}^{0} \boldsymbol{f}(\boldsymbol{a},\ 0, 1, \ldots) \boldsymbol{da}$$

where f is the standardized conditional density with any additional skew and shape parameters. Half-life and unconditional variance follow from the persistence parameter are calculated as in Section 3.1.

v. The asymmetric power ARCH model (APARCH) of Ding et al. (1993) is successful to capture leverage effect of financial time series, in addition to capturing the effect that the sample autocorrelation of absolute returns was usually larger than the sample autocorrelation of squared returns.So, APARCH(1,1) is

$$\sigma_t^\delta = \alpha_0 + \alpha_1 \left[\left| a_{t-1} \right| - \xi_1 a_{t-1} \right]^\delta + \beta_1 \sigma_{t-1}^\delta \quad (3.18)$$

where $-1 < \xi_1 < 1$ is leverage term and $\delta \in \mathbb{R}^+$ is a Box-Cox transformation of σ_t. The persistence of the APARCH process is $\beta_1 + \xi_1 \kappa_1$ where κ_1 is the expected value of the standardized residuals a_t under the Box–Cox transformation of the term including leverage term. Hence, it is computed by

$$\kappa_1 = E[|a| - \xi_1 a]^\delta = \int_{-\infty}^{\infty} [|a| - \xi_1 a]^\delta f(a, 0, 1, \ldots)\, da \quad (3.19)$$

One can easily obtain the extensions of the standard GARCH model using APARCH model. For instance, when $\delta = 2$ and $\xi_1 = 0$ APARCH model is reduced to a standard GARCH model.

vi. The Absolute Value GARCH (AVGARCH) model of Taylor (1986) and Schwert (1990) is a particular case of APARCH model. APARCH(1,1) model is reduced to a AVGARCH (1, 1) model of for when $\delta = 1$ and $\xi_1 = 0$.

vii. The Threshold GARCH (TGARCH) model of Zakoian (1994) is another particular case of the APARCH model when $\delta = 1$.

viii. The ALL GARCH (1, 1) model due to Hentschel (1995) has:

$$\sigma_t^\delta = \alpha_0 + \alpha_1 \sigma_{t-1}^\delta [|a_{t-1} - \eta_1| - \xi_1 (a_{t-1} - \eta_1)]^\delta + \beta_1 \sigma_{t-1}^\delta \quad (3.20)$$

where $\alpha_0 > 0, \alpha_1 \geq 0, \beta_1 > 0, -1 < \xi_1 < 1, \delta \in \mathbb{R}^+$ and $-\infty < \eta_1 < \infty$. The persistence of the model is equal to where is given in Equation (3.19).

ix. The Nonlinear GARCH (NGARCH) model due to Higgins and Bera (1992) is a particular case of the ALLGARCH model when $\xi_1 = 0$ and $\eta_1 = 0$.

x. The Nonlinear Asymmetric GARCH (NAGARCH) model of Engle and Ng (1993) for $\delta = 2$ and $\xi_1 = 0$ is another reduced form of the ALLGARCH model.

COGARCH MODEL

This section that contains the derivation of the Continuous-Time GARCH (COGARCH) process and its second-order moment properties, is constructed on the studies of Klüppelberg et al (2004 and 2011) and Arı (2019).

Klüppelberg et al. (2004) preserved the structure and basic features of the discrete-time GARCH model and built the COGARCH model by replacing the innovations in this model with the increments obtained from the Lévy process. That is, they created a continuous-time analogue using the discrete-time GARCH (1,1) model. The basic idea is to use increments derived from Lévy processes instead of the white noise process present in the discrete-time model. So, the continuous time GARCH process $(G_t)_{t\geq 0}$ can be obtained by replacing the $\beta = \alpha_0, \eta = -\log\beta_1$, and $\varphi = \alpha_1/\beta_1$ in the discrete time GARCH(1,1) model, then the continuous volatility process can be expressed as follows

$$(G_t)_{t\geq 0} = \int_0^t \sigma_{s-} dL_s \quad (4.1)$$

and the variance process $\sigma^2 = (\sigma_t^2)_{t\geq 0}$ are defined by the stochastic differential equations

$$dG_t = \sigma_{t-} dL_t \quad (4.2)$$

$$d\sigma_t^2 = (\beta - \eta\sigma_{t-})dt + \varphi\sigma_{t-} d[L,L]_t^d \quad (4.3)$$

for $t \geq 0$ and $G_0 = 0, \beta > 0, \eta > 0, \varphi \geq 0$ and $[L,L]_t^d$ is the discrete part of the quadratic variation Lévy process.

The definition shows that the size of the jumps of the process G is the same as L, and the size of the jumps is $\Delta G_t = \sigma_t \Delta L_t$ for $t \geq 0$. So, it is observed that ΔL_t has same behaviour as innovations in case of discrete GARCH models.

One can use several frequentists and Bayesian methods for the parameter estimation of COGARCH and its extensions models. Maller et al. (2008) proposed an approximation in which the model is fitted by deriving a pseudo-maximum likelihood (PML) function. In this method, PML function is maximized numerically in order to estimate the corresponding parameters. one of the advantages of the method is that it can be applied for either equally or unequally spaced time data. They suppose that the observations are $G(t_i)$ where $0 = t_0 < t_1 < \dots < t_N = T$, on the integrated COGARCH $(G_t)_{t\geq 0} = \int_0^t \sigma_{s-} dL_s$ and assumed to be stationary. The $\{t_i\}$ are assumed fixed time points. Let $a_i = Y_i = G(t_i) - G(t_{i-1})$ denote the observed returns and the difference between the observations is $\Delta t_i := t_i - t_{i-1}$. So, the observed return can be written as following $Y_i = \int_0^t \sigma_{s-} dL_s$ where L is a Lévy process with $E[L(1)] = 0$ and $E[L^2(1)] = 1$.

The purpose is to estimate (β, η, φ) from the observed $Y_1, Y_2, \dots, Y_N$ using pseudo-maximum likelihood (PML) method. Y_i is conditionally independent of $Y_{i-1}, Y_{i-2}, \dots$ given information set F_{t-1} since σ is Markovian. So, $E(Y_i|F_{t-1}) = 0$ for the conditional expectation of Y_i, and, for the conditional variance,

$$\rho_i^2 =, E(Y_i^2|F_{t-1}) = \left(\sigma^2(t_{i-1}) - \frac{\beta}{\eta-\varphi}\right)\left(\frac{e^{(\eta-\varphi)\Delta t_i}}{\eta-\varphi}\right) + \frac{\beta\Delta t_i}{\eta-\varphi} \tag{4.4}$$

$E(\sigma^2(0)) = \beta/(\eta-\varphi)$, with $\eta > \varphi$ and $E[L^2(1)] = 1$ satisfy the stationarity of the model. The pseudo-maximum likelihood function for $Y_1, Y_2, \dots, Y_N$ can be written as following with the assumption of Y_i, are conditionally $N(0,\rho_i^2)$

$$L = L(\beta, \eta, \varphi) = \sum_{i=1}^{N}\left(-\frac{1}{2}log(2\pi) - \frac{1}{2}ln\rho_i^2 - \frac{1}{2}\frac{Y_i^2}{\rho_i^2}\right) \tag{4.5}$$

Above equation (2.5) needs a calculable quantity for ρ_i^2. Hence $\sigma^2(t_{i-1})$ should be substituted by $\sigma_i^2 = \beta\Delta t_i + e^{-\eta\Delta t_i}\sigma_{i-1}^2 + \varphi e^{-\eta\Delta t_i}Y_i^2$. After substituting σ_i^2 for $\sigma^2(t_{i-1})$ and resulting modified ρ_i^2, pseudo-maximum likelihood function can be found for fitting a GARCH model to the unequally spaced series. The recursion of σ_i^2 can be easily done taking $\sigma^2(0) = \beta/(\eta-\varphi)$ as an initial value. The maximization of $L = L(\beta, \eta, \varphi)$ gives PMLEs of (β, η, φ). In this study, PMLE method is applied for parameter estimation of COGARCH model.

DATA

The Bitcoin versus USD exchange rate (BTC-USD) data is downloaded from "finance.yahoo.com" via "quantmod" package (Ryan et al., 2018) in R. The dataset consists of the daily log-returns of the BTC-USD exchange rate between 2015-01-02 and 2020-05-01. The mentioned models in Section 3.3 and Section 4. are fitted to log-returns of the BTC-USD of which time series plot is given in Figure 1. using the "rugarch" package (Ghalanos, 2020a) and later a COGARCH(1,1) by R package "yuima" by Iacus et al. (2015).

The descriptive statistics of log-returns of BTC-USD exchange rate is given below Table 1.

FINDINGS

The discrete-time GARCH-type models are compared according to the Akaike information criterion $(AIC = 2k - 2lnL(\hat{\omega}))$, Bayesian information criterion $(BIC = kln(n) - 2lnL(\hat{\omega}))$, Shibata information criterion $(SIC = ln(n+k) - 2lnL(\hat{\omega}))$ and Hannan–Quinn information criterion $(HQC = 2kln(ln(n)) - 2lnL(\hat{\omega}))$ where n: number of observations, k: number of parameters and $L(\hat{\omega})$ maximum log-likelihood of estimated parameter set $\hat{\omega}$. The model comparison results are given in Table $A1$ at Appendix A. According to the comparison results, it is understood that the jsu-ALLGARCH(1,1) model is the discrete-time volatility model that best fits the BTC-USD log returns. The ALLGARCH (1, 1) model is

$$\sigma_t^\delta = \alpha_0 + \alpha_1\sigma_{t-1}^\delta\left[\left|a_{t-1} - \eta_1\right| - \xi_1(a_{t-1} - \eta_1)\right]^\delta + \beta_1\sigma_{t-1}^\delta$$

with Johnson's SU distributed innovations. So, the log likelihood function is

$$L = ln(\underline{x}|\omega) = nln(\varphi_1) - nln(\sigma) - \frac{n}{2}ln(2\pi) + \sum_{t=p+q+1}^{T}\left[g\left(\frac{a_t}{\sigma}\right) - \frac{1}{2}\left(\varphi_2 + \varphi_1 g\left(\frac{a_t}{\sigma}\right)\right)^2\right]$$

where φ_1 and φ_2 are skew and shape parameters respectively. One can easily estimate the parameters of the jsu-ALLGARCH(1,1) by following the MLE method given in Section 3.2. The estimation of the model parameters and the time-series plot of the volatility are given in Table 2 and Figure 2 respectively.

The persistence of jsu-ALLGARCH(1,1) volatility is equal to 0.9840795 that is calculated using to $\beta_1 + \xi_1\kappa_1$ where κ_1 is given in Equation (3.19). The half-life is 43.1905 that is the number of days the volatility takes for half of the expected reversion back towards to the expected variance value. The asymmetry parameter η_1 for rotation is non-significant but the other asymmetry parameter ξ_1 that is used in persistence calculation is statistically significant. The conditional sigma power parameter δ is statistically significant.

The all discrete-time GARCH-type models assume that there is no autocorrelation between standardized residuals and no remaining ARCH effect on standardized residuals. The Ljung-Box test on standard residuals assesses the dependence of the first moments with a time lag. In other words, it tests the presence of autocorrelation between the residuals. The Ljung-Box test and the ARCH-LM test on the squares of standardized residuals evaluate the dependence of the second moments with

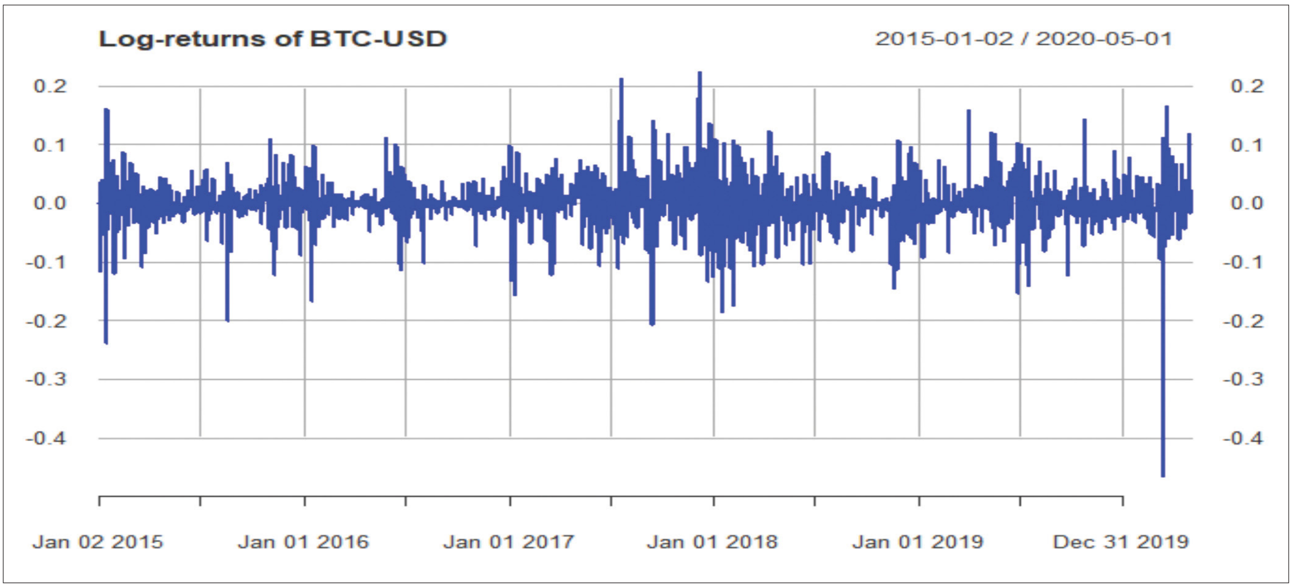

Figure 1. The plot of log-returns of BTC-USD

Table 1. Descriptive Statistics

Descriptive Statistics of Log-Returns of BTC-USD (sample size)					
min	*max*	*range*	*sum*	*median*	*mean*
-4.66E-01	2.23E-01	6.90E-01	7.26E-16	2.15E-04	3.73E-19
SE.mean	*CI.mean.0.95*	*var*	*std.dev*	*kurtosis*	*skew*
9.11E-04	1.79E-03	1.62E-03	4.02E-02	1.34E+01	-1.00E+00

Table 2. The Parameter Estimation of jsu-ALLGARCH(1,1) Model

Optimal Parameters				
parameter	estimate	Std.Error	t value	Pr(>|t|)
	0.001614	0.001248	1.29307	0.19599
	0.174181	0.01727	10.08586	0
	0.8668	0.011607	74.678	0
	0.037569	0.133442	0.28154	0.7783
	-0.223334	0.034805	-6.41667	0
	0.734313	0.269369	2.72605	0.00641
	-0.033581	0.029939	-1.12165	0.26201
	1.004228	0.04972	20.19762	0

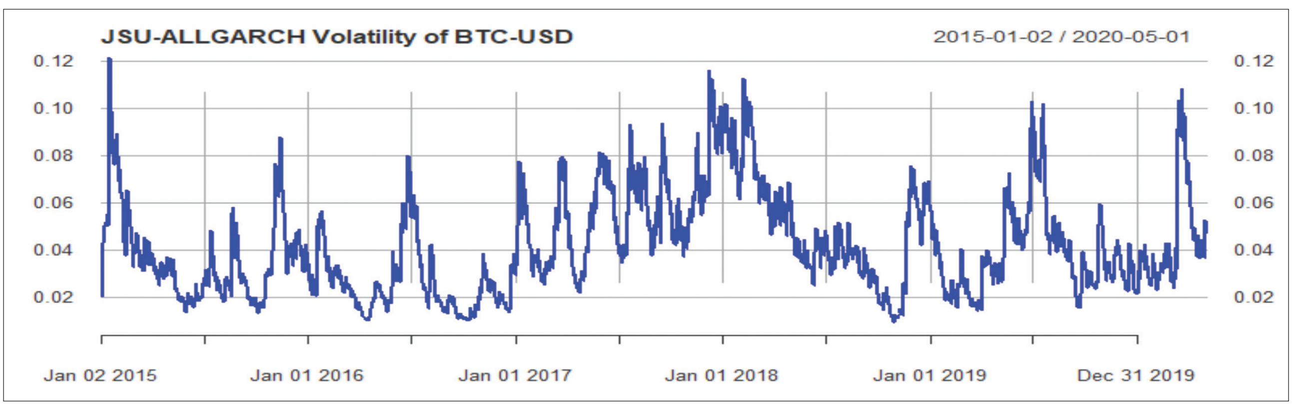

Figure 2. Time-series Plot of the JSU-ALLGARCH Volatility

Table 3. The Parameter Estimation of COGARCH(1,1) Model

	Coefficients:			Var-Cov Matrix			
	initial	Estimate	Std.Error		a0	a1	b1
a0	7.61E-05	0.001666256	5.05E-09	a0	2.55E-17	5.15E-15	7.95E-15
a1	1.71E-01	1.022156927	1.08E-01	a1	5.15E-15	1.17E-02	1.10E-02
b1	8.08E-01	1.053472221	1.04E-01	b1	7.95E-15	1.10E-02	1.09E-02

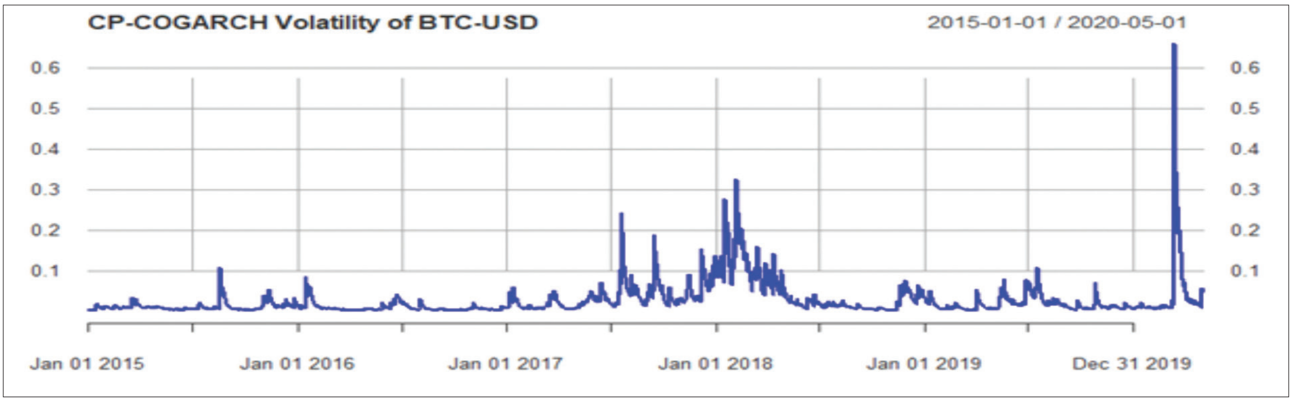

Figure 3. Time-series Plot of the COGARCH Volatility

a time lag. Findings obtained by the mentioned test are given in Appendix B in the tables Table B1, Table B2 and Table B3 respectively. According to the results, there is no autocorrelation by the weighted Ljung-Box test on standard residuals. Further, the Ljung-Box test and the ARCH-LM test on standardized squared residuals are supporting each other by concluding that there is no autocorrelation and ARCH effect on standardized squared residuals.

The Sign Bias Test proposed by Engle and Ng (1993) to verify whether previous positive and negative shocks have a different impact on heteroscedasticity The Sign-Bias Test that is designed for detecting asymmetry in the conditional variance (leverage effect) requires a finite fourth moment.. Specifically, it examines whether the standardized squared residual is predictable using (dummy) variables indicative of certain information. Three dummy variables are used in the Sign Bias Test to test for the impact of positive & negative shocks on volatility not predicted by the model, the effect of large and small negative shocks.The null hypothesis for these tests is those additional parameters corresponding to the additional (dummy) variables are zero. The most important point of these tests is that in the case of the null hypotheses are rejected, the coefficients of the additional parameters are non-zero indicating misspecification of the model. The result of the test given in Appendix B in Table B4 shows that there is no misspecification in the model and there are no leverage effects remained in the residuals.

The adjusted Pearson goodness-of-fit test for the GARCH diagnostics compares the empirical distribution of the standardized residuals with the selected theoretical distribution. The null hypothesis is that the empirical and theoretical distribution is identical, in other words, the conditional distribution is chosen appropriately. The test results in Appendix B in Table B5 indicates that the conditional distribution of the innovations is proper.

The Nyblom Stability Test that is used to determine the structural break in time series examines the constancy of all parameters in the model. However, Nyblom Stability Test does not give any information about the type and the date of structural change. The structural break/change means that the relationship between variables changes over time. The null hypothesis of the test is that the parameter values are constant and have zero variance. The test results in Appendix B in Table B6 indicate that some parameters are constant over time individually but there is a structural change in time series.

After this point, the outputs of the COGARCH process are discussed. As in the work of Klüpellberg et al. (2004), the Gaussian white noise process of the discrete model GARCH (1,1) is replaced by the compound Poisson Lévy increments to obtain a continuous process. The parameter estimation of the discrete-time GARCH model, in which innovations have a normal distribution, is used as the initial values of the COGARCH parameters. COGARCH parameter values and initial values found by the PML estimation method are given in Table 3. The plot of volatility obtained from COGARCH process is given in Figure 3.

Table 4. The Descriptive Statistics of the Lévy Increments

n	Min.	1st Q	Median	Mean	3rd Q	Max.	Std. Dev.
1947	-5.5499	-0.18091	0.003318	-0.00052	0.196352	3.480641	0.463035

Table 5. The Prediction Comparison of The Volatility Models

Model	MSE	RMSE	MAE
JSU-ALLGARCH	0.00204	0.045165	0.040054
CP-COGARCH	0.000596	0.024409	0.012804

The Lévy increments of the COGARCH process are derived from the Compound Poisson distribution and the statistics for these increments can be found in Table 8. One can obtain these increments also using distributions such as normal, variance gamma, and normal inverse Gaussian. This application is done by using the R-based software "yuimagui" developed by Iacus and Yoshida (2018).

The delta value shows the difference between the observations while estimating the parameters of the continuous process. Usually, the delta value is used as in daily data sets. The reason for this is that there is an average of 252 working days a year. However, was chosen since the larger delta value gave better prediction results in this study. The descriptive statistics of the increments is given at the following table.

The diagnostics test shows that the process is strictly stationary and the unconditional first moment of the Variance process exists. Moreover, the variance is a positive process with pseudo-log-likelihood value -3710.498.

Finally, the predictions of the discrete and continuous models are compared according to the mean square error (MSE), root mean square error (RMSE) and mean absolute error (MAE) measures, and it is concluded that the CP-COGARCH (1,1) model performs better. The comparison measures are given in Table 5.

CONCLUSION

In this study, Bitcoin volatility, which is accepted as the origin of crypto coins, has been examined. Volatility is modeled on BTC-USD exchange rate data. In volatility modeling, ten different discrete-time GARCH models with eight different innovation distributions and compound Poisson COGARCH models were used. The structure and properties of these models are given in detail. Among the discrete-time GARCH models, the most appropriate model was selected according to the information criteria and the JSU-ALLGARCH model was found to be the best-fitting model. The order of all compared models is determined to be one for convenience and simplicity. Parameter estimation of the models was made by pseudo maximum likelihood method. In the study, the most widely used models in the "rugarch" package and in the literature were preferred. Anyone can include more extensions of the GARCH model in the analysis. In the COGARCH model, innovations are derived only with compound Poisson increments. While Lévy increments derived from Variance Gamma and normal-inverse Gaussian distributions can be used, parameter estimates can also be made by the method of moments and other Bayesian methods. In future studies, Exponential COGARCH and GJR-COGARCH models, which are the derivatives of the COGARCH model, can also be used.

In the comparison of predictions made according to various measurements, it has been revealed that the continuous model has lower error values and performs better than the discrete model. It is an expected result that continuous models can make better predictions with their flexibility in high-frequency data. However, to reach this inference, a simulation study must be done. It can also be compared using value-at-risk and option pricing calculations. All R codes used in this study can be downloaded from the website https://math-stat.net/garch-and-cogarch-modelling.htm (Arı, 2020).

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Appendix A. Model Comparison

Table A1. The Comparison of GARCH-type Models with Order 1

#	Model	AIC	BIC	SIC	HQC	
1	jsu ALLGARCH11	-4.1874	-4.1645	-4.1874	-4.1789	4084.39
2	jsu AVGARCH11	-4.1873	-4.1673	-4.1873	-4.1800	4083.36
3	std ALLGARCH11	-4.1858	-4.1657	-4.1858	-4.1784	4081.85
4	sstd ALLGARCH11	-4.1858	-4.1657	-4.1858	-4.1784	4081.85
5	std AVGARCH11	-4.1855	-4.1683	-4.1855	-4.1792	4080.58
6	sstd AVGARCH11	-4.1855	-4.1683	-4.1855	-4.1792	4080.58
7	nig AVGARCH11	-4.1835	-4.1635	-4.1835	-4.1761	4079.63
8	nig ALLGARCH11	-4.1832	-4.1603	-4.1832	-4.1748	4080.34
9	jsu TGARCH11	-4.1831	-4.1660	-4.1832	-4.1768	4078.28
10	jsu APARCH11	-4.1821	-4.1621	-4.1821	-4.1747	4078.28
11	jsu EGARCH11	-4.1813	-4.1641	-4.1813	-4.1749	4076.46
12	jsu NGARCH11	-4.1811	-4.1639	-4.1811	-4.1748	4076.28
13	std TGARCH11	-4.1808	-4.1665	-4.1808	-4.1756	4075.03
14	sstd TGARCH11	-4.1808	-4.1665	-4.1808	-4.1756	4075.03
15	nig TGARCH11	-4.1800	-4.1628	-4.1800	-4.1737	4075.21
16	std APARCH11	-4.1798	-4.1626	-4.1798	-4.1735	4075.05
17	sstd APARCH11	-4.1798	-4.1626	-4.1798	-4.1735	4075.05
18	nig APARCH11	-4.1790	-4.1589	-4.1790	-4.1716	4075.23
19	nig NGARCH11	-4.1786	-4.1614	-4.1786	-4.1723	4073.84
20	std NGARCH11	-4.1785	-4.1642	-4.1785	-4.1732	4072.78
21	sstd NGARCH11	-4.1785	-4.1642	-4.1785	-4.1732	4072.78
22	std EGARCH11	-4.1785	-4.1642	-4.1785	-4.1732	4072.76
23	sstd EGARCH11	-4.1785	-4.1642	-4.1785	-4.1732	4072.76
24	nig EGARCH11	-4.1784	-4.1612	-4.1784	-4.1721	4073.69
25	ged ALLGARCH11	-4.1691	-4.1491	-4.1691	-4.1617	4065.63
26	nig NAGARCH11	-4.1684	-4.1513	-4.1685	-4.1621	4063.98
27	jsu NAGARCH11	-4.1682	-4.1510	-4.1682	-4.1619	4063.71
28	nig IGARCH11	-4.1675	-4.1560	-4.1675	-4.1633	4061.06
29	nig GJRGARCH11	-4.1675	-4.1503	-4.1675	-4.1612	4063.05
30	ged EGARCH11	-4.1673	-4.1530	-4.1673	-4.1620	4061.85
31	ged NGARCH11	-4.1671	-4.1528	-4.1671	-4.1618	4061.65
32	jsu GJRGARCH11	-4.1669	-4.1498	-4.1670	-4.1606	4062.52
33	sged EGARCH11	-4.1666	-4.1494	-4.1666	-4.1603	4062.18
34	sged NGARCH11	-4.1664	-4.1492	-4.1664	-4.1601	4062.01
35	nig GARCH11	-4.1662	-4.1519	-4.1663	-4.1610	4060.84
36	jsu IGARCH11	-4.1662	-4.1548	-4.1662	-4.1620	4059.83
37	jsu GARCH11	-4.1649	-4.1506	-4.1649	-4.1597	4059.56
38	ged IGARCH11	-4.1611	-4.1525	-4.1611	-4.1580	4053.87
39	sged IGARCH11	-4.1604	-4.1490	-4.1604	-4.1562	4054.16
40	ged NAGARCH11	-4.1603	-4.1460	-4.1603	-4.1550	4055.03
41	ged GARCH11	-4.1600	-4.1486	-4.1600	-4.1558	4053.77
42	ged GJRGARCH11	-4.1599	-4.1455	-4.1599	-4.1546	4054.62

43	std NAGARCH11	-4.1597	-4.1454	-4.1597	-4.1544	4054.46
44	sstd NAGARCH11	-4.1597	-4.1454	-4.1597	-4.1544	4054.46
45	sged NAGARCH11	-4.1596	-4.1424	-4.1596	-4.1533	4055.37
46	sged GARCH11	-4.1593	-4.1450	-4.1593	-4.1540	4054.06
47	sged GJRGARCH11	-4.1592	-4.1420	-4.1592	-4.1529	4054.97
48	std GJRGARCH11	-4.1581	-4.1438	-4.1581	-4.1529	4052.93
49	sstd GJRGARCH11	-4.1581	-4.1438	-4.1581	-4.1529	4052.93
50	std IGARCH11	-4.1574	-4.1488	-4.1574	-4.1542	4050.23
51	sstd IGARCH11	-4.1574	-4.1488	-4.1574	-4.1542	4050.23
52	std GARCH11	-4.1561	-4.1446	-4.1561	-4.1519	4049.93
53	sstd GARCH11	-4.1561	-4.1446	-4.1561	-4.1519	4049.93
54	snorm EGARCH11	-3.8211	-3.8068	-3.8211	-3.8158	3724.84
55	snorm APARCH11	-3.8160	-3.7988	-3.8160	-3.8097	3720.88
56	snorm GJRGARCH11	-3.8159	-3.8016	-3.8159	-3.8107	3719.81
57	snorm ALLGARCH11	-3.8157	-3.7956	-3.8157	-3.8083	3721.56
58	snorm GARCH11	-3.8151	-3.8036	-3.8151	-3.8109	3717.99
59	snorm NAGARCH11	-3.8150	-3.8007	-3.8150	-3.8098	3718.92
60	snorm NGARCH11	-3.8146	-3.8003	-3.8146	-3.8093	3718.48
61	norm EGARCH11	-3.8145	-3.8031	-3.8145	-3.8103	3717.44
62	snorm IGARCH11	-3.8136	-3.8050	-3.8136	-3.8105	3715.55
63	snorm TGARCH11	-3.8112	-3.7969	-3.8112	-3.8060	3715.22
64	snorm AVGARCH11	-3.8104	-3.7933	-3.8105	-3.8041	3715.47
65	norm GJRGARCH11	-3.8088	-3.7974	-3.8088	-3.8046	3711.90
66	norm APARCH11	-3.8085	-3.7942	-3.8086	-3.8033	3712.61
67	norm NAGARCH11	-3.8081	-3.7966	-3.8081	-3.8039	3711.17
68	norm ALLGARCH11	-3.8076	-3.7904	-3.8076	-3.8013	3712.70
69	norm GARCH11	-3.8048	-3.7962	-3.8048	-3.8017	3706.99
70	norm AVGARCH11	-3.8047	-3.7904	-3.8047	-3.7994	3708.89
71	norm IGARCH11	-3.8043	-3.7986	-3.8043	-3.8022	3705.47
72	norm NGARCH11	-3.8038	-3.7924	-3.8038	-3.7996	3707.03
73	norm TGARCH11	-3.8032	-3.7917	-3.8032	-3.7989	3706.38
74	sged AVGARCH11	3.9718	3.9919	3.9718	3.9792	-3859.57
75	sged TGARCH11	3.9939	4.0111	3.9939	4.0002	-3882.08
76	sged APARCH11	3.9949	4.0150	3.9949	4.0023	-3882.08
77	ged TGARCH11	3.9952	4.0095	3.9952	4.0005	-3884.36
78	ged AVGARCH11	4.8044	4.8215	4.8043	4.8107	-4671.04
79	ged APARCH11	5.2909	5.3081	5.2909	5.2973	-5144.74
80	sged ALLGARCH11	na	na	na	na	na

Table B1. Weighted Ljung-Box Test on Standardized Residuals

Weighted Ljung-Box Test on Standardized Residuals		
	statistic	p-value
Lag[1]	3.09	0.07878
Lag[2*(p+q)+(p+q)-1][2]	3.464	0.10511
Lag[4*(p+q)+(p+q)-1][5]	6.241	0.0789
df=0		

Table B2. Weighted Ljung-Box Test on Standardized Squared Residuals

Weighted Ljung-Box Test on Standardized Squared Residuals			
	statistic	p-value	df
Lag[1]	0.02735	0.8686	2
Lag[2*(p+q)+(p+q)-1][5]	1.01777	0.8556	
Lag[4*(p+q)+(p+q)-1][9]	2.16508	0.8844	

Table B3. Weighted ARCH LM Tests

Weighted ARCH LM Tests					
		Statistic	Shape	Scale	P-Value
ARCH	Lag[3]	0.2575	0.5	2	0.6118
ARCH	Lag[5]	2.1955	1.44	1.667	0.4296
ARCH	Lag[7]	2.5342	2.315	1.543	0.6054

Table B4. Sign Bias Test

Sign Bias Test		
	t-value	prob sig
Sign Bias	1.2971	0.1947
Negative Sign Bias	1.0711	0.2843
Positive Sign Bias	0.6655	0.5058
Joint Effect	3.8822	0.2745

Table B5. Adjusted Pearson Goodness-of-Fit Test

Adjusted Pearson Goodness-of-Fit Test			
	group	statistic	p-value(g-1)
1	20	17.22	0.5748
2	30	27.68	0.5348
3	40	35.97	0.6089
4	50	49.33	0.46

Table B6. Nyblom stability test

	Nyblom stability test
Parameters	Individual Stats:
omega	0.24151
alpha1	0.2009
beta1	0.26743
eta11	0.02601
eta21	0.11419
lambda	0.17937
skew	0.36294
shape	0.24832
Joint Stat:	1.9578
Asymptotic Critical Values	10% 5%
Joint Stat:	1.89 2.11
Individual Stat:	0.35 0.47