



## FINITE BLASCHKE PRODUCTS AND THE GOLDEN RATIO

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**ABSTRACT.** Geometric properties of finite Blaschke products have been intensively studied by many different aspects. In this paper, our aim is to study geometric properties of finite Blaschke products related to the golden ratio  $\alpha = \frac{1+\sqrt{5}}{2}$ . Mainly, we focus on the relationships between the zeros of canonical finite Blaschke products of lower degree and the golden ratio. We show that the geometric notions such as "golden triangle", "golden ellipse" and "golden rectangle" are closely related to the geometry of finite Blaschke products.

### 1. INTRODUCTION

The golden ratio  $\alpha = \frac{1+\sqrt{5}}{2}$  is the positive root of the quadratic equation  $x^2 - x - 1 = 0$ . So we have

$$\alpha^2 = \alpha + 1. \quad (1)$$

It is well-known that the golden ratio is almost everywhere in nature and science [12]. This ratio appears in modern research in many fields. For example, in [19], the golden ratio is used in graphs; in [10], it is proved that in any dimension all solutions between unity and the golden ratio to the optimal spherical code problem for  $N$  spheres are also solutions to the corresponding DLP (the densed local packing problem) problem. Some geometric applications of the golden ratio and its generalizations have been used to introduce new types of manifolds (see, for example, [2], [3], [11], [13], [14] and the references therein).

The rational function

$$B(z) = \beta \prod_{i=1}^n \frac{z - a_i}{1 - \bar{a}_i z}$$

is called a finite Blaschke product of degree  $n$  for the unit disc where  $|\beta| = 1$  and  $|a_i| < 1$ ,  $1 \leq i \leq n$ . Finite Blaschke products and geometric properties of them have

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been extensively studied by many different aspects (see, for example, [4], [5], [6], [7], [8], [9], [15], [16], [17], [18]). Mainly, in this paper, we study on the connection between geometric properties of Blaschke products and the golden ratio.

It is known that Blaschke products of the following form are called as canonical:

$$B(z) = z \prod_{j=1}^{n-1} \frac{z - a_j}{1 - \bar{a}_j z}, |a_j| < 1 \text{ for } 1 \leq j \leq n-1. \quad (2)$$

Note that the canonical Blaschke products correspond to finite Blaschke products vanishing at the origin. It is well-known that every Blaschke product  $B$  of degree  $n$  with  $B(0) = 0$ , is associated with a unique Poncelet curve (for more details, see [4], [5] and [8]). From [4] we know that the Poncelet curve associated with a Blaschke product of degree 3 is an ellipse.

In this paper, we investigate the relationships between the zeros of canonical finite Blaschke products of lower degree and the golden ratio. We see that some geometric notions such as "golden triangle", "golden ellipse" and "golden rectangle" are closely related to the geometry of finite Blaschke products.

## 2. BLASCHKE PRODUCTS OF DEGREE TWO

Let  $AB$  be a line segment and  $C$  be a point on the line segment  $AB$  such that  $AC$  is the greater part of  $AB$ . Recall that we say the point  $C$  divides the line segment  $AB$  in the golden ratio if  $\frac{AC}{BC} = \alpha$  [12].

In this section we consider a finite Blaschke product  $B$  of degree two of the form

$$B_a(z) = z \frac{z - a}{1 - \bar{a}z}, \quad (3)$$

with  $a \neq 0$ ,  $|a| < 1$ . From [4], we know that there exist two distinct points  $z_1$  and  $z_2$  on  $\partial\mathbb{D}$  that  $B_a(z)$  maps to  $\lambda$ , for any point  $\lambda$  on the unit circle  $\partial\mathbb{D}$ , and that the line joining  $z_1$  and  $z_2$  passes through  $a$ , the nonzero zero of  $B_a$ . Conversely, let  $L$  be any line through the point  $a$ , then for the points  $z_1$  and  $z_2$  at which  $L$  intersects  $\partial\mathbb{D}$  we have  $B_a(z_1) = B_a(z_2)$ .

Now we ask the following questions:

1) Does the point  $a$  divide the line segment  $[z_1, z_2]$  joining  $z_1$  and  $z_2$  in the golden ratio?

2) If it does, what is the number of these line segments?

The answers of these questions are given in the following theorem.

**Theorem 1.** *Let  $B_a(z) = z \frac{z-a}{1-\bar{a}z}$  be a Blaschke product with  $a \neq 0$ ,  $|a| < 1$ . There are infinitely many values of  $a$  such that there is a line segment with endpoints on the unit circle divided by  $a$  in the golden ratio. Furthermore the number of such line segments is at most two for a fixed  $a$ .*

*Proof.* Let  $a$  be a fixed point such that  $a \neq 0$ ,  $|a| < 1$  and consider the finite Blaschke product  $B_a(z) = z \frac{z-a}{1-\bar{a}z}$ . The ratio of the length of the longer part to length of the smaller part of the segment  $[z_1, z_2]$  divided by the point  $a$  gives rise

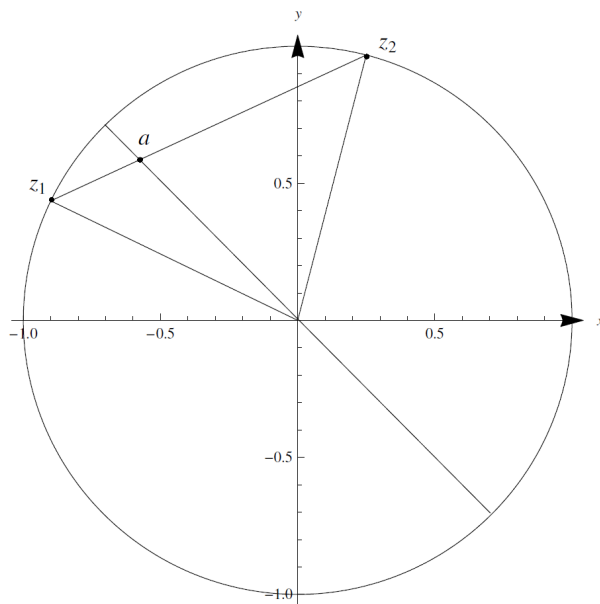


FIGURE 1. A Blaschke product of degree 2

to a continuous function of the angle  $\theta$  between the segments  $[0, a]$  and  $[z_1, z_2]$ . For  $\theta = 0$  the ratio is  $\frac{1+|a|}{1-|a|}$  and for  $\theta = \frac{\pi}{2}$  the ratio is 1. Applying the well known secant property of a circle to Figure 1, it should be

$$(1 - |a|)(1 + |a|) = l\alpha$$

where  $l$  is the length of the segment  $[z_1, a]$  and  $l\alpha$  is the length of the segment  $[a, z_2]$  [1]. Then we get

$$l = \sqrt{\frac{1 - |a|^2}{\alpha}} \quad (4)$$

Since nonlinear three distinct points determine a triangle, if the points  $0, z_1, z_2$  form a triangle it should be

$$0 < l + l\alpha < 2. \quad (5)$$

If we substitute the equation (4) in (5), we get

$$\sqrt{\frac{1 - |a|^2}{\alpha}}(\alpha + 1) < 2.$$

Then we get

$$\frac{1 + |a|}{1 - |a|} > \alpha.$$

If  $\alpha = \frac{1+|a|}{1-|a|}$ , then the line passing through the points  $z_1, z_2$  and  $a$  is the diameter of the unit circle.

For this reason, as long as  $\frac{1+|a|}{1-|a|} \geq \alpha$ , there is a segment divided by  $a$  in the golden ratio. Now we find the number of the segments divided by  $a$  in the golden ratio for a such  $a$ .

Let  $a$  be chosen such that  $\frac{1+|a|}{1-|a|} \geq \alpha$  and  $z_1$  be chosen such that the point  $a$  divides the line segment  $[z_1, z_2]$  in the golden ratio. Then by definition we have

$$\frac{|z_2 - a|}{|z_1 - a|} = \alpha. \quad (6)$$

Using the fact that  $|z| = 1$  for  $z \in \partial\mathbb{D}$ , we can write

$$B(z) = \frac{z - a}{\bar{z} - \bar{a}}, \quad z \in \partial\mathbb{D}.$$

Also, we know that  $B(z_1) = B(z_2)$  and so we obtain

$$\frac{(z_1 - a)}{(\bar{z}_1 - \bar{a})} = \frac{z_2 - a}{\bar{z}_2 - \bar{a}} \quad (7)$$

From the equation (6), we have

$$\frac{(z_2 - a)(\bar{z}_2 - \bar{a})}{(z_1 - a)(\bar{z}_1 - \bar{a})} = \alpha^2 \quad (8)$$

and from the equation (7), we find

$$\bar{z}_2 - \bar{a} = \frac{(z_2 - a)(\bar{z}_1 - \bar{a})}{(z_1 - a)} \quad (9)$$

After substitute (9) into (8), we get the equation

$$-z_2^2 + 2az_2 + \alpha a^2 - 2a\alpha^2 z_1 + \alpha^2 z_1^2 = 0. \quad (10)$$

Clearly the last equation (10) has at most two roots with respect to  $z_2$ . Hence there are at most two line segments  $[z_1, z_2]$  passing through the point  $a$  and divided by  $a$  in the golden ratio. This fact can be also seen by some geometric arguments.  $\square$

**Example 2.** Let us consider the Blaschke product  $B(z) = z \frac{z - \frac{1}{2}}{1 - \frac{1}{2}z}$ . Let  $z_1$  and  $z_2$  be two distinct points satisfying  $B(z_1) = B(z_2)$ . If the point  $a = \frac{1}{2}$  divides the line segment  $[z_1, z_2]$  in the golden ratio, from the common solutions of the equations (8) and (7), we obtain Figure 2. There the dashed line segments show the line segments which are divided by the point  $a = \frac{1}{2}$  in the golden ratio.

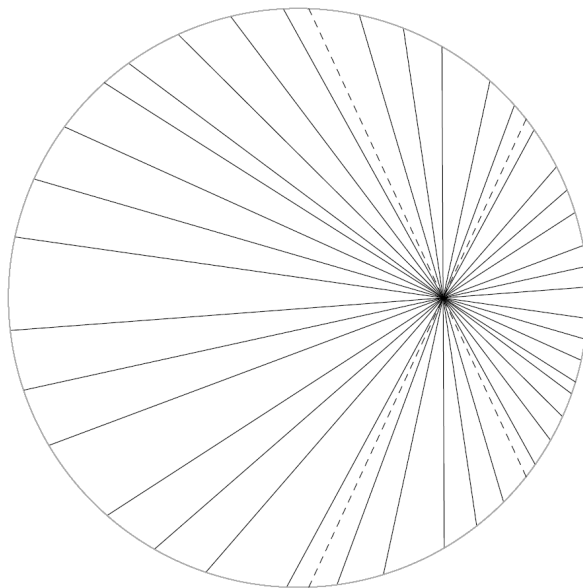


FIGURE 2. Blaschke product of degree 2 with  $a = \frac{1}{2}$ .

### 3. BLASCHKE PRODUCTS OF DEGREE THREE

In this section, we consider a finite Blaschke product  $B$  of degree three of the form

$$B(z) = z \frac{(z - a_1)(z - a_2)}{(1 - \bar{a}_1 z)(1 - \bar{a}_2 z)},$$

with distinct zeros at the points  $0$ ,  $a_1$  and  $a_2$ . It is well-known that for any specified point  $\lambda$  of the unit circle  $\partial\mathbb{D}$ , there exist 3 distinct points  $z_1$ ,  $z_2$  and  $z_3$  of  $\partial\mathbb{D}$  such that  $B(z_1) = B(z_2) = B(z_3) = \lambda$ .

We know the following theorem for a Blaschke product of degree three.

**Theorem 3.** (See [4] Theorem 1) *Let  $B$  be a Blaschke product of degree three with distinct zeros at the points  $0$ ,  $a_1$  and  $a_2$ . For  $\lambda$  on the unit circle, let  $z_1$ ,  $z_2$  and  $z_3$  denote the points mapped to  $\lambda$  under  $B$ . Then the lines joining  $z_j$  and  $z_k$  for  $j \neq k$  are tangent to the ellipse  $E$  with equation*

$$|z - a_1| + |z - a_2| = |1 - \bar{a}_1 a_2|. \quad (11)$$

*Conversely, every point on  $E$  is the point of tangency of a line segment joining two distinct points  $z_1$  and  $z_2$  on the unit circle for which  $B(z_1) = B(z_2)$ .*

The ellipse  $E$  in (11) is called a Blaschke 3-ellipse associated with the Blaschke product  $B(z)$  of degree 3. There are many studies on the ellipse  $E$  given in (11)

(see [5], [6], [7], [9], [16] and [17] for more details). For any  $\lambda \in \partial\mathbb{D}$ , we know that  $E$  circumscribed in the triangle  $\Delta(z_1, z_2, z_3)$ , where  $z_1, z_2$  and  $z_3$  are the points mapped to  $\lambda$  under  $B$ .

A golden triangle is an isosceles triangle such that the ratio of one its lateral sides to the base is the golden ratio  $\alpha = \frac{1+\sqrt{5}}{2}$ . A golden ellipse is an ellipse such that the ratio of the major axis to the minor axis is the golden ratio  $\frac{1+\sqrt{5}}{2}$  (see [12] for more details).

We have the following questions:

- 1) Are there any Blaschke 3-ellipses which are circumscribed (at least) one golden triangle?
- 2) Can a Blaschke 3-ellipse be a golden ellipse? If so, what is the number of these ellipses?

We begin with answering of the first question.

**Theorem 4.** *There are infinitely many golden triangles whose three vertices lie on the unit circle.*

*Proof.* Without loss of generality, let  $x$  and  $y$  be chosen so that  $x, y > 0$  and such that the triangle with vertices at the points  $1, -x + iy, -x - iy$  is inscribed in the unit circle. We try to determine the values of  $x$  and  $y$  such that  $x^2 + y^2 = 1$ . By the definition of a golden triangle it is sufficient to show that there are values of  $x$  and  $y$  on the unit circle such that

$$2\alpha y = \sqrt{y^2 + (x+1)^2}. \quad (12)$$

Squaring both sides of (12) and using the fact that  $x^2 + y^2 = 1$ , we obtain  $2y^2\alpha^2 = x + 1$ . Then we have

$$2(1 - x^2)\alpha^2 - x - 1 = 0$$

and so

$$2x^2\alpha^2 + x + (1 - 2\alpha^2) = 0.$$

Solving this quadric equation for  $x$  and  $y$ , we obtain  $x = 0,809017$  and  $y = 0,587785$  where  $y = \sqrt{1 - x^2}$ . So we have one golden triangle such that its vertices are on the unit circle. Then there are infinitely many golden triangles with vertices on the unit circle by rotation.  $\square$

Now we can construct some examples using some results from [5] and [9]. Recall that two sets  $\{z_1, z_2, \dots, z_n\}$  and  $\{w_1, w_2, \dots, w_n\}$  of points from  $\partial\mathbb{D}$  are interspersed if  $0 \leq \arg(z_1) < \arg(w_1) < \dots < \arg(z_n) < \arg(w_n) < 2\pi$  (see [5] for more details).

From [6], we know that the ellipses inscribed in triangles with vertices on the unit circle are precisely Blaschke 3-ellipses.

**Example 5.** *Let  $\Delta(z_1, z_2, z_3)$  be a golden triangle on the unit circle. From Theorem 2.1 in [9], we know that the Steiner ellipse  $E$  inscribed in this golden triangle has*

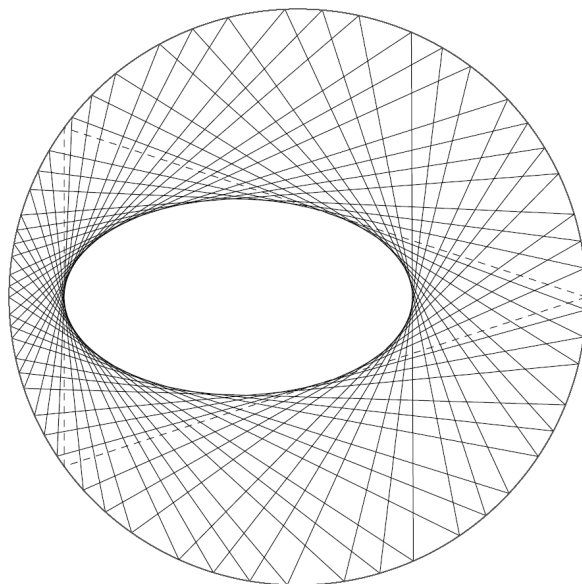


FIGURE 3. Blaschke product  $B$  of degree 3 whose Poncelet curve inscribed in (at least) one golden triangle. The dashed triangle is the golden triangle.

foci  $a_1$  and  $a_2$  with the following equation :

$$a_1 = \frac{1}{3}(z_1 + z_2 + z_3) + \sqrt{\left(\frac{1}{3}(z_1 + z_2 + z_3)\right)^2 - \frac{1}{3}(z_1z_2 + z_1z_3 + z_2z_3)}$$

and

$$a_2 = \frac{1}{3}(z_1 + z_2 + z_3) - \sqrt{\left(\frac{1}{3}(z_1 + z_2 + z_3)\right)^2 - \frac{1}{3}(z_1z_2 + z_1z_3 + z_2z_3)}.$$

Then this Steiner ellipse  $E$  is the Poncelet curve of the Blaschke product  $B(z) = z \frac{z-a_1}{1-\bar{a}_1z} \frac{z-a_2}{1-\bar{a}_2z}$ .

**Example 6.** Let  $z_1, z_2, z_3$  and  $w_1, w_2, w_3$  be triples of points which form the golden triangles  $\Delta(z_1, z_2, z_3)$  and  $\Delta(w_1, w_2, w_3)$  on the unit circle so that  $\{z_1, z_2, z_3\}$  and  $\{w_1, w_2, w_3\}$  are interspersed sets of the points. From Corollary 10 on page 97 in [5], we know that there exists a Blaschke product  $B$  of degree 3 which maps 0 to 0 such that  $B(z_j) = B(z_k)$  and  $B(w_j) = B(w_k)$  for all  $j$  and  $k$  ( $1 \leq j, k \leq 3$ ). Since we can choose the triples  $z_1, z_2, z_3$  and  $w_1, w_2, w_3$  by infinitely many different ways then clearly there are infinitely many Blaschke ellipses each of which has at least two golden triangle circumscribing them and having the vertices on the unit circle.

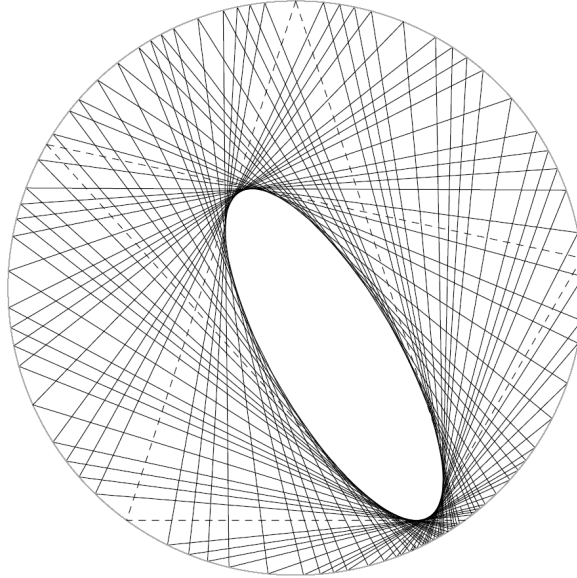


FIGURE 4. Blaschke product  $B$  of degree 3 whose Poncelet curve inscribed in (at least) two golden triangles. The dashed triangles are the golden triangles.

We have seen examples of Blaschke products of degree three of which Poncelet curves inscribed in at least one or two golden triangles.

Now we consider the answer of our second question.

**Theorem 7.** *There are infinitely many golden ellipses which are Blaschke ellipses in the unit disc.*

*Proof.* Let us take a golden ellipse with equation  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  in the unit disc. Then by definition  $\frac{a}{b} = \alpha$  and so  $a = b\alpha$ . Recall that we have the equation  $a^2 = b^2 + c^2$  where the point  $c$  is the positive focus of the ellipse. So this ellipse has foci  $-c$  and  $c$ . By the last equation and  $\alpha^2 = \alpha + 1$ , we find  $b^2\alpha = c^2$ . Combining  $a = b\alpha$  and  $b^2\alpha = c^2$  we find  $a = \pm c\sqrt{\alpha}$ . Now we consider the Blaschke product associated with this ellipse. If this ellipse is a Blaschke ellipse, it must be  $2a = 1 + c^2$  by the definition of a Blaschke ellipse. Hence we find  $c^2 \pm 2\sqrt{\alpha}c + 1 = 0$ . As these equations have only one positive root  $c = \frac{1}{2}(-\sqrt{2(-1 + \sqrt{5})} + \sqrt{2(1 + \sqrt{5})})$ , there is one golden ellipse which is a Blaschke ellipse. Since every rotation of this golden ellipse is again golden, clearly we have infinitely many golden Blaschke ellipses in the unit disc.  $\square$

We give the following definition.



**Definition 8.** Let  $B$  be a finite Blaschke product of degree  $n$  of the canonical form. If the Poncelet curve associated with  $B$  is an ellipse and this ellipse is a golden ellipse, then  $B$  is called as a golden Blaschke product.

**Example 9.** Let us consider the Blaschke product

$$B_1(z) = z \frac{(z - a_1)(z - a_2)}{(1 - \bar{a}_1 z)(1 - \bar{a}_2 z)},$$

where

$$a_1 = \frac{1}{2}(-\sqrt{2(-1 + \sqrt{5})} + \sqrt{2(1 + \sqrt{5})})$$

and  $a_2 = -a_1$ . By the proof of Theorem 7, we know that the Blaschke 3-ellipse  $E$  associated with  $B_1$  is a golden ellipse. So  $B_1(z)$  is a golden Blaschke product. The image of this golden Blaschke ellipse under the rotation transformation  $f(z) = (\frac{1}{2} + i\frac{\sqrt{3}}{2})z$  is another golden Blaschke ellipse. Clearly we find the equation of  $f(E)$  as

$$\left| z - \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)a_1 \right| + \left| z - \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)a_2 \right| = |1 - \bar{a}_1 a_2|.$$

More precisely, this image ellipse  $f(E)$  is the Poncelet curve of the following Blaschke product :

$$B_2(z) = z \frac{(z - (\frac{1}{2} + i\frac{\sqrt{3}}{2})a_1)(z - (\frac{1}{2} + i\frac{\sqrt{3}}{2})a_2)}{(1 - (\frac{1}{2} - i\frac{\sqrt{3}}{2})\bar{a}_1 z)(1 - (\frac{1}{2} - i\frac{\sqrt{3}}{2})\bar{a}_2 z)}.$$

#### 4. BLASCHKE PRODUCTS OF DEGREE FOUR

A golden rectangle is a rectangle such that the ratio of the length  $x$  of the longer side to the length  $y$  of the shorter side is the golden ratio  $\frac{1+\sqrt{5}}{2}$  (see [12] for more details).

We give the following theorem.

**Theorem 10.** There are infinitely many golden rectangles whose four vertices lie on the unit circle.

*Proof.* Without loss of generality, let  $x$  and  $y$  be chosen so that  $x, y > 0$  and such that the rectangle with vertices at the points  $x + iy, x - iy, -x - iy, -x + iy$  is inscribed in the unit circle. We try to determine the values of  $x$  and  $y$  such that  $x^2 + y^2 = 1$ . So, it is sufficient to show that there are values of  $x$  and  $y$  on the unit circle such that

$$2x = 2\alpha y.$$

We get  $x = \alpha y$  and using the facts that  $x^2 + y^2 = 1$  and  $\alpha^2 = \alpha + 1$  we obtain  $y^2(\alpha^2 + 1) = 1$  and hence

$$y = \frac{1}{\sqrt{\alpha + 2}} = 0.525731 \text{ and } x = \frac{\alpha}{\sqrt{\alpha + 2}} = 0.850651.$$

So, we have one golden rectangle such that its vertices are on the unit circle. Then, there are infinitely many golden triangles with vertices on the unit circle by rotation.  $\square$

**Example 11.** Let  $z_1, z_2, z_3, z_4$  and  $w_1, w_2, w_3, w_4$  be eight points which form the golden rectangles  $(z_1, z_2, z_3, z_4)$  and  $(w_1, w_2, w_3, w_4)$  on the unit circle so that  $\{z_1, z_2, z_3, z_4\}$  and  $\{w_1, w_2, w_3, w_4\}$  are interspersed sets of the points. From Corollary 10 on page 97 in [5], we know that there exists a Blaschke product  $B$  of degree 4 which maps 0 to 0 such that  $B(z_j) = B(z_k)$  and  $B(w_j) = B(w_k)$  for all  $j$  and  $k$  ( $1 \leq j, k \leq 4$ ). Then clearly there are infinitely many Poncelet curves associated with a finite Blaschke product of degree 4 each of which has at least two golden rectangle circumscribing them and having the vertices on the unit circle.

Using the following lemmas, we construct examples of finite Blaschke products of degree 4 whose Poncelet curves are ellipses and each of them have at least one golden rectangle.

**Lemma 12.** (See [7] Lemma 5) For any quadrilateral that is inscribed in the unit circle, an ellipse is inscribed in it if and only if the ellipse is associated with the composition of two Blaschke products of degree 2.

**Lemma 13.** (See [7] Lemma 6) For four mutually distinct points  $z_1, \dots, z_4$  on the unit circle ( $0 \leq \arg z_1 < \arg z_2 < \arg z_3 < \arg z_4 < 2\pi$ ), there exists an ellipse that is inscribed in the quadrilateral with vertices  $z_1, \dots, z_4$ . Moreover, for each quadrilateral, inscribed ellipses form a real-valued one-parameter family.

Now we give the following theorem.

**Theorem 14.** Let  $Q$  be any golden rectangle inscribed in the unit circle. Then there is at least one ellipse  $E$  inscribed in  $Q$  such that  $E$  is a Poncelet curve of a finite Blaschke product  $B$  of degree 4.

*Proof.* Let  $Q$  be any golden rectangle with the vertices  $z_1, z_2, z_3, z_4$  on the unit circle. By Lemma 13 there exists an ellipse  $E$  inscribed in  $Q$ . We know that the two foci  $a$  and  $b$  of an ellipse inscribed in any rectangle whose vertices are  $z_1, z_2, z_3, z_4$  satisfy the equations

$$\begin{aligned} & [((-z_2 + z_1)z_3 - z_1z_2)z_4 + z_1z_2z_3]\bar{a} + z_2z_4 - z_1z_3]a^2 \\ & - [z_1z_2z_3z_4(z_4 - z_3 + z_2 - z_1)\bar{a}^2 - (z_3 + z_1)(z_4 + z_2)(z_2z_4 - z_1z_3)\bar{a} \\ & + z_2z_4(z_4 + z_2) - z_1z_3(z_1 + z_3)]a + z_1z_2z_3z_4(z_2z_4 - z_1z_3)\bar{a}^2 \\ & - [(z_2^2z_3 + z_1z_2^2)z_4^2 - z_1^2z_3^2z_4 - z_1^2z_2z_3^2]\bar{a} \\ & + (z_2z_4 - z_1z_3)(z_2z_4 + z_1z_3) = 0 \end{aligned}$$

and

$$\begin{aligned} & (z_4 - z_3 + z_2 - z_1)ab - (z_2z_4 - z_1z_3)(a + b) \\ & + [(z_2 - z_1)z_3 + z_1z_2]z_4 - z_1z_2z_3 = 0 \end{aligned}$$

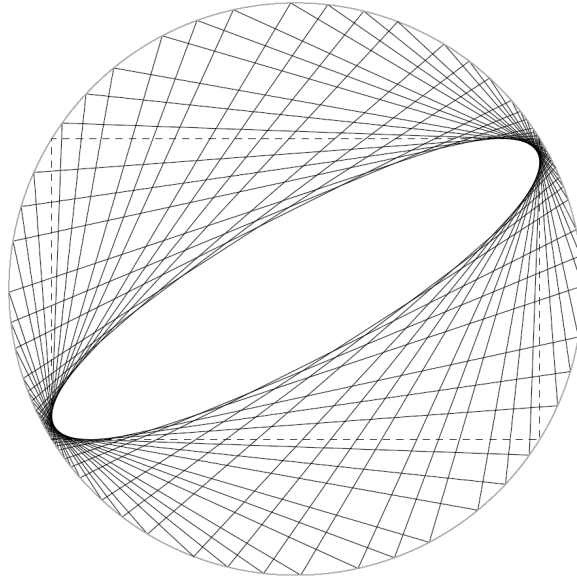


FIGURE 5. Blaschke product  $B$  of degree 4 whose Poncelet curve is an ellipse inscribed in (at least) one golden rectangle. The dashed rectangle is the golden rectangle.

given in [7]. Then by the proof of Lemma 12,  $E$  has the following equation

$$E : |z - a| + |z - b| = |1 - \bar{a}b| \sqrt{\frac{|a|^2 + |b|^2 - 2}{|a|^2 |b|^2 - 1}}$$

and  $E$  is the Poncelet curve of the finite Blaschke product  $B$  of the following form:

$$B(z) = z \frac{z - \beta}{1 - \bar{\beta}z} \frac{z^2 + (\bar{\beta}\alpha - \beta)z - \alpha}{1 - (-\bar{\alpha}\beta + \bar{\beta})z - \bar{\alpha}z^2},$$

where  $\alpha = -ab$  and  $\beta = \frac{a+b-ab(\bar{a}+\bar{b})}{1-|ab|^2}$ .  $\square$

## 5. BLASCHKE PRODUCTS OF HIGHER DEGREE

We know that regular pentagon and regular decagon have the same properties of the golden ratio among polygons (see [12] for more details). It is not known the equation of the Poncelet curves of Blaschke products of degree 5 or 10, so we cannot obtain similar theorems to the ones given in the previous sections. In these two cases, by the similar arguments used in the Example 6 and Example 11,

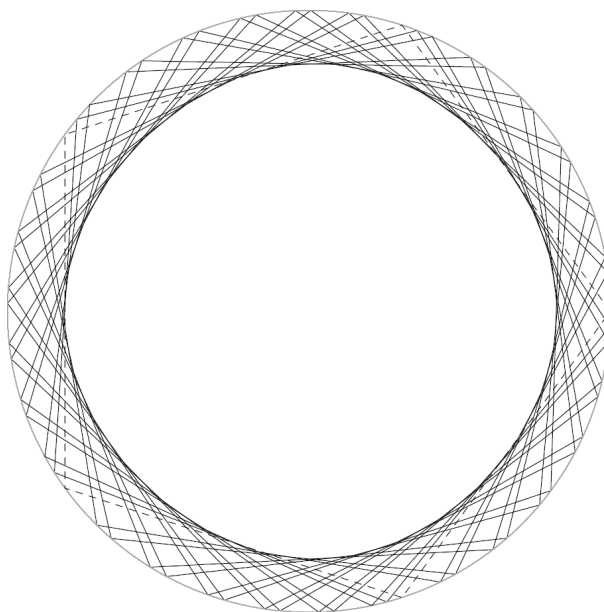


FIGURE 6. Blaschke product  $B$  of degree 5 whose Poncelet curve inscribed in (at least) one golden pentagon. The dashed pentagon is the golden pentagon.

we can obtain finite Blaschke products of degree 5 and 10 whose Poncelet curves circumscribed by at least two regular pentagon and regular decagon, respectively.

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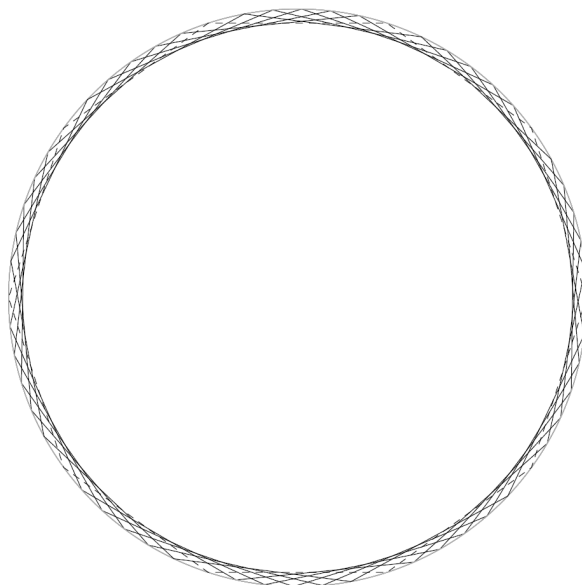


FIGURE 7. Blaschke product  $B$  of degree 10 whose Poncelet curve inscribed in (at least) one golden decagon. The dashed decagon is the golden decagon.

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