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Uniform Convergence of Generalized Fourier Series of Hahn-Sturm-Liouville Problem

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Abstract

In this work, we consider the Hahn-Sturm-Liouville boundary value problem defined by

$$\begin{aligned}&(Ly)(x) := \tfrac{1}{r(x)}\left[-q^{-1}D_{-\omega q^{-1},q^{-1}}(p(x)D_{\omega,q}y(x)) + v(x)y(x)\right]\\ &= \lambda y(x),\ x \in J^0_{\omega_0,a} = \{x : x = \omega_0 + (a-\omega_0)q^n,\ n = 1,2,...\}\end{aligned} \tag{0.1}$$

with the boundary conditions

$$\begin{aligned}&y(\omega_0) - h_1 p(\omega_0) D_{-\omega q^{-1},q^{-1}}y(\omega_0) = 0,\\ &y(a) + h_2 p\left(h^{-1}(a)\right) D_{-\omega q^{-1},q^{-1}}y(a) = 0,\end{aligned} \tag{0.2}$$

where $q \in (0,1)$, $\omega > 0$, $h_1, h_2 > 0$, λ is a complex eigenvalue parameter, p, v, r are real-valued continuous functions at ω_0, defined on $J_{\omega_0,h^{-1}(a)}$ and $p(x) > 0$, $r(x) > 0$, $v(x) > 0$, $x \in J_{\omega_0,h^{-1}(a)}$, $h^{-1}(a) = q^{-1}(a-\omega) > a$, $h^{-1}(\omega_0) = \omega_0$, $J_{\omega_0,a} = \{x : x = \omega_0 + (a-\omega_0)q^n,$ $n = 0,1,2...\} \cup \{\omega_0\}$. The existence of a countably infinite set of eigenvalues and eigenfunctions is proved and a uniformly convergent expansion formula in the eigenfunctions is established.

Keywords: *Hahn's Sturm-Liouville equation, Green's function, Parseval equality, eigenfunction expansion*
2010 Mathematics Subject Classification: *39A13, 39A12, 39A70, 34L10, 47B25, 47B37*

1. Introduction

In 1910, F. H. Jackson [22] introduced the quantum difference operator D_q. It is defined by

$$D_q f(x) := \frac{f(qx) - f(x)}{(qx) - x},\ x \neq 0,$$

where $q \in (0,1)$. Later Wolfgang Hahn extends this operator to his quantum difference operator $D_{\omega,q}$ defined by

$$D_{\omega,q} f(x) = \frac{f(\omega + qx) - f(x)}{\omega + (q-1)x},$$

where $q \in (0,1)$ and $\omega > 0$ (see [16], [17]). It has important applications in the construction of families of orthogonal polynomials, approximation problems [4],[12], [26], [27], [33], [8]. In [1], [5], a proper inverse of $D_{\omega,q}$ and the associated integral calculus was given. Clearly, the Hahn difference operator $D_{\omega,q}$ is also generalized to the forward difference operator (see [23], [24]). In [18], Hamza et al. established the theory of linear Hahn difference equations. They also investigate the existence and uniqueness of solution for the initial value problems for Hahn difference equations in addition, they proved Gronwall's and Bernoulli's inequalities with respect to the Hahn difference operator and investigated the mean value theorems for this calculus. Later, Hamza and Makharesh [19] studied Leibniz's rule and Fubini's theorem associated with Hahn difference operator. Sitthiwirattham [34] investigated the nonlocal boundary value problem for nonlinear

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Hahn difference equation. In [7], the authors introduce a couple of sampling theorems of Lagrange-type interpolation for $\omega, q-$integral transforms, whose kernels are either solutions or Green's function of the $\omega, q-$Hahn-Sturm-Liouville problem.
On the other hand, spectral expansions theorems play an important role in the study of partial differential equation because if we solve a partial differential equation by the Fourier method then we consider the problem of expanding an arbitrary function as a series of eigenfunctions. Hence it has been extensively studied by many authors with various methods (see [39, 38, 30, 9, 32, 10, 14, 15, 13, 20, 2, 3, 21, 31, 40, 41, 28, 36, 37, 35, 11, 29]). In [35], Steklov established uniformly convergent eigenfunction expansions for the regular Sturm-Liouville problem. In [13], the authors studied an uniformly convergent eigenfunction expansions for the Sturm-Liouville problem with impulse. In [15] and [20], the authors established uniformly convergent eigenfunction expansions for the Sturm-Liouville problem on time scales.
Sturm–Liouville problems play a major role in many physical phenomena. If these problems involve non-differentiable functions, then we follow different approaches. The Hahn calculus is one of these approaches. Recently, in [6], the regular Hahn-Sturm-Liouville problem

$$
\begin{aligned}
&-q^{-1}D_{-\omega q^{-1},q^{-1}}D_{\omega,q}y(x)+p(x)y(x)=\lambda y(x),\\
&\qquad a_1y(\omega_0)+a_2D_{-\omega q^{-1},q^{-1}}y(\omega_0)=0,\\
&\qquad\quad b_1y(b)+b_2D_{-\omega q^{-1},q^{-1}}y(b)=0,
\end{aligned}
$$

is studied where $\omega_0\le x\le b,\ \alpha\in\mathbb{C},\ a_i,b_i\in\mathbb{R}:=(-\infty,\infty),\ i=1,2,$ and $p(.)$ is a real-valued continuous function at ω_0 defined on $[\omega_0,b]$. Annaby et al. [6] define a Hilbert space of $\omega, q-$square summable functions. They discussed the formulation of the self-adjoint operator and the properties of the eigenvalues and the eigenfunctions. Furthermore, they constructed the Green's function and gave an eigenfunction expansion theorem. This yields mean square convergent expansions in eigenfunctions.
However, we note that among the existing literature, no one has studied the uniformity convergence of eigenfunction expansions. In this paper, we shall study the uniform convergence of generalized Fourier series expansions for the Hahn-Sturm-Liouville boundary value problem by using Steklov's method [35, 15, 13].

2. Notation and basic results

In this section, our aim is to present some basic concepts concerning the theory of Hahn calculus. For more details, the reader may want to consult [5], [16], [17], [6]. Throughout the paper, we let $q\in(0,1)$ and $\omega>0$.
Define $\omega_0:=\omega/(1-q)$ and let I be a real interval containing ω_0.

Definition 2.1 ([16], [17]). *Let $f:I\to\mathbb{R}$ be a function. The Hahn difference operator is defined by*

$$
D_{\omega,q}f(x)=\begin{cases}\frac{f(\omega+qx)-f(x)}{\omega+(q-1)x}, & x\neq\omega_0,\\ f'(\omega_0), & x=\omega_0,\end{cases}\tag{2.1}
$$

provided that f is differentiable at ω_0. In this case, we call $D_{\omega,q}f$, the $\omega,q-$derivative of f.

Remark 2.2. *The Hahn difference operator unifies two well known operators. When $q\to1$, we get the forward difference operator, which is defined by*

$$
\Delta_\omega f(x):=\frac{f(\omega+x)-f(x)}{(\omega+x)-x},\ x\in\mathbb{R}.
$$

When $\omega\to0$, we get the Jackson $q-$difference operator, which is defined by

$$
D_qf(x):=\frac{f(qx)-f(x)}{(qx)-x},\ x\neq0.
$$

Furthermore, under appropriate conditions, we have

$$
\lim_{\substack{q\to1\\ \omega\to0}}D_{\omega,q}f(x)=f'(x).
$$

In what follows, we present some important properties of the $\omega,q-$derivative.

Theorem 2.3 ([5]). *Let $f,g:I\to\mathbb{R}$ be $\omega,q-$differentiable at $x\in I$ and $h(x):=\omega+qx$, then we have for all $x\in I:$*

i) $D_{\omega,q}(af+bg)(x)=aD_{\omega,q}f(x)+bD_{\omega,q}g(x),\ a,b\in I,$
ii) $D_{\omega,q}(fg)(x)=D_{\omega,q}(f(x))g(x)+f(\omega+xq)D_{\omega,q}g(x),$
iii) $D_{\omega,q}\left(\frac{f}{g}\right)(x)=\frac{D_{\omega,q}(f(x))g(x)-f(x)D_{\omega,q}g(x)}{g(x)g(\omega+xq)},$
iv) $D_{\omega,q}f\left(h^{-1}(x)\right)=D_{-\omega q^{-1},q^{-1}}f(x),\ h^{-1}(x)=q^{-1}(x-\omega).$

The concept of the $\omega,q-$integral of the function f can be defined as follows.

Definition 2.4 (Jackson-Nörlund Integral [5]). *Let $f:I\to\mathbb{R}$ be a function and $a,b,\omega_0\in I$. We define $\omega,q-$integral of the function f from a to b by*

$$
\int_a^b f(x)\,d_{\omega,q}(x):=\int_{\omega_0}^b f(x)\,d_{\omega,q}(x)-\int_{\omega_0}^a f(x)\,d_{\omega,q}(x),
$$

where

$$
\int_{\omega_0}^x f(t)\,d_{\omega,q}(t):=((1-q)x-\omega)\sum_{n=0}^{\infty}q^nf\left(\omega\frac{1-q^n}{1-q}+xq^n\right),x\in I
$$

provided that the series converges at $x=a$ and $x=b$. In this case, f is called $\omega,q-$integrable on $[a,b]$.

The following properties of $\omega,q-$integration can be found in [5].

Lemma 2.5 ([6]). *Let $f,g:I\to\mathbb{R}$ be $\omega,q-$integrable on I, $a,b,c\in I$, $a<c<b$ and $\alpha,\beta\in\mathbb{R}$. Then the following formulas hold:*

i) $\int_a^b\{\alpha f(x)+\beta g(x)\}d_{\omega,q}(x)=\alpha\int_a^b f(x)d_{\omega,q}(x)+\beta\int_a^b g(x)d_{\omega,q}(x)$,

ii) $\int_a^a f(x)d_{\omega,q}(x)=0$,

iii) $\int_a^b f(x)d_{\omega,q}(x)=\int_a^c f(x)d_{\omega,q}(x)+\int_c^b f(x)d_{\omega,q}(x)$,

iv) $\int_a^b f(x)d_{\omega,q}(x)=-\int_b^a f(x)d_{\omega,q}(x)$.

Next, we present the $\omega,q-$integration by parts.

Lemma 2.6 ([5]). *Let $f,g:I\to\mathbb{R}$ be $\omega,q-$integrable on I, $a,b\in I$, and $a<b$. Then the following formula holds:*

$$\int_a^b f(x)D_{\omega,q}g(x)d_{\omega,q}(x)+\int_a^b g(\omega+qx)D_{\omega,q}f(x)d_{\omega,q}(x)$$
$$=f(b)g(b)-f(a)g(a). \tag{2.2}$$

The next result is the fundamental theorem of Hahn calculus.

Theorem 2.7 ([5]). *Let $f:I\to\mathbb{R}$ be continuous at ω_0. Define*

$$F(x):=\int_{\omega_0}^x f(t)d_{\omega,q}(t),\ x\in I.$$

Then F is continuous at ω_0. Moreover, $D_{\omega,q}F(x)$ exists for every $x\in I$ and $D_{\omega,q}F(x)=f(x)$. Conversely,

$$\int_a^b D_{\omega,q}f(x)d_{\omega,q}(x)=f(b)-f(a).$$

Let $L^2_{\omega,q}((\omega_0,a),r)$, $\omega_0<a<\infty$ be the space of all complex-valued functions defined on $[\omega_0,a]$ such that

$$\|f\|:=\left(\int_{\omega_0}^a|f(x)|^2r(x)d_{\omega,q}x\right)^{1/2}<\infty,$$

where r is a real-valued continuous function at ω_0 defined on $[\omega_0,a]$ and $r(x)>0$ for all $x\in[\omega_0,a]$. The space $L^2_{\omega,q}((\omega_0,a),r)$ is a separable Hilbert space with the inner product

$$(f,g):=\int_{\omega_0}^a f(x)\overline{g(x)}r(x)d_{\omega,q}x,\ f,g\in L^2_{\omega,q}((\omega_0,a),r)$$

(see [5]).

The $\omega,q-$Wronskian of $y(.)$, $z(.)$ is defined to be

$$W_{\omega,q}(y,z)(x):=y(x)D_{\omega,q}z(x)-z(x)D_{\omega,q}y(x),\ x\in[\omega_0,a]. \tag{2.3}$$

Now, we recall that the following well-known theorems and definition.

Theorem 2.8 (Hilbert-Schmidt). *Let A be a compact self-adjoint operator mapping a Hilbert space H into itself. Then there is an orthonormal system $\varphi_1,\varphi_2,...$ of eigenvectors of A, with corresponding nonzero eigenvalues $\lambda_1,\lambda_2,...$, such that every element $x\in H$ has a unique representation of the form*

$$x=\sum_n c_n\varphi_n+x',$$

where x' satisfies the condition $Ax'=0$. Moreover

$$Ax=\sum_n\lambda_n c_n\varphi_n,$$

and

$$\lim_{n\to\infty}\lambda_n=0$$

in the case where there are infinitely many nonzero eigenvalues ([25]).

Theorem 2.9 ([31]). *If*

$$\sum_{i,k=1}^{\infty}|a_{ik}|^2<\infty \tag{2.4}$$

then the operator A defined by the formula

$$A\{x_i\}=\{y_i\}\ (i\in\mathbb{N}:=(1,2,3,...)),$$

where

$$y_i=\sum_{k=1}^{\infty}a_{ik}x_k\ (i\in\mathbb{N}) \tag{2.5}$$

is compact in the sequence space l^2.

Definition 2.10. *A complex-valued function $M(x,t)$ of two variables with $\omega_0\le x,t\le a$ is called the $\omega,q-$Hilbert-Schmidt kernel in the space $L^2_{\omega,q}((\omega_0,a),r)$ if*

$$\int_{\omega_0}^a\int_{\omega_0}^a|M(x,t)|^2r(x)r(t)d_{\omega,q}xd_{\omega,q}t<\infty.$$

3. Main Results

Consider the Hahn-Sturm-Liouville difference equation

$$(Ly)(x) := \frac{1}{r(x)}\left[-q^{-1}D_{-\omega q^{-1},q^{-1}}(p(x)D_{\omega,q}y(x))+v(x)y(x)\right]$$
$$= \lambda y(x),\ x\in J^0_{\omega_0,a} = \{x : x = \omega_0 + (a-\omega_0)q^n,\ n\in\mathbb{N}\} \tag{3.1}$$

with the boundary conditions

$$y(\omega_0) - h_1 p(\omega_0) D_{-\omega q^{-1},q^{-1}} y(\omega_0) = 0,$$
$$y(a) + h_2 p\left(h^{-1}(a)\right) D_{-\omega q^{-1},q^{-1}} y(a) = 0, \tag{3.2}$$

where $h_1, h_2 > 0$, λ is a complex eigenvalue parameter, p, v, r are real-valued continuous functions at ω_0, defined on $J_{\omega_0,h^{-1}(a)}$ and $p(x) > 0$, $r(x) > 0$, $v(x) > 0$, $x \in J_{\omega_0,h^{-1}(a)}$, $h^{-1}(a) = q^{-1}(a-\omega) > a$, $h^{-1}(\omega_0) = \omega_0$, $J_{\omega_0,a} = \{x : x = \omega_0 + (a-\omega_0)q^n,\ n = 0,1,2,...\} \cup \{\omega_0\}$.
Next denote by $\mathscr{D}$ the linear set of all functions $y \in L^2_{\omega,q}(J_{\omega_0,a}, r)$ such that y and $pD_{\omega,q}y$ are continuous functions at ω_0 defined on $J_{\omega_0,h^{-1}(a)}$, $Ly \in L^2_{\omega,q}(J_{\omega_0,a}, r)$ and satisfying the boundary conditions (3.2).
Now we define the operator $T : \mathscr{D} \subset L^2_{\omega,q}(J_{\omega_0,a}, r) \to L^2_{\omega,q}(J_{\omega_0,a}, r)$ as follows. The domain of definition of T is $\mathscr{D}$ and we put $Ty = Ly$ for $y \in \mathscr{D}$.

Theorem 3.1. *For any y and z in $\mathscr{D}$, the following relations hold:*

(i) *the operator T is self-adjoint,*
(ii) *the operator T is positive, i.e.,*

$(Ty, y) > 0,\ y \in \mathscr{D},\ y \neq 0.$

Proof. (i) For all $y, z \in \mathscr{D}$, it follows from the formula (2.2) that

$$\begin{aligned}
&(Ty,z)\\
&= \int_{\omega_0}^{a}\left[-q^{-1}D_{-\omega q^{-1},q^{-1}}(p(x)D_{\omega,q}y(x))+v(x)y(x)\right]\overline{z(x)}d_{\omega,q}x\\
&= \int_{\omega_0}^{a} -q^{-1}D_{-\omega q^{-1},q^{-1}}(p(x)D_{\omega,q}y(x))\overline{z(x)}d_{\omega,q}x\\
&+\int_{\omega_0}^{a} v(x)y(x)\overline{z(x)}d_{\omega,q}x\\
&= -q^{-1}\int_{\omega_0}^{a} D_{\omega,q}\left[p\left(h^{-1}(x)\right)D_{\omega,q}y\left(h^{-1}(x)\right)\right]\overline{z(x)}d_{\omega,q}x\\
&+\int_{\omega_0}^{a} v(x)y(x)\overline{z(x)}d_{\omega,q}x\\
&= -q^{-1}\left[p\left(h^{-1}(x)\right)D_{\omega,q}y\left(h^{-1}(x)\right)\overline{z(x)}\right]\Big|_{\omega_0}^{a}\\
&+q^{-1}\int_{\omega_0}^{a} D_{\omega,q}y(x)p(x)\overline{D_{\omega,q}z(x)}d_{\omega,q}x+\int_{\omega_0}^{a} v(x)y(x)\overline{z(x)}d_{\omega,q}x\\
&= -q^{-1}\left[p\left(h^{-1}(x)\right)D_{\omega,q}y\left(h^{-1}(x)\right)\overline{z(x)}\right]\Big|_{\omega_0}^{a}\\
&+q^{-1}\left[y(x)p(x)\overline{D_{\omega,q}z(x)}\right]\Big|_{\omega_0}^{a}\\
&-q^{-1}\int_{\omega_0}^{a} y(h(x))D_{\omega,q}\left(p(x)\overline{D_{\omega,q}z(x)}\right)d_{\omega,q}x\\
&+\int_{\omega_0}^{a} v(x)y(x)\overline{z(x)}d_{\omega,q}x\\
&= q^{-1}\left[y(x)p(x)\overline{D_{\omega,q}z(x)}-p\left(h^{-1}(x)\right)D_{\omega,q}y\left(h^{-1}(x)\right)\overline{z(x)}\right]\Big|_{\omega_0}^{a}\\
&-q^{-1}\int_{\omega_0}^{h(a)} y(u)D_{\omega,q}\left(p\left(h^{-1}(u)\right)\overline{D_{\omega,q}z\left(h^{-1}(u)\right)}\right)d_{\omega,q}u\\
&+\int_{\omega_0}^{a} v(x)y(x)\overline{z(x)}d_{\omega,q}x\\
&= q^{-1}\left[y(x)p(x)\overline{D_{\omega,q}z(x)}-p\left(h^{-1}(x)\right)D_{\omega,q}y\left(h^{-1}(x)\right)\overline{z(x)}\right]\Big|_{\omega_0}^{a}\\
&+q^{-1}(a-h(a))y(a)\overline{D_{-\omega q^{-1},q^{-1}}z(a)}\\
&-q^{-1}\int_{\omega_0}^{a} y(x)\overline{D_{-\omega q^{-1},q^{-1}}z(x)}d_{\omega,q}x+\int_{\omega_0}^{a} v(x)y(x)\overline{z(x)}d_{\omega,q}x
\end{aligned}$$

$$= q^{-1}\left[\begin{array}{c} y(x)\,p\left(h^{-1}(x)\right)\overline{D_{-\omega q^{-1},q^{-1}}z(x)} \\ -p\left(h^{-1}(x)\right)D_{-\omega q^{-1},q^{-1}}y\left(h^{-1}(x)\right)\overline{z(x)} \end{array}\right]|_{\omega_0}^{a}$$

$$+\int_{\omega_0}^{a} y(x)\left[-q^{-1}\overline{D_{-\omega q^{-1},q^{-1}}z(x)}+v(x)\overline{z(x)}\right]d_{\omega,q}x.$$

Then by the boundary conditions (3.2), we have

$$(Ty,z)=(y,Tz). \tag{3.3}$$

Obviously, the domain $\mathscr{D}$ of definition of T is a linear dense subset in $L^2_{\omega,q}(J_{\omega_0,a},r)$. Hence we get the desired result.
(ii) For $y\in\mathscr{D}$, $y\neq 0$, we have

$$(Ty,y)$$

$$=\int_{\omega_0}^{a}\left[-q^{-1}D_{-\omega q^{-1},q^{-1}}(p(x)D_{\omega,q}y(x))+v(x)y(x)\right]\overline{y(x)}d_{\omega,q}x$$

$$=\int_{\omega_0}^{a}-q^{-1}D_{-\omega q^{-1},q^{-1}}(p(x)D_{\omega,q}y(x))\overline{y(x)}d_{\omega,q}x$$

$$+\int_{\omega_0}^{a}v(x)\left|y(x)\right|^2 d_{\omega,q}x$$

$$=-q^{-1}\int_{\omega_0}^{a}D_{\omega,q}\left[p\left(h^{-1}(x)\right)D_{\omega,q}y\left(h^{-1}(x)\right)\right]\overline{y(x)}d_{\omega,q}x$$

$$+\int_{\omega_0}^{a}v(x)\left|y(x)\right|^2 d_{\omega,q}x$$

$$=-q^{-1}\left[p\left(h^{-1}(x)\right)D_{\omega,q}y\left(h^{-1}(x)\right)\overline{y(x)}\right]|_{\omega_0}^{a}$$

$$+q^{-1}\int_{\omega_0}^{a}p(x)\left|D_{\omega,q}y(x)\right|^2 d_{\omega,q}x+\int_{\omega_0}^{a}v(x)\left|y(x)\right|^2 d_{\omega,q}x$$

$$=q^{-1}h_1\left|p(\omega_0)D_{\omega,q}y(\omega_0)\right|^2+q^{-1}h_2\left|p\left(h^{-1}(a)\right)D_{\omega,q}y\left(h^{-1}(a)\right)\right|^2$$

$$+q^{-1}\int_{\omega_0}^{a}p(x)\left|D_{\omega,q}y(x)\right|^2 d_{\omega,q}x+\int_{\omega_0}^{a}v(x)\left|y(x)\right|^2 d_{\omega,q}x>0,$$

since $h_1,h_2>0$ and $p(x)$ and $v(x)$ are positive on the set $J_{\omega_0,a}$. □

Let $u(x)$ and $\chi(x)$ the solutions of the equation $Ty=0$ satisfying the initial conditions

$$u(\omega_0)=h_1,\ p(\omega_0)D_{-\omega q^{-1},q^{-1}}u(\omega_0)=1,$$
$$\chi(a)=-h_2,\ p\left(h^{-1}(a)\right)D_{-\omega q^{-1},q^{-1}}\chi(a)=1.$$

Lemma 3.2. *Zero is not an eigenvalue of the operator T.*

Proof. Let $y\in L^2_{\omega,q}(J_{\omega_0,a},r)$ and $Ty=0$. Then

$$-q^{-1}D_{-\omega q^{-1},q^{-1}}(p(x)D_{\omega,q}y(x))+v(x)y(x)=0,$$

and $y(x)=c_1u(x)+c_2\chi(x)$. Substituting this in the boundary conditions (3.2) we find that $c_1=c_2=0$; i.e., $y=0$. □

It follows from Lemma 3.2, there exist the inverse operator T^{-1}. In order to describe the operator T^{-1} we use the Green's function method. Let

$$G(x,t)=-\frac{1}{pW_{\omega,q}(u,\chi)}\left\{\begin{array}{ll} u(x)\chi(t), & \omega_0\le x\le t\le a \\ u(t)\chi(x), & \omega_0\le t\le x\le a. \end{array}\right. \tag{3.4}$$

Let $K:L^2_{\omega,q}(J_{\omega_0,a},r)\to L^2_{\omega,q}(J_{\omega_0,a},r)$ be the integral operator defined by the formula

$$(Kf)(x)=\int_{\omega_0}^{a}G(x,t)f(t)r(t)d_{\omega,q}t\ \left(f\in L^2_{\omega,q}(J_{\omega_0,a},r)\right). \tag{3.5}$$

Then we have

Theorem 3.3. *The Green's function $G(x,t)$ defined by the formula (3.4) is a $\omega,q-$Hilbert-Schmidt kernel.*

Proof. By the upper half of the formula (3.4), we have

$$\int_{\omega_0}^{a} r(x)\, d_{\omega,q}x \int_{\omega_0}^{x} |G(x,t)|^2 r(t)\, d_{\omega,q}t < \infty,$$

and by the lower half of (3.4), we have

$$\int_{\omega_0}^{a} r(x)\, d_{\omega,q}x \int_{x}^{a} |G(x,t)|^2 r(t)\, d_{\omega,q}t < \infty$$

since the inner integral exists and is a linear combination of the products $u(.)\chi(.)$ and these products belong to $L^2_{\omega,q}(J_{\omega_0,a},r) \times L^2_{\omega,q}(J_{\omega_0,a},r)$ because each of the factors belongs to $L^2_{\omega,q}(J_{\omega_0,a},r)$. Then, we obtain

$$\int_{\omega_0}^{a}\int_{\omega_0}^{a} |G(x,t)|^2 r(x)\, r(t)\, d_{\omega,q}x d_{\omega,q}t < \infty. \tag{3.6}$$

□

Theorem 3.4. *The operator K defined by the formula (3.5) is compact in the space $L^2_{\omega,q}(J_{\omega_0,a},r)$.*

Proof. Let $\phi_i = \phi_i(t)$ $(i \in \mathbb{N})$ be a complete, orthonormal basis of $L^2_{\omega,q}(J_{\omega_0,a},r)$. Since $G(x,t)$ is a $\omega,q-$Hilbert-Schmidt kernel, one can define

$$x_i = (f,\phi_i) = \int_{\omega_0}^{a} f(t)\,\overline{\phi_i(t)}\, r(t)\, d_{\omega,q}t,$$

$$y_i = (g,\phi_i) = \int_{\omega_0}^{a} g(t)\,\overline{\phi_i(t)}\, r(t)\, d_{\omega,q}t,$$

$$a_{ik} = \int_{\omega_0}^{a}\int_{\omega_0}^{a} G(x,t)\,\overline{\phi_i(x)\phi_k(t)}\, r(x)\, r(t)\, d_{\omega,q}x d_{\omega,q}t \ (i \in \mathbb{N}).$$

Then, $L^2_{\omega,q}(J_{\omega_0,a},r)$ is mapped isometrically l^2. Consequently, our integral operator transforms into the operator defined by the formula (2.5) in the space l^2 by this mapping and the condition (3.6) is translated into the condition (2.4). By Theorem 2.9, this operator is compact. Therefore, the original operator K is compact. □

It is evident that $K = T^{-1}$. In $L^2_{\omega,q}(J_{\omega_0,a},r)$, the completeness of the system of all eigenvectors of T is equivalent to the completeness of those for K. By virtue of Theorem 2.8, we get the following result.

Theorem 3.5. *For the problem (3.1), (3.2), there exists an orthonormal system $\{\psi_k\}$ eigenvectors corresponding to eigenvalues $\{\lambda_k\}$ $(k \in \mathbb{N})$. The system $\{\psi_k\}$ forms an orthonormal basis for the Hilbert space $L^2_{\omega,q}(J_{\omega_0,a},r)$. Any function $f \in L^2_{\omega,q}(J_{\omega_0,a},r)$ can be expanded in eigenfunctions ψ_k in the form*

$$f(x) = \sum_{k=1}^{\infty} c_k \psi_k(x), \tag{3.7}$$

where

$$c_k = \int_{\omega_0}^{a} f(x)\, \psi_k(x)\, r(x)\, d_{\omega,q}x.$$

Thus, we have

$$\lim_{N\to\infty} \int_{\omega_0}^{a} \left| f(x) - \sum_{k=1}^{N} c_k \psi_k(x) \right|^2 r(x)\, d_{\omega,q}x = 0, \tag{3.8}$$

i.e., the sum (3.7) converges to the function f in mean square metric of the space $L^2_{\omega,q}(J_{\omega_0,a},r)$. Furthermore, by (3.8), we deduce that

$$\int_{\omega_0}^{a} |f(x)|^2 r(x)\, d_{\omega,q}x = \sum_{k=1}^{N} |c_k|^2 \tag{3.9}$$

which is called the Parseval equality.

The main result of this paper is the following statement.

Theorem 3.6. *Let $f : J_{\omega_0,h^{-1}(a)} \to \mathbb{R}$ be a continuous function at ω_0, such that it has a continuous $\omega,q-$derivative at ω_0 and satisfying the boundary conditions (3.2). Then the series*

$$f(x) = \sum_{k=1}^{\infty} c_k \psi_k(x), \tag{3.10}$$

where

$$c_k = \int_{\omega_0}^{a} f(x)\psi_k(x) r(x) d_{\omega,q}x,$$

converges uniformly to the function f on $J_{\omega_0,a}$.

Proof. Consider the functional

$$\begin{aligned} S(y) &:= q^{-1}h_1 \left|p(\omega_0) D_{\omega,q}y(\omega_0)\right|^2 \\ &\quad + q^{-1}h_2 \left|p\left(h^{-1}(a)\right) D_{\omega,q}y\left(h^{-1}(a)\right)\right|^2 \\ &\quad + q^{-1}\int_{\omega_0}^{a} p(x)\left|D_{\omega,q}y(x)\right|^2 d_{\omega,q}x \\ &\quad + \int_{\omega_0}^{a} v(x)\left|y(x)\right|^2 d_{\omega,q}x \end{aligned} \tag{3.11}$$

so that we have $S(y) \geq 0$. Substituting

$$y = f(x) - \sum_{k=1}^{N} c_k \psi_k(x)$$

into (3.11), we obtain

$$S\left(f(x) - \sum_{k=1}^{N} c_k\psi_k(x)\right)$$

$$= q^{-1}h_1\left[p(\omega_0) D_{\omega,q}f(\omega_0) - \sum_{k=1}^{N} c_k p(\omega_0) D_{\omega,q}\psi_k(\omega_0)\right]^2 + q^{-1}h_2$$

$$\times \left[p\left(h^{-1}(a)\right) D_{\omega,q}f\left(h^{-1}(a)\right) - \sum_{k=1}^{N} c_k\left(p\left(h^{-1}(a)\right) D_{\omega,q}\psi_k(a)\right)\right]^2$$

$$+q^{-1}\int_{\omega_0}^{a} p(x)\left(D_{\omega,q}f(x) - \sum_{k=1}^{N} c_k D_{\omega,q}\psi_k(x)\right)^2 d_{\omega,q}x$$

$$+\int_{\omega_0}^{a} v(x)\left(f(x) - \sum_{k=1}^{N} c_k\psi_k(x)\right)^2 d_{\omega,q}x$$

$$= q^{-1}h_1\left[p(\omega_0) D_{\omega,q}f(\omega_0)\right]^2 + q^{-1}h_2\left[p\left(h^{-1}(a)\right) D_{\omega,q}f\left(h^{-1}(a)\right)\right]^2$$

$$-2q^{-1}\sum_{k=1}^{N} c_k \begin{bmatrix} -h_1 p(\omega_0) D_{\omega,q}f(\omega_0) D_{\omega,q}\psi_k(\omega_0) \\ -h_2 p\left(h^{-1}(a)\right) D_{\omega,q}f\left(h^{-1}(a)\right) D_{\omega,q}\psi_k(a) \end{bmatrix} - q^{-1}$$

$$\times \sum_{k,m=1}^{N} c_k c_m \begin{bmatrix} -h_1 p(\omega_0) D_{\omega,q}\psi_k(\omega_0) p\left(\omega_0 D_{\omega,q}\psi_m(\omega_0)\right) \\ -h_2 p\left(h^{-1}(a)\right) D_{\omega,q}\psi_k(a) p\left(h^{-1}(a) D_{\omega,q}\psi_m(a)\right) \end{bmatrix}$$

$$+q^{-1}\int_{\omega_0}^{a} p(x)(D_{\omega,q}f(x))^2 d_{\omega,q}x + \int_{\omega_0}^{a} v(x) f^2(x) d_{\omega,q}x$$

$$-2q^{-1}\sum_{k=1}^{N} c_k\left[\int_{\omega_0}^{a} p(x) D_{\omega,q}f(x) D_{\omega,q}\psi_k(x) d_{\omega,q}x\right]$$

$$-2\sum_{k=1}^{N} c_k\left[\int_{\omega_0}^{a} v(x) f(x)\psi_k(x) d_{\omega,q}x\right]$$

$$+q^{-1}\sum_{k,m=1}^{N} c_k c_m \int_{\omega_0}^{a} p(x) D_{\omega,q}\psi_k(x) D_{\omega,q}\psi_m(x) d_{\omega,q}x$$

$$+\sum_{k,m=1}^{N} c_k c_m \int_{\omega_0}^{a} v(x)\psi_k(x)\psi_m(x) d_{\omega,q}x.$$

Applications of (3.2) and $\omega,q-$integration by parts yield

$$q^{-1}\int_{\omega_0}^{a}p(x)D_{\omega,q}\psi_k(x)D_{\omega,q}f(x)\,d_{\omega,q}x$$

$$+\int_{\omega_0}^{a}v(x)f(x)\psi_k(x)\,d_{\omega,q}x$$

$$=q^{-1}p\left(h^{-1}(a)\right)D_{\omega,q}\psi_k\left(h^{-1}(a)\right)f(a)-q^{-1}p(\omega_0)D_{\omega,q}\psi_k(\omega_0)f(\omega_0)$$

$$-q^{-1}\int_{\omega_0}^{a}f(x)D_{-\omega q^{-1},q^{-1}}\left(p(x)D_{\omega,q}\psi_k(x)\right)d_{\omega,q}x$$

$$+\int_{\omega_0}^{a}v(x)f(x)\psi_k(x)\,d_{\omega,q}x$$

$$=-q^{-1}h_2p\left(h^{-1}(a)\right)D_{-\omega q^{-1},q^{-1}}f(a)\,p\left(h^{-1}(a)\right)D_{-\omega q^{-1},q^{-1}}\psi_k(a)$$

$$-q^{-1}h_1p(\omega_0)D_{-\omega q^{-1},q^{-1}}f(\omega_0)\,p(\omega_0)D_{\omega,q}\psi_k(\omega_0)$$

$$+\int_{\omega_0}^{a}f(x)\left[-q^{-1}D_{-\omega q^{-1},q^{-1}}\left(p(x)D_{\omega,q}\psi_k(x)\right)+v(x)\psi_k(x)\right]d_{\omega,q}x$$

$$=-h_2q^{-1}p\left(h^{-1}(a)\right)D_{-\omega q^{-1},q^{-1}}f(a)\,p\left(h^{-1}(a)\right)D_{-\omega q^{-1},q^{-1}}\psi_k(a)$$

$$-h_1q^{-1}p(\omega_0)D_{-\omega q^{-1},q^{-1}}f(\omega_0)\,p(\omega_0)D_{\omega,q}\psi_k(\omega_0)+\lambda_kc_k,$$

and

$$q^{-1}\int_{\omega_0}^{a}p(x)D_{\omega,q}\psi_k(x)D_{\omega,q}\psi_m(x)\,d_{\omega,q}x$$

$$+\int_{\omega_0}^{a}v(x)\psi_k(x)\psi_m(x)\,d_{\omega,q}x$$

$$=q^{-1}p\left(h^{-1}(a)\right)D_{\omega,q}\psi_m\left(h^{-1}(a)\right)\psi_k(a)$$

$$-q^{-1}p(\omega_0)D_{\omega,q}\psi_m(\omega_0)\psi_k(\omega_0)$$

$$+\int_{\omega_0}^{a}\psi_k(x)\left[-q^{-1}D_{-\omega q^{-1},q^{-1}}\left(p(x)D_{\omega,q}\psi_m(x)\right)+v(x)\psi_m(x)\right]d_{\omega,q}x$$

$$=q^{-1}\psi_k(a)\,p\left(h^{-1}(a)\right)D_{-\omega q^{-1},q^{-1}}\psi_m(a)$$

$$-q^{-1}\psi_k(\omega_0)\,p(\omega_0)D_{-\omega q^{-1},q^{-1}}\psi_k(\omega_0)$$

$$+\lambda_k\int_{\omega_0}^{a}\psi_k(x)\psi_m(x)r(x)d_{\omega,q}x$$

$$=-q^{-1}h_1p\left(h^{-1}(a)\right)D_{-\omega q^{-1},q^{-1}}\psi_k(a)\,p\left(h^{-1}(a)\right)D_{-\omega q^{-1},q^{-1}}\psi_m(a)$$

$$-q^{-1}h_2p(\omega_0)D_{-\omega q^{-1},q^{-1}}\psi_k(\omega_0)\,p(\omega_0)D_{-\omega q^{-1},q^{-1}}\psi_m(\omega_0)+\lambda_k\delta_{km},$$

where

$$\delta_{km}:=\begin{cases}1, & \text{if } k=m\\ 0, & \text{if } k\neq m.\end{cases}$$

Thus, we have

$$S\left(f(x)-\sum_{k=1}^{N}c_k\psi_k(x)\right)$$

$$=q^{-1}h_1\left[p(\omega_0)D_{\omega,q}f(\omega_0)\right]^2+q^{-1}h_2\left[p\left(h^{-1}(a)\right)D_{\omega,q}f\left(h^{-1}(a)\right)\right]^2$$

$$+q^{-1}\int_{\omega_0}^{a}p(x)\left(D_{\omega,q}f(x)\right)^2d_{\omega,q}x+\int_{\omega_0}^{a}v(x)f^2(x)\,d_{\omega,q}x-q^{-1}\sum_{k=1}^{N}\lambda_kc_k^2.$$

Since the functional S is nonnegative for all N, we obtain the inequality

$$\sum_{k=1}^{\infty}\lambda_k c_k^2$$

$$\leq h_1\left[p(\omega_0)D_{\omega,q}f(\omega_0)\right]^2+h_2\left[p\left(h^{-1}(a)\right)D_{\omega,q}f\left(h^{-1}(a)\right)\right]^2$$

$$+\int_{\omega_0}^{a}p(x)\left(D_{\omega,q}f(x)\right)^2 d_{\omega,q}x+q\int_{\omega_0}^{a}v(x)f^2(x)\,d_{\omega,q}x. \tag{3.12}$$

Therefore, the convergence of the series

$$\sum_{k=1}^{\infty}\lambda_k c_k^2$$

follows.

Now, we shall show that the series

$$\sum_{k=1}^{\infty}\left|c_k\psi_k(x)\right| \tag{3.13}$$

is uniformly convergent on $J_{\omega_0,a}$. Since $T\psi_k=\lambda_k\psi_k$, we have

$$\psi_k(x)=\lambda_k(T^{-1}\psi_k)(x)$$

$$=\lambda_k\int_{\omega_0}^{a}G(x,t)\,\psi_k(t)\,r(t)d_{\omega,q}t.$$

If we rewrite the series (3.13), we get

$$\sum_{k=1}^{\infty}\left|c_k\psi_k(x)\right|=\sum_{k=1}^{\infty}\lambda_k\left|c_k\Upsilon_k(x)\right|, \tag{3.14}$$

where

$$\Upsilon_k(x)=\int_{\omega_0}^{a}G(x,t)\,\psi_k(t)\,r(t)d_{\omega,q}t.$$

This can be regarded as the Fourier coefficients of $G(x,t)$ as a function of t. It follows from (3.12) that

$$\sum_{k=1}^{\infty}\lambda_k\Upsilon_k^2(x)$$

$$\leq h_1\left[p(\omega_0)D_{\omega,q}G(x,\omega_0)\right]^2+h_2\left[p\left(h^{-1}(a)\right)D_{\omega,q}G\left(x,h^{-1}(a)\right)\right]^2$$

$$+\int_{\omega_0}^{a}p(t)\left(D_{\omega,q}G(x,t)\right)^2 d_{\omega,q}t+q\int_{\omega_0}^{a}v(t)\,G^2(x,t)\,d_{\omega,q}t.$$

Obviously, all the functions appearing under the integral sign are bounded. Then we have

$$\sum_{k=1}^{\infty}\lambda_k\Upsilon_k^2(x)\leq C,$$

where C is a constant. Applying the Cauchy-Schwartz inequality to the series (3.14), we deduce that

$$\sum_{k=\alpha}^{\alpha+\beta}\lambda_k\left|c_k\Upsilon_k(x)\right|\leq\sqrt{\sum_{k=\alpha}^{\alpha+\beta}\lambda_k c_k^2}\sqrt{\sum_{k=\alpha}^{\alpha+\beta}\lambda_k\Upsilon_k^2(x)}$$

$$\leq\sqrt{C}\sqrt{\sum_{k=\alpha}^{\alpha+\beta}\lambda_k c_k^2}. \tag{3.15}$$

By virtue of (3.12) and (3.15), we conclude that the series (3.13) is uniformly convergent on $J_{\omega_0,a}$. Since

$$\left|\sum_{k=1}^{\infty}c_k\psi_k(x)\right|\leq\sum_{k=1}^{\infty}\left|c_k\psi_k(x)\right|,$$

the series (3.10) is also uniformly convergent on $J_{\omega_0,a}$.
Let

$$f_1(x) = \sum_{k=1}^{\infty} c_k \psi_k(x). \tag{3.16}$$

Since the series (3.16) is uniformly convergent on $J_{\omega_0,a}$, we obtain

$$\int_{\omega_0}^{a} f_1(x)\psi_k(x) r(x) d_{\omega,q}x = c_k \ (k \in \mathbb{N}).$$

Consequently, the Fourier coefficients of f and f_1 are the same. Applying the Parseval equality (3.9) to the function $f - f_1$, we obtain $f - f_1 = 0$, since the Fourier coefficients of the function $f - f_1$ are zero. This finishes the proof. □

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