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On the Dissipative Extensions of the Conformable Fractional Sturm-Liouville Operator

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Abstract. In this work, we consider singular conformable fractional Sturm-Liouville operators defined by the expression

$$\varrho(y) = -T_\alpha^2 y(t) + \frac{\xi^2 - \frac{1}{4}}{t^2} y(t) + p(t)y(t),$$

where $0 < t < \infty$, $\xi \geq 1$ and $p(.)$ is real-valued functions defined on $[0, \infty)$ and satisfy the condition $p(.) \in L^1_{\alpha,loc}(0, \infty)$. We construct a space of boundary values for minimal symmetric singular conformable fractional Sturm-Liouville operators in limit-circle case at singular end point. Finally, we give a description of all maximal dissipative, accumulative and self-adjoint extensions of conformable fractional Sturm-Liouville operators with the help of boundary conditions.

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1. Introduction

The theory of extensions of symmetric operators is one of the main research areas of operator theory. The extensions theory developed originally by J. von Neumann [21]. Later, in [17], Rofe-Beketov investigated self adjoint extensions of a symmetric operator in terms of abstract boundary conditions with aid of linear relations. Bruk [4] and Kochubei [10] are introduced the notion of a space of boundary values. They described all maximal dissipative, accumulative, self-adjoint extensions of symmetric operators. For a more comprehensive discussion of extension theory of symmetric operators, the reader is referred to [7].

Fractional differential equations are used today in many fields such as transmission line theory, signal processing, chemical analysis, heat transfer, hydraulics of dams, material science, temperature field problems oil strata, diffusion problems, waves in liquids and gases, Schrödinger equation and fractal equation [3, 9, 11–13, 15, 16, 18–20, 22].

Recently Khalil et al. gave a new definition of the fractional derivative and fractional integral, viewed as the natural extension of the classical derivative using the limit form [8]. This new definition draws attention with its classical derivative compatibility and the product rule and division rule, which is not provided for other fractional derivatives, is provided for this new fractional derivative definition. This definition, which is called a conformable fractional derivative, has attracted a great deal of attention due to these features and many studies have been done in a short time. In their study, Khalil and his colleagues expressed Rolle's theorem and mean value theorem for functions that

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can be differentiated as a fractional-order by proving that this new fractional derivative definition provides the product rule and division rule. Later in [1], the conformable fractional derivative analysis developed by Abdeljawad; For this new definition, the concepts of left and right conformable fractional derivatives, fractional chain rule and Gronwall inequality, high order fractional integral definition for $\alpha > 1$, fractional power serial expansion and Laplace transform.

In this study, we give a description of all maximal dissipative, self adjoint and other extensions of conformable fractional singular Sturm-Liouville operators with the help of boundary conditions. A similar way was employed earlier in the classical Sturm-Liouville operator case in [2, 5–7].

In what follows, we state some concepts for conformable fractional equations that will be related to our results.

Definition 1.1 ([1]). Assume α be a positive number with $0 < \alpha < 1$. A function $x : (0, b) \longrightarrow \mathbb{R} : =(-\infty, \infty)$ the conformable fractional derivative of order α of x at $t > 0$ was defined by

$$T_\alpha x(t) = \lim_{\varepsilon\to 0} \frac{x\left(t + \varepsilon t^{1-\alpha}\right) - x(t)}{\varepsilon},$$

and the fractional derivative at 0 is defined

$$(T_\alpha x)(0) = \lim_{t\to 0} T_\alpha x(t).$$

Definition 1.2 ([1]). The conformable fractional integral starting from 0 of a function x of order $0 < \alpha \leq 1$ is defined by

$$(I_\alpha x)(t) = \int_0^t s^{\alpha-1} x(s) ds = \int_0^t x(s) d_\alpha s.$$

Lemma 1.3 ([1]). *Assume that x is a continuous function on $(0, b)$ and $0 < \alpha < 1$. Then, we have*

$$T_\alpha I_\alpha x(t) = x(t),$$

for all $t > 0$.

Theorem 1.4 ([1]). *Let $x, y : [0, b] \to \mathbb{R}$ be two functions such that x and y are conformable fractional differentiable. Then, we have*

$$\int_0^b y(t) T_\alpha(x)(t) d_\alpha t + \int_0^b x(t) T_\alpha(y)(t) d_\alpha t = x(b) y(b) - x(0) y(0).$$

Let us denote by $L^2_\alpha(0, \infty)$, the space of all complex-valued functions defined on $[0, \infty)$ such that

$$\|y\| := \sqrt{\int_0^\infty |y(t)|^2 d_\alpha t} < \infty.$$

This space is a Hilbert space with the inner product

$$(x, y) := \int_0^\infty x(t)\overline{y(t)} d_\alpha t, \text{ where } x, y \in L^2_\alpha(0, \infty).$$

The conformable α-Wronskian of x and y is defined by

$$W_\alpha[x, y]_t = x(t) T_\alpha y(t) - y(t) T_\alpha x(t), \text{ where } t \in [0, \infty).$$

2. Main Results

Here we study conformable fractional singular Sturm-Liouville differential expression

$$\varrho(y) = -T^2_\alpha y(t) + \frac{\xi^2 - \frac{1}{4}}{t^2} y(t) + p(t) y(t), \ 0 < t < \infty, \tag{2.1}$$

where $\xi \geq 1$ and $p(.)$ is real-valued functions defined on $[0, \infty)$ and satisfy the condition $p(.) \in L^1_{\alpha,loc}(0, \infty)$.

Let us denote by R_0 the closure of the minimal symmetric operator [14] generated by (2.1). We denote by D the set of all the functions $y(.)$ from $L^2_\alpha(0, \infty)$ whose first conformable fractional derivatives are locally absolutely continuous in $[0,\infty)$ and $\varrho(y) \in L^2_\alpha(0,\infty)$; D is the domain of the maximal operator R generated by expression ϱ,and $R = R_0^*$ [2]. For defect index $(0,0)$, the operator R_0 is self-adjoint, i.e., $R_0^* = R_0 = R$.

In this work, we assume that the symmetric operator R_0 has defect index $(1,1)$.

Let $u_1(t)$ and $u_2(t)$ denote the solutions of $\varrho(y) = 0$ satisfying the initial conditions

$$u_1(1) = 1,\ T_\alpha u_1(1) = 0,\ u_2(1) = 0,\ T_\alpha u_2(1) = 1.$$

Clearly, $u_1(t)$ and $u_2(t)$ are linearly independent and their Wronskians are equal to one;

$$W_\alpha[u_1,u_2]_t = W_\alpha[u_1,u_2]_1 = 1\ (1 \le t \le \infty).$$

We recall that a triple $(\mathbf{H},\Upsilon_1,\Upsilon_2)$, where $\mathbf{H}$ is a Hilbert space and Υ_1 and Υ_2 are linear maps of $D(A^*)$ into $\mathbf{H}$ is called the space of boundary values (SBV) of closed symmetric operator A in the Hilbert space H with equal defect indices, if the following two conditions hold;

(1) For every $f,h \in D(A^*)$

$$(A^*f,h)_H - (f,A^*h)_H = (\Upsilon_1 f,\Upsilon_2 h)_{\mathbf{H}} - (\Upsilon_2 f,\Upsilon_1 h)_{\mathbf{H}},$$

2. For every $h_1,h_2 \in H$, there is a vector $f \in D(A^*)$ such that $\Upsilon_1 f = h_1$, $\Upsilon_2 f = h_2$.

We consider the following linear maps of D into $\mathbb{C}$.

$$\Upsilon_1 f = W_\alpha[f,u_1]_\infty,\ \Upsilon_2 f = W_\alpha[f,u_2]_\infty,\ f \in D. \tag{2.2}$$

Theorem 2.1. *The triple $(\mathbb{C},\Upsilon_1,\Upsilon_2)$ defined by (2.2) is the space of boundary values of the operator R_0.*

In order to check the first condition of (SBV), we first prove the following lemma.

Lemma 2.2. *For arbitrary functions $y(.),\ w(.) \in D$, we have*

$$W_\alpha[y,\overline{w}]_t = W_\alpha[y,u_1]_t.W_\alpha[\overline{w},u_2]_t - W_\alpha[y,u_2]_t.W_\alpha[\overline{w},u_1]_t,\ 0 \le t \le \infty.$$

Proof. Observing that $u_1(t)$ and $u_2(t)$ are real functions, we have,

$$\begin{aligned} &W_\alpha[y,u_1]_t.W_\alpha[\overline{w},u_2]_t - W_\alpha[y,u_2]_t.W_\alpha[\overline{w},u_1]_t \\ &= (y(t)T_\alpha u_1(t) - T_\alpha y(t)u_1(t))\left(\overline{w(t)}T_\alpha u_2(t) - \overline{T_\alpha w(t)}u_2(t)\right) \\ &= (y(t)T_\alpha u_2(t) - T_\alpha y(t)u_2(t))\left(\overline{w(t)}T_\alpha u_1(t) - \overline{T_\alpha w(t)}u_1(t)\right) \\ &= y(t)\overline{T_\alpha w(t)} - T_\alpha y(t)\overline{w(t)} = W_\alpha[y,\overline{w}]_t,\ 0 \le t \le \infty. \end{aligned}$$

□

Since the operator R_0 having defect index (1,1) for $\xi \ge 1$, we get the following Lagrange formula:

$$(Ry,w)_{L^2_\alpha(0,\infty)} - (y,Rw)_{L^2_\alpha(0,\infty)} = W_\alpha[y,\overline{w}]_\infty, \tag{2.3}$$

for every $y(.),w(.) \in D$.

It follows from Lemma 2.2 and (2.3) that

$$\begin{aligned} &(\Upsilon_1 y,\Upsilon_2 w)_{\mathbb{C}} - (\Upsilon_2 y,\Upsilon_1 w)_{\mathbb{C}} \\ &= W_\alpha[y,u_1]_t.W_\alpha[\overline{w},u_2]_t - W_\alpha[y,u_2]_t.W_\alpha[\overline{w},u_1]_t = W_\alpha[y,\overline{w}]_\infty. \end{aligned}$$

Therefore, the first requirement of the SBV is fulfilled. The second requirement is proved by the use of the following lemma.

Lemma 2.3. *For any complex δ_0,δ_1, there is a function $y \in D$ satisfying*

$$W_\alpha[y,u_1]_\infty = \delta_0,\ W_\alpha[y,u_2]_\infty = \delta_1.$$

Proof. Let us denote by R_1 the closure of the minimal symmetric operator generated by $\varrho(y)$ in $1 \le t \le \infty$. For any complex numbers ρ_0,ρ_1,δ_0 and $\delta_{1,}$ there is function $y_1(.) \in D(R_1^*)$ which satisfies the following conditions

$$y(1) = \rho_0,\ T_\alpha y(1) = \rho_1,\ W_\alpha[y,u_1]_\infty = \beta_0,\ W_\alpha[y,u_2]_\infty = \beta_1. \tag{2.4}$$

Now, let us prove these relations. We consider a function $f(.) \in L^2_\alpha(1,\infty)$ satisfying

$$(f,u_1)_{L^2_\alpha(1,\infty)} = \delta_0 + \rho_1,\ (f,u_2)_{L^2_\alpha(1,\infty)} = \delta_1 - \rho_0. \tag{2.5}$$

Let $y_1(t)$ denote the solution equation $\varrho(y) = f(t),\ (1 < t < \infty)$ satisfying the initial conditions

$$y(1) = \rho_0,\ T_\alpha y(1) = \rho_1.$$

This solution can be written as

$$y_1(t) = \rho_0 u_1(t) + \rho_1 u_2(t) + \int_1^t \{u_1(t)u_2(v) - u_1(v)u_2(t)\} f(v) d_\alpha v.$$

This expression shows that $y_1(.) \in D(R_1^*)$. Let us apply Lagrange's formula to the functions $y_1(t)$ and $u_j(t)$ $(j = 1, 2)$

$$\begin{aligned}(f, u_j)_{L^2_\alpha(1,\infty)} &= (\varrho(y_1), u_j)_{L^2_\alpha(1,\infty)} \\ &= W_\alpha[y_1, u_j]_\infty - W_\alpha[y_1, u_j]_1 + (y, \varrho(u_j))_{L^2_\alpha(1,\infty)}.\end{aligned} \tag{2.6}$$

If we let

$$\varrho(u_j) = 0,\ y_1(1) = \rho_0,\ T_\alpha y_1(1) = \rho_1.$$

in (2.6), we find

$$W_\alpha[y_1, u_j]_1 = \begin{cases} -\rho_1, & j = 1 \\ \rho_0, & j = 2, \end{cases}$$

and

$$\begin{aligned}(f, u_1)_{L^2_\alpha(1,\infty)} &= W_\alpha[y_1, u_1]_\infty + \rho_1 \\ (f, u_2)_{L^2_\alpha(1,\infty)} &= W_\alpha[y_1, u_2]_\infty - \rho_0.\end{aligned}$$

From (2.5) we obtain

$$W_\alpha[y_1, u_1]_\infty = \delta_0,\ W_\alpha[y_1, u_2]_\infty = \delta_1.$$

Hence, we have proved that there exists a function $y_1(.) \in D(R_1^*)$ which satisfies (2.4).

For any complex numbers ρ_0 and ρ_1, let

$$y_2(t) = \rho_0 u_1(t) + \rho_1 u_2(t)\ (0 < t \le 1).$$

Then, let us define

$$y(t) = \begin{cases} y_2(t), & 0 < t \le 1 \\ y_1(t), & 1 \le t < \infty. \end{cases}$$

It is clearly that $y \in D$. With respect to the condition (2.4) we obtain

$$W_\alpha[y, u_1]_\infty = W_\alpha[y_1, u_1]_\infty = \delta_0,\ W_\alpha[y, u_2]_\infty = W_\alpha[y_1, u_2]_\infty = \delta_1.$$

Hence, Lemma 2.3 and Theorem 2.1 are proved. □

Recall that a linear operator S (with dense domain $D(S)$) acting in some Hilbert space $\mathcal{H}$ is called *dissipative (accumulative)* if $\operatorname{Im}(Sf, f) \ge 0$ ($\operatorname{Im}(Sf, f) \le 0$) for all $f \in D(S)$ and *maximal dissipative (maximal accumulative)* if it does not have a proper dissipative (accumulative) extension

Using Theorem 2.1 and Theorem 1.6 [7], we can state the following theorem.

Theorem 2.4. *For every number $b \in \mathbb{C},\ \operatorname{Im} b \ge 0$ or $b = \infty$, the restriction of R to the set of functions $y \in D$ satisfying either*

$$W_\alpha[y, u_1]_\infty - bW_\alpha[y, u_2]_\infty = 0, \tag{2.7}$$

or

$$W_\alpha[y, u_1]_\infty + bW_\alpha[y, u_2]_\infty = 0, \tag{2.8}$$

is respectively the maximal dissipative and accumulative extension of the symmetric operator R_0. Conversely, every maximally dissipative (accumulative) extension of R_0 is the restriction of R to the set of functions $y(.) \in D$ satisfying (2.7), (2.8). If $\operatorname{Im} b = 0$ or $b = \infty$, the conditions (2.7) and (2.8) define self-adjoint extensions of the symmetric operator R_0. For $b = \infty$, the conditions (2.7), (2.8) should be replaced by $W_\alpha[y, u_2]_\infty = 0$.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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