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Research Article

Bi-periodic r-Fibonacci sequence and bi-periodic r-Lucas sequence of type s

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Abstract

In the present paper, for a positive integer r, we study bi-periodic r-Fibonacci sequence and its family of companion sequences, bi-periodic r-Lucas sequence of type s with $1 \leq s \leq r$, which extend the classical Fibonacci and Lucas sequences. Afterwards, we establish the link between the bi-periodic r-Fibonacci sequence and its companion sequences. Furthermore, we give their properties as linear recurrence relations, generating functions, explicit formulas and Binet forms.

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1. Introduction

Recently, many authors have studied generalizations of Fibonacci and Lucas sequences. Edson and Yayenie [6] defined the bi-periodic Fibonacci sequence $(p_n)_n$ by

$$p_n = \begin{cases} ap_{n-1} + p_{n-2}, & if \quad n \equiv 0 \pmod{2}, \\ bp_{n-1} + p_{n-2}, & if \quad n \equiv 1 \pmod{2}, \end{cases}$$

for $n \geq 2$ with initial conditions $p_0 = 0, p_1 = 1$ and nonzero real numbers a, b. Bilgici [4] defined its companion sequence the bi-periodic Lucas sequence $(q_n)_n$ by

$$q_n = \begin{cases} bq_{n-1} + q_{n-2}, & if \quad n \equiv 0 \pmod{2}, \\ aq_{n-1} + q_{n-2}, & if \quad n \equiv 1 \pmod{2}, \end{cases}$$

for $n \geq 2$ with initial conditions $q_0 = 2, q_1 = a$.

For positive integer r and positive real numbers a, b, Yazlik et *al.* [12] introduced the sequences $(f_n)_n$ and $(l_n)_n$ as follows:

$$f_n = \begin{cases} af_{n-1} + f_{n-r-1}, & if \quad n \equiv 0 \pmod{2}, \\ bf_{n-1} + f_{n-r-1}, & if \quad n \equiv 1 \pmod{2}, \end{cases}$$

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and

$$l_n = \begin{cases} bl_{n-1} + l_{n-r-1}, & if \quad n \equiv 0 \,(\text{mod } 2), \\ al_{n-1} + l_{n-r-1}, & if \quad n \equiv 1 \,(\text{mod } 2), \end{cases}$$

for $n \geq r+1$ with initial conditions $f_0 = 0, f_1 = 1, f_2 = a, ..., f_r = a^{\lfloor r/2 \rfloor} b^{\lfloor (r-1)/2 \rfloor}$ and $l_0 = r+1$, $l_1 = a, l_2 = ab, ..., l_r = a^{\lfloor (r+1)/2 \rfloor} b^{\lfloor r/2 \rfloor}$, respectively.
It is clear to see that when $a = b = 1$ and $r = 1$, the sequences $(f_n)_n$ and $(l_n)_n$ reduce to the Fibonacci and Lucas sequences, respectively.

Raab [8] introduced the generalized r-Fibonacci sequence, for a positive integer r and real numbers x and y, by

$$T_n^{(r)} = xT_{n-1}^{(r)} + yT_{n-r-1}^{(r)},$$

for $n \geq r+1$ and initial conditions $T_0^{(r)} = 0, T_k^{(r)} = x^{k-1}$ with $1 \leq k \leq r$. When $x = y = 1$, the numbers $T_n^{(r)}$ reduce to the r-Fibonacci numbers.
Abbad et *al.* [1] defined its family of companion sequences; the r-Lucas sequences of type s, for a positive integers r, s with $1 \leq s \leq r$ and real numbers x and y, by

$$Z_n^{(r,s)} = xZ_{n-1}^{(r,s)} + yZ_{n-r-1}^{(r,s)},$$

for $n \geq r+1$ and initial conditions $Z_0^{(r)} = s+1, Z_k^{(r)} = x^k$ with $1 \leq k \leq r$.

Our study consists of two aspects. The first one, is to introduce the parameters c and d in the expression of the recurrence sequences given by Yazlik et *al.* in [12]. The second one, is to define a family of companion sequences as introduced in [1] for the bi-periodic case.

The outline of this paper is as follows. In Section 2, we give the expression of the bi-periodic r-Fibonacci sequence $(U_n^{(r)})_n$ and its linear recurrence relation. Then, we introduce a family of its companion sequences indexed by the parameter s; with $1 \leq s \leq r$; named the bi-periodic r-Lucas sequence of type s, $(V_n^{(r,s)})_n$. After that, we express $V_n^{(r,s)}$ in terms of $U_n^{(r)}$ and s. Section 3 is devoted to the generating functions of the bi-periodic r-Fibonacci sequence and its companion sequences. In Section 4, we propose an explicit formulas, which generalize the results given in [10, 11]. In Section 5, we give the Binet forms of $U_n^{(r)}$ and $V_n^{(r,s)}$. Finally, in Section 6, we present some examples for different values of r and s.

2. The bi-periodic r-Fibonacci and r-Lucas sequences

In this section, we define bi-periodic r-Fibonacci sequence $(U_n^{(r)})_n$ and we introduce the family of its companion sequences, bi-periodic r-Lucas sequence of type s, $(V_n^{(r,s)})_n$, then we express $V_n^{(r,s)}$ in terms of $U_n^{(r)}$ and we give their linear recurrence relations.

Definition 2.1. For nonzero real numbers a, b, c, d and positive integer r, bi-periodic r-Fibonacci sequence $(U_n^{(r)})_n$ is defined by

$$U_n^{(r)} = \begin{cases} aU_{n-1}^{(r)} + cU_{n-r-1}^{(r)}, & if \quad n \equiv 0 \,(\text{mod } 2), \\ bU_{n-1}^{(r)} + dU_{n-r-1}^{(r)}, & if \quad n \equiv 1 \,(\text{mod } 2), \end{cases} \tag{2.1}$$

for $n \geq r+1$ with initial conditions $U_0^{(r)} = 0, U_1^{(r)} = 1, U_2^{(r)} = a, \ldots, U_r^{(r)} = a^{\lfloor r/2 \rfloor} b^{\lfloor (r-1)/2 \rfloor}$.

We give the first values of the bi-periodic r-Fibonacci sequence.

(1) For $r=1$,

$$\begin{aligned} U_0^{(1)} &= 0, U_1^{(1)}=1, U_2^{(1)}=a, U_3^{(1)}=ab+d, U_4^{(1)}=a^2b+a(d+c),\\ U_5^{(1)} &= a^2b^2+ab(2d+c)+d^2, U_6^{(1)}=a^3b^2+a^2b(2d+2c)+a(d^2+dc+c^2). \end{aligned}$$

(2) For $r=2$,

$$\begin{aligned} U_0^{(2)} &= 0, U_1^{(2)}=1, U_2^{(2)}=a, U_3^{(2)}=ab, U_4^{(2)}=a^2b+c, U_5^{(2)}=a^2b^2+(bc+ad),\\ U_6^{(2)} &= a^3b^2+a(2bc+ad). \end{aligned}$$

The bi-periodic r-Fibonacci sequence can be expressed by the following linear recurrence relation.

Theorem 2.2. *Let a,b,c,d be nonzero real numbers and r be a positive integer. The bi-periodic r-Fibonacci sequence satisfies the following linear recurrence relation:*
For $n\geq 2r+2$,

$$U_n^{(r)}=abU_{n-2}^{(r)}+(a^{\xi(r+1)}d+b^{\xi(r+1)}c)U_{n-r-1-\xi(r+1)}^{(r)}-(-1)^{r+1}cdU_{n-2r-2}^{(r)}, \tag{2.2}$$

with initial conditions $U_0^{(r)}=0, U_1^{(r)}=1, U_2^{(r)}=a,\ldots,U_r^{(r)}=a^{\lfloor r/2\rfloor}b^{\lfloor (r-1)/2\rfloor}$,
for $r+1\leq m\leq 2r+1$,

$$U_m^{(r)}=\begin{cases} a^{\lfloor\frac{m}{2}\rfloor}b^{\lfloor\frac{m-1}{2}\rfloor}+\left(\lfloor\frac{m-r}{2}\rfloor d+\left\lfloor\frac{m-r-1}{2}\right\rfloor c\right)a^{\lfloor\frac{m-r-1}{2}\rfloor}b^{\lfloor\frac{m-r-2}{2}\rfloor}, & \text{if } r \text{ is odd},\\[2ex] a^{\lfloor\frac{m}{2}\rfloor}b^{\lfloor\frac{m-1}{2}\rfloor}+\lfloor\frac{m-r}{2}\rfloor a^{\lfloor\frac{m-r-2}{2}\rfloor}b^{\lfloor\frac{m-r-1}{2}\rfloor}c \\ \quad+\left\lfloor\frac{m-r-1}{2}\right\rfloor a^{\lfloor\frac{m-r}{2}\rfloor}b^{\lfloor\frac{m-r-3}{2}\rfloor}d, & \text{if } r \text{ is even}, \end{cases} \tag{2.3}$$

where $\xi(k)=2(k/2-\lfloor k/2\rfloor)$ is the parity function.

Proof. Note that $\xi(n+m)=\xi(n)+\xi(m)-2\xi(n)\xi(m)$.
Formula (2.1) can be rewritten as

$$\begin{aligned} U_n^{(r)} =& a^{1-\xi(n)}b^{\xi(n)}U_{n-1}^{(r)}+c^{1-\xi(n)}d^{\xi(n)}U_{n-r-1}^{(r)}\\ =& a^{1-\xi(n)}b^{\xi(n)}\left(a^{\xi(n)}b^{1-\xi(n)}U_{n-2}^{(r)}+c^{\xi(n)}d^{1-\xi(n)}U_{n-r-2}^{(r)}\right)\\ &+c^{1-\xi(n)}d^{\xi(n)}\left(a^{\xi(n+r)}b^{1-\xi(n+r)}U_{n-r-2}^{(r)}+c^{\xi(n+r)}d^{1-\xi(n+r)}U_{n-2r-2}^{(r)}\right)\\ =& abU_{n-2}^{(r)}+\left(a^{1-\xi(n)}b^{\xi(n)}c^{\xi(n)}d^{1-\xi(n)}+c^{1-\xi(n)}d^{\xi(n)}a^{\xi(n+r)}b^{1-\xi(n+r)}\right)U_{n-r-2}^{(r)}\\ &+c^{1-\xi(n)}d^{\xi(n)}c^{\xi(n+r)}d^{1-\xi(n+r)}U_{n-2r-2}^{(r)}. \end{aligned}$$

When r is odd, we get

$$\begin{aligned} U_n^{(r)} &= abU_{n-2}^{(r)}+\left(a^{1-\xi(n)}b^{\xi(n)}c^{\xi(n)}d^{1-\xi(n)}+c^{1-\xi(n)}d^{\xi(n)}a^{1-\xi(n)}b^{\xi(n)}\right)U_{n-r-2}^{(r)}\\ &\quad+c^{1-\xi(n)}d^{\xi(n)}c^{1-\xi(n)}d^{\xi(n)}U_{n-2r-2}^{(r)}\\ &= abU_{n-2}^{(r)}+a^{1-\xi(n)}b^{\xi(n)}(c+d)U_{n-r-2}^{(r)}+c^{2(1-\xi(n))}d^{2\xi(n)}U_{n-2r-2}^{(r)}\\ &= abU_{n-2}^{(r)}+(c+d)\left(U_{n-r-1}^{(r)}-c^{1-\xi(n)}d^{\xi(n)}U_{n-2r-2}^{(r)}\right)+c^{2(1-\xi(n))}d^{2\xi(n)}U_{n-2r-2}^{(r)}\\ &= abU_{n-2}^{(r)}+(c+d)U_{n-r-1}^{(r)}+\left(c^{2(1-\xi(n))}d^{2\xi(n)}-(c+d)c^{1-\xi(n)}d^{\xi(n)}\right)U_{n-2r-2}^{(r)}\\ &= abU_{n-2}^{(r)}+(c+d)U_{n-r-1}^{(r)}+\left(c^{2(1-\xi(n))}d^{2\xi(n)}-c^{2-\xi(n)}d^{\xi(n)}-c^{1-\xi(n)}d^{1+\xi(n)}\right)U_{n-2r-2}^{(r)}\\ &= abU_{n-2}^{(r)}+(c+d)U_{n-r-1}^{(r)}-cdU_{n-2r-2}^{(r)}, \end{aligned}$$

when r is even, we get

$$\begin{aligned} U_n^{(r)} =& abU_{n-2}^{(r)} + \left(a^{1-\xi(n)}b^{\xi(n)}c^{\xi(n)}d^{1-\xi(n)} + c^{1-\xi(n)}d^{\xi(n)}a^{\xi(n)}b^{1-\xi(n)}\right) U_{n-r-2}^{(r)} \\ &+ c^{1-\xi(n)}d^{\xi(n)}c^{\xi(n)}d^{1-\xi(n)}U_{n-2r-2}^{(r)} \\ =& abU_{n-2}^{(r)} + (ad+bc)U_{n-r-2}^{(r)} + cdU_{n-2r-2}^{(r)}. \end{aligned}$$

□

Now, we introduce a family of companion sequences related to the bi-periodic r-Fibonacci sequence, called bi-periodic r-Lucas sequence of type s, $(V_n^{(r,s)})_n$.

Definition 2.3. For nonzero real numbers a, b, c, d and integers r, s such that $1 \leq s \leq r$, bi-periodic r-Lucas sequence of type s is defined by

$$V_n^{(r,s)} = \begin{cases} bV_{n-1}^{(r,s)} + dV_{n-r-1}^{(r,s)}, & if \quad n \equiv 0 \pmod 2, \\ aV_{n-1}^{(r,s)} + cV_{n-r-1}^{(r,s)}, & if \quad n \equiv 1 \pmod 2, \end{cases}$$

for $n \geq r+1$ with initial conditions $V_0^{(r,s)} = s+1$, $V_1^{(r,s)} = a, V_2^{(r,s)} = ab, \ldots, V_r^{(r,s)} = a^{\lfloor (r+1)/2 \rfloor} b^{\lfloor r/2 \rfloor}$.

We give the first values of the bi-periodic r-Lucas sequence of type s.

(1) For $r = s = 1$,

$V_0^{(1,1)} = 2, V_1^{(1,1)} = a, V_2^{(1,1)} = ab + 2d, V_3^{(1,1)} = a^2b + 2ad + ac,$
$V_4^{(1,1)} = a^2b^2 + 3abd + abc + 2d^2, V_5^{(1,1)} = a^3b^2 + 3a^2bd + 2a^2bc + 2ad^2 + 2adc + ac^2.$

(2) For $r = 2$ and $s \in \{1, 2\}$,

$V_0^{(2,s)} = s+1, V_1^{(2,s)} = a, V_2^{(2,s)} = ab, V_3^{(2,s)} = a^2b + (s+1)c, V_4^{(2,s)} = a^2b^2 + (s+1)bc + ad,$
$V_5^{(2,s)} = a^3b^2 + (s+2)abc + a^2d.$

The bi-periodic r-Fibonacci sequence $(U_n^{(r)})_n$ and the bi-periodic r-Lucas sequence of type s, $(V_n^{(r,s)})_n$ can be seen as a generalization of the Fibonacci and Lucas sequences, we list some particular cases.

- For $a = b = c = d = 1$ and $r = s = 1$, we get the classical Fibonacci and Lucas sequences.
- For $a = b = 2$, $c = d = 1$ and $r = s = 1$, we get the classical Pell and Pell-Lucas sequences.
- For a, b nonzero real numbers, $c = d = 1$ and $r = s = 1$, we get the bi-periodic Fibonacci and bi-periodic Lucas sequences.
- For a, b nonzero real numbers, $c = d = 2$ and $r = s = 1$, we get the Jacobsthal and the Jacobsthal-Lucas sequences.
- For $a = b$, $c = d$ nonzero real numbers, we get the r-Fibonacci sequence and the r-Lucas sequence of type s.
  For more details on these sequences, we refer the reader to [1, 4, 6, 8, 12].

Each sequence in the family of companion sequences, the bi-periodic r-Lucas sequence of type s, satisfies the following linear recurrence relation.

Theorem 2.4. *Let a,b,c,d be nonzero real numbers and r,s be integers such that $1 \leq s \leq r$. The bi-periodic r-Lucas sequence of type s satisfies the following linear recurrence relation:*
For $n \geq 2r+2$,

$$V_n^{(r,s)} = abV_{n-2}^{(r,s)} + (a^{\xi(r+1)}d + b^{\xi(r+1)}c)V_{n-r-1-\xi(r+1)}^{(r,s)} - (-1)^{r+1}cdV_{n-2r-2}^{(r,s)}, \tag{2.4}$$

with initial conditions $V_0^{(r,s)} = s+1, V_1^{(r,s)} = a, V_2^{(r,s)} = ab, \ldots, V_r^{(r,s)} = a^{\lfloor (r+1)/2 \rfloor} b^{\lfloor r/2 \rfloor}$, for $r+1 \leq m \leq 2r+1$,

$$V_m^{(r,s)} = \begin{cases} a^{\lfloor \frac{m+1}{2} \rfloor} b^{\lfloor \frac{m}{2} \rfloor} + \left(\left(s + \left\lfloor \frac{m-r+1}{2} \right\rfloor\right) d + \lfloor \frac{m-r}{2} \rfloor c\right) a^{\lfloor \frac{m-r}{2} \rfloor} b^{\lfloor \frac{m-r-1}{2} \rfloor}, & \text{if } r \text{ is odd,} \\ a^{\lfloor \frac{m+1}{2} \rfloor} b^{\lfloor \frac{m}{2} \rfloor} + \left(s + \left\lfloor \frac{m-r+1}{2} \right\rfloor\right) a^{\lfloor \frac{m-r-1}{2} \rfloor} b^{\lfloor \frac{m-r}{2} \rfloor} c & \\ \quad + \lfloor \frac{m-r}{2} \rfloor a^{\lfloor \frac{m-r+1}{2} \rfloor} b^{\lfloor \frac{m-r-2}{2} \rfloor} d, & \text{if } r \text{ is even.} \end{cases} \tag{2.5}$$

Proof. The proof is done using Definition 2.3. □

Theorem 2.5. *Let r and s be positive integers, such that $1 \leq s \leq r$, the bi-periodic r-Fibonacci sequence and the bi-periodic r-Lucas sequence of type s satisfy the following relationship*

$$V_n^{(r,s)} = \begin{cases} U_{n+1}^{(r)} + sdU_{n-r}^{(r)}, & n \geq r, \quad \text{if } r \text{ is odd,} \\ U_{n+1}^{(r)} + scbU_{n-r-1}^{(r)} + scdU_{n-2r-1}^{(r)}, & n \geq 2r+1, \quad \text{if } r \text{ is even.} \end{cases} \tag{2.6}$$

Proof. We prove the theorem by induction on n, using Definition 2.3 and relations (2.3), (2.5) in Theorem 2.2 and Theorem 2.4 respectively.

□

3. The generating functions

In this section, we give the generating functions of the bi-periodic r-Fibonacci sequence and the bi-periodic r-Lucas sequence of type s.

Theorem 3.1. *Let r be a positive integer, the generating function of $(U_n^{(r)})_n$ is*

$$G(x) = \frac{x + ax^2 + (-1)^{\xi(r)}cx^{r+2}}{1 - abx^2 - (a^{\xi(r+1)}d + b^{\xi(r+1)}c)x^{r+1+\xi(r+1)} - (-1)^r cdx^{2r+2}}. \tag{3.1}$$

Proof. The formal power series representation of the generating function for $(U_n{}^{(r)})_n$ gives

$$G(x) = \frac{\sum_{k=0}^{2r+1} U_k^{(r)} x^k - abx^2 \sum_{k=0}^{2r-1} U_k^{(r)} x^k - (a^{\xi(r+1)}d + b^{\xi(r+1)}c)x^{r+1+\xi(r+1)} \sum_{k=0}^{r-\xi(r+1)} U_k^{(r)} x^k}{1 - abx^2 - (a^{\xi(r+1)}d + b^{\xi(r+1)}c)x^{r+1+\xi(r+1)} - (-1)^r cdx^{2r+2}}.$$

Indeed, we suppose that r is odd, we write

$$G(x) = \sum_{k \geqslant 0} U_k^{(r)} x^k$$

then

$$-abx^2 G(x) = -ab \sum_{k \geqslant 0} U_k^{(r)} x^{k+2}$$

$$(-(d+c))x^{r+1} G(x) = -(d+c) \sum_{k \geqslant 0} U_k^{(r)} x^{k+r+1}$$

$$(cd)x^{2r+2} G(x) = cd \sum_{k \geqslant 0} U_k^{(r)} x^{k+2r+2}$$

The relation (2.2) in Theorem 2.2 gives

$$\begin{aligned}(1-abx^2-(d+c)x^{r+1}+cdx^{2r+2})G(x) =&U_0^{(r)}+U_1^{(r)}x^1+\cdots+U_{2r+1}^{(r)}x^{2r+1}\\ &-abU_0^{(r)}x^2-abU_1^{(r)}x^3-\cdots-abU_{2r-1}^{(r)}x^{2r+1}\\ &-(d+c)U_0^{(r)}x^{r+1}-(d+c)U_1^{(r)}x^{r+2}-\dots\\ &-(d+c)U_r^{(r)}x^{2r+1}\\ =&\sum_{k=0}^{2r+1}U_k^{(r)}x^k-abx^2\sum_{k=0}^{2r-1}U_k^{(r)}x^k\\ &-(d+c)x^{r+1}\sum_{k=0}^{r}U_k^{(r)}x^k,\end{aligned}$$

using relation (2.3) given in Theorem 2.2, we obtain

$$G(x)=\frac{x+ax^2-cx^{r+2}}{1-abx^2-(d+c)x^{r+1}+cdx^{2r+2}}.$$

Similary, if r is even, we get

$$G(x)=\frac{x+ax^2+cx^{r+2}}{1-abx^2-(ad+bc)x^{r+2}-cdx^{2r+2}}.$$

□

Remark 3.2. If we take $r=1$, we obtain the generating function of the bi-periodic Fibonacci sequence given by Sahin [9].

The following theorem express the generating function of $(V_n^{(r,s)})_n$.

Theorem 3.3. *Let r and s be positive integers, such that $1\leq s\leq r$, the generating function of $(V_n^{(r,s)})_n$ is*

$$H(x)=\frac{(s+1)+ax-absx^2+(-1)^{\xi(r)}(s+1)cx^{r+1}+(-1)^{\xi(r+1)}adsx^{r+2}}{1-abx^2-(a^{\xi(r+1)}d+b^{\xi(r+1)}c)x^{r+1+\xi(r+1)}-(-1)^rcdx^{2r+2}}. \tag{3.2}$$

Proof. For odd r, relation (2.6) gives

$$\begin{aligned}H(x)&=\sum_{n\geqslant 0}V_n^{(r,s)}x^n\\ &=\sum_{n\geqslant 0}U_{n+1}^{(r)}x^n+sd\sum_{n\geqslant r}U_{n-r}^{(r)}x^n\\ &=\frac{1}{x}\sum_{n\geqslant 0}U_{n+1}^{(r)}x^{n+1}+sdx^r\sum_{n\geqslant r}U_{n-r}^{(r)}x^{n-r}\\ &=\frac{1}{x}\sum_{n\geqslant 0}U_{n}^{(r)}x^{n}+sdx^r\sum_{n\geqslant 0}U_{n}^{(r)}x^{n}\\ &=\frac{1+ax-cx^{r+1}}{1-abx^2-(d+c)x^{r+1}+cdx^{2r+2}}+\frac{sd(x^{r+1}+ax^{r+2}-cx^{2r+2})}{1-abx^2-(d+c)x^{r+1}+cdx^{2r+2}}\\ &=\frac{1+ax-cx^{r+1}+sdx^{r+1}+sadx^{r+2}-scdx^{2r+2}}{1-abx^2-(d+c)x^{r+1}+cdx^{2r+2}}\\ &=\frac{(s+1)+ax-absx^2-(s+1)cx^{r+1}+adsx^{r+2}}{1-abx^2-(d+c)x^{r+1}+cdx^{2r+2}}.\end{aligned}$$

For even r, relation (2.6) gives

$$\begin{aligned}
H(x) &= \sum_{n\geqslant 0} V_n^{(r,s)} x^n \\
&= \sum_{n\geqslant 0} U_{n+1}^{(r)} x^n + scb \sum_{n\geqslant r+1} U_{n-r-1}^{(r)} x^n + scd \sum_{n\geqslant 2r+1} U_{n-2r-1}^{(r)} x^n \\
&= \frac{1}{x} \sum_{n\geqslant 0} U_{n+1}^{(r)} x^{n+1} + scbx^{r+1} \sum_{n\geqslant r+1} U_{n-r-1}^{(r)} x^{n-r-1} + scdx^{2r+1} \sum_{n\geqslant 2r+1} U_{n-2r-1}^{(r)} x^{n-2r-1} \\
&= \frac{1}{x} \sum_{n\geqslant 0} U_n^{(r)} x^n + scbx^{r+1} \sum_{n\geqslant 0} U_n^{(r)} x^n + scdx^{2r+1} \sum_{n\geqslant 0} U_n^{(r)} x^n \\
&= (\frac{1}{x} + scbx^{r+1} + scdx^{2r+1}) \sum_{n\geqslant 0} U_n^{(r)} x^n \\
&= \frac{(\frac{1}{x} + scbx^{r+1} + scdx^{2r+1})(x + ax^2 + cx^{r+2})}{1 - abx^2 - (ad+bc)x^{r+2} - cdx^{2r+2}} \\
&= \frac{1 + ax + cx^{r+1} + s - sabx^2 - sadx^{r+2} + scx^{r+1}(abx^2 + cbx^{r+2} + adx^{r+2} + cdx^{2r+2})}{1 - abx^2 - (ad+bc)x^{r+2} - cdx^{2r+2}} \\
&= \frac{(s+1) + ax - absx^2 + (s+1)cx^{r+1} - adsx^{r+2}}{1 - abx^2 - (ad+bc)x^{r+2} - cdx^{2r+2}}.
\end{aligned}$$

□

Remark 3.4. If we take $r = 1$ and $c = d = 1$, we obtain the generation function of the bi-periodic Lucas sequence given by Bilgici [4].

4. Explicit formulas

In this section, we will state explicit formulas for $(U_n^{(r)})_n$ and $(V_n^{(r,s)})_n$, to generalize the explicit formulas of bi-periodic Fibonacci and Lucas sequences.
We use the following notation for the multinomial coefficient, given in [2], for all $k, k_1, k_2, \ldots, k_m \in \mathbb{Z}$,

$$\binom{k}{k_1, k_2, \ldots, k_m} = \begin{cases} \dfrac{k!}{k_1!k_2!\ldots k_m!} & if \quad k_1 + k_2 + \cdots + k_m = k \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Belbachir and Bencherif [3], gave a formula expressing the general terms of a linear recurrence sequence cited in the following lemma.

Lemma 4.1 ([3])**.** *Let $(u_n)_{n>-m}$ the sequence of elements over an unitary ring $\mathcal{A}$, defined by*

$$\begin{cases} u_{-j} &= 0 & 1 \leq j \leq m-1, \\ u_0 &= 1 & \\ u_n &= a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_m u_{n-m} & n \geq 1. \end{cases} \tag{4.1}$$

Then for all integers $n > -m$,

$$u_n = \sum_{k_1+2k_2+\cdots+mk_m=n} \binom{k_1 + k_2 + \cdots + k_m}{k_1, k_2, \ldots, k_m} a_1^{k_1} a_2^{k_2} \cdots a_m^{k_m}. \tag{4.2}$$

Using this lemma, we give an explicit formula of the bi-periodic r-Fibonacci sequence.

Theorem 4.2. *For any integer $r \geq 1$, we have*

$$U_{n+1}^{(r)} = \begin{cases} \displaystyle\sum_{2i+(r+1)t+(r+1)k=n} \binom{i+t}{t}\binom{t}{k}(ab)^i(c+d)^{t-k}(-cd)^k, & \text{if } r \text{ is odd,} \\ \displaystyle\sum_{2i+(r+2)t+rk=n} \binom{i+t}{t}\binom{t}{k}(ab)^i(ad+bc)^{t-k}(cd)^k, & \text{if } r \text{ is even.} \end{cases}$$

Proof. Considering the sequence $W_n^{(r)} = U_{n+1}^{(r)}$, then $W_0^{(r)} = 1, W_{-j}^{(r)} = 0$ for $1 \leq j \leq 2r+1$, relation (2.2) gives

$$W_n^{(r)} = abW_{n-2}^{(r)} + (a^{\xi(r+1)}d + b^{\xi(r+1)}c)W_{n-r-1-\xi(r+1)}^{(r)} - (-1)^{r+1}cdW_{n-2r-2}^{(r)}. \tag{4.3}$$

If r is odd, formula (4.3) reduces to

$$W_n^{(r)} = abW_{n-2}^{(r)} + (c+d)W_{n-r-1}^{(r)} - cdW_{n-2r-2}^{(r)}. \tag{4.4}$$

Using Lemma 4.1, we get

$$\begin{aligned} W_{n+1}^{(r)} &= \sum_{2i+(r+1)j+2(r+1)k=n} \binom{i+j+k}{i,j,k}(ab)^i(c+d)^j(-cd)^k \\ &= \sum_{2i+(r+1)(j+k)+(r+1)k=n} \binom{i+j+k}{j+k}\binom{j+k}{k}(ab)^i(c+d)^j(-cd)^k \\ &= \sum_{2i+(r+1)t+(r+1)k=n} \binom{i+t}{t}\binom{t}{k}(ab)^i(c+d)^{t-k}(-cd)^k. \end{aligned}$$

If r is even, formula (4.3) reduces to

$$W_n^{(r)} = abW_{n-2}^{(r)} + (ad+bc)W_{n-r-2}^{(r)} + cdW_{n-2r-2}^{(r)}. \tag{4.5}$$

Using Lemma 4.1, we get

$$\begin{aligned} W_{n+1}^{(r)} &= \sum_{2i+(r+2)j+2(r+1)k=n} \binom{i+j+k}{i,j,k}(ab)^i(ad+bc)^j(cd)^k \\ &= \sum_{2i+(r+2)(j+k)+rk=n} \binom{i+j+k}{j+k}\binom{j+k}{k}(ab)^i(ad+bc)^j(cd)^k \\ &= \sum_{2i+(r+2)t+rk=n} \binom{i+t}{t}\binom{t}{k}(ab)^i(ad+bc)^{t-k}(cd)^k. \end{aligned}$$

□

Now, we give an analogous result for the bi-periodic r-Lucas sequence of type s.

Theorem 4.3. *For any positive integers r and s, such that $1 \leq s \leq r$, we have*

$$\begin{aligned} V_n^{(r,s)} = &\sum_{2i+(r+1)t+(r+1)k=n} \binom{i+t}{t}\binom{t}{k}(ab)^i(c+d)^{t-k}(-cd)^k \\ &+ sd \sum_{2i+(r+1)t+(r+1)k=n-r-1} \binom{i+t}{t}\binom{t}{k}(ab)^i(c+d)^{t-k}(-cd)^k, \end{aligned}$$

if r is odd.

$$V_n^{(r,s)} = \sum_{2i+(r+2)t+rk=n} \binom{i+t}{t}\binom{t}{k}(ab)^i(ad+bc)^{t-k}(cd)^k$$

$$+ sbc \sum_{2i+(r+2)t+rk=n-r-2} \binom{i+t}{t}\binom{t}{k}(ab)^i(ad+bc)^{t-k}(cd)^k$$
$$+ scd \sum_{2i+(r+2)t+rk=n-2r-2} \binom{i+t}{t}\binom{t}{k}(ab)^i(ad+bc)^{t-k}(cd)^k,$$

if r is even.

Proof. We get the proof by using Theorem 2.5. □

Remark 4.4. Theorems 4.2 and 4.3 generalize the explicit formulas given in [10,11].

5. The Binet forms

In order to obtain the Binet forms of the bi-periodic r-Fibonacci sequence and the bi-periodic r-Lucas sequence of type s, we first express the characteristic polynomial. Considering relations (2.2) and (2.4), we get the characteristic polynomial of $(U_n^{(r)})_n$ and $(V_n^{(r,s)})_n$

$$y^{2r+2} - aby^{2r} - (a^{\xi(r+1)}d + b^{\xi(r+1)}c)y^{r+\xi(r)} - (-1)^{\xi(r)}cd, \tag{5.1}$$

putting $x = y^2$, we obtain

$$x^{r+1} - abx^r - (a^{\xi(r+1)}d + b^{\xi(r+1)}c)x^{\lfloor \frac{r+1}{2} \rfloor} - (-1)^{\xi(r)}cd. \tag{5.2}$$

Before stating the main theorems of this section, the following lemma will be useful.

Lemma 5.1. *Let $\mathbf{K}$ a field and $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = \sum_{i=0}^{n} a_ix^i \in \mathbf{K}[\mathbf{x}]$, a split polynomial on $\mathbf{K}$ with n roots, $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbf{K}$. The polynomial $P(x)$ can be written as $P(x) = a_n(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$ and*

$$\sigma_p = \sum_{1\le i_1<i_2<\cdots<i_p\le p+1} \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_p} = (-1)^p \frac{a_{n-p}}{a_n}. \tag{5.3}$$

For any i, j, we put $\sigma_j = \alpha_i\tilde{\sigma}^i_{j-1} + \tilde{\sigma}^i_j$, where $\tilde{\sigma}^i_j = \sum_{\substack{1\le k_1<k_2<\cdots<k_{r+1-j}\le r+1 \\ k_1,k_2,\ldots,k_{r+1-j}\neq i}} \alpha_{k_1}\alpha_{k_2}\ldots\alpha_{k_{r+1-j}}$.

Theorem 5.2. *Let $\alpha_1, \alpha_2, \ldots, \alpha_{r+1}$ be the distinct roots of the characteristic polynomial* (5.2) *associated with $(U_n^{(r)})_n$ and $(V_n^{(r,s)})_n$. We have*

$$U_n^{(r)} = \sum_{i=1}^{r+1} \frac{\left(\sum_{j=1}^{r}(-1)^j\tilde{\sigma}^i_j\ U^{(r)}_{2r-2j+\xi(n)} + U^{(r)}_{2r+\xi(n)}\right)}{\prod_{\substack{1\le k\le r+1 \\ k\neq i}}(\alpha_i-\alpha_k)}\alpha_i^{\lfloor n/2\rfloor},$$

and

$$V_n^{(r,s)} = \begin{cases} \sum_{i=1}^{r+1} \frac{\left(\sum_{j=1}^{r}(-1)^j\tilde{\sigma}^i_j\ U^{(r)}_{2r-2j+\xi(n+1)} + U^{(r)}_{2r+\xi(n+1)}\right)}{\prod_{\substack{1\le k\le r+1 \\ k\neq i}}(\alpha_i-\alpha_k)}\left(\alpha_i^{\lfloor (n+1)/2\rfloor} + sd\alpha_i^{\lfloor (n-r)/2\rfloor}\right), & \text{if } r \text{ is odd,} \\ \sum_{i=1}^{r+1} \frac{\left(\sum_{j=1}^{r}(-1)^j\tilde{\sigma}^i_j\ U^{(r)}_{2r-2j+\xi(n+1)} + U^{(r)}_{2r+\xi(n+1)}\right)}{\prod_{\substack{1\le k\le r+1 \\ k\neq i}}(\alpha_i-\alpha_k)} & \\ \quad\times\left(\alpha_i^{\lfloor (n+1)/2\rfloor} + scb\alpha_i^{\lfloor (n-r-1)/2\rfloor} + scd\alpha_i^{\lfloor (n-2r-1)/2\rfloor}\right), & \text{if } r \text{ is even,} \end{cases}$$

with initial conditions

$$U_0^{(r)} = 0, U_1^{(r)} = 1, U_2^{(r)} = a, \dots, U_r^{(r)} = a^{\lfloor r/2 \rfloor} b^{\lfloor (r-1)/2 \rfloor},$$
$$V_0^{(r,s)} = s+1, V_1^{(r,s)} = a, V_2^{(r,s)} = ab, \dots, V_r^{(r,s)} = a^{\lfloor (r+1)/2 \rfloor} b^{\lfloor r/2 \rfloor}.$$

Proof. As mentioned in [5], the general term of $(U_n^{(r)})_n$ is given by $U_n^{(r)} = \sum_{i=1}^{r+1} b_{i,n} \alpha_i^{\lfloor n/2 \rfloor}$, where $b_{i,n}$'s are rational numbers. The system can be solved by Cramer's rule with Vandermonde determinant, for more details, we refer to [7]. Using the initial terms of the sequence $(U_n^{(r)})_n$, for $n = 0, 2, 4, \dots, 2r$, we get

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{r+1} \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \cdots & \alpha_{r+1}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_1^r & \alpha_2^r & \alpha_3^r & \cdots & \alpha_{r+1}^r \end{pmatrix}^{-1} \begin{pmatrix} U_0^{(r)} \\ U_2^{(r)} \\ U_4^{(r)} \\ \vdots \\ U_{2r}^{(r)} \end{pmatrix} = \begin{pmatrix} b_{1,n} \\ b_{2,n} \\ b_{3,n} \\ \vdots \\ b_{r+1,n} \end{pmatrix},$$

and for $n = 1, 3, 5, \dots, 2r+1$, we get

$$\begin{pmatrix} \sqrt{\alpha_1} & \sqrt{\alpha_2} & \sqrt{\alpha_3} & \cdots & \sqrt{\alpha_{r+1}} \\ \sqrt{\alpha_1}^3 & \sqrt{\alpha_2}^3 & \sqrt{\alpha_3}^3 & \cdots & \sqrt{\alpha_{r+1}}^3 \\ \sqrt{\alpha_1}^5 & \sqrt{\alpha_2}^5 & \sqrt{\alpha_3}^5 & \cdots & \sqrt{\alpha_{r+1}}^5 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{\alpha_1}^{2r+1} & \sqrt{\alpha_2}^{2r+1} & \sqrt{\alpha_3}^{2r+1} & \cdots & \sqrt{\alpha_{r+1}}^{2r+1} \end{pmatrix}^{-1} \begin{pmatrix} U_1^{(r)} \\ U_3^{(r)} \\ U_5^{(r)} \\ \vdots \\ U_{2r+1}^{(r)} \end{pmatrix} = \begin{pmatrix} b_{1,n} \\ b_{2,n} \\ b_{3,n} \\ \vdots \\ b_{r+1,n} \end{pmatrix},$$

it results that $b_{i,n} = \dfrac{\sum_{j=1}^{r} (-1)^j \sum_{\substack{1 \le k_1 < k_2 < \dots < k_{r+1-j} \le r+1 \\ k_1, k_2, \dots, k_{r+1-j} \neq i}} \alpha_{k_1} \alpha_{k_2} \dots \alpha_{k_{r+1-j}} U_{2r-2j+\xi(n)}^{(r)} + U_{2r+\xi(n)}^{(r)}}{\prod_{\substack{1 \le k \le r+1 \\ k \neq i}} (\alpha_i - \alpha_k)}$,

using Lemma 5.1, we obtain $b_{i,n} = \dfrac{\sum_{j=1}^{r} (-1)^j \tilde{\sigma}_j^i U_{2r-2j+\xi(n)}^{(r)} + U_{2r+\xi(n)}^{(r)}}{\prod_{\substack{1 \le k \le r+1 \\ k \neq i}} (\alpha_i - \alpha_k)}$,

which gives $U_n^{(r)} = \sum_{i=1}^{r+1} \dfrac{\sum_{j=1}^{r} (-1)^j \tilde{\sigma}_j^i \ U_{2r-2j+\xi(n)}^{(r)} + U_{2r+\xi(n)}^{(r)}}{\prod_{\substack{1 \le k \le r+1 \\ k \neq i}} (\alpha_i - \alpha_k)} \alpha_i^{\lfloor n/2 \rfloor}$.

Using relation (2.6) in Theorem 2.5 for odd r, we get

$$\begin{aligned} V_n^{(r,s)} =& U_{n+1}^{(r)} + sdU_{n-r}^{(r)} \\ =& \sum_{i=1}^{r+1} \frac{\sum_{j=1}^{r} (-1)^j \tilde{\sigma}_j^i \ U_{2r-2j+\xi(n+1)}^{(r)} + U_{2r+\xi(n+1)}^{(r)}}{\prod_{\substack{1 \le k \le r+1 \\ k \neq i}} (\alpha_i - \alpha_k)} \alpha_i^{\lfloor (n+1)/2 \rfloor} \\ &+ sd \sum_{i=1}^{r+1} \frac{\sum_{j=1}^{r} (-1)^j \tilde{\sigma}_j^i \ U_{2r-2j+\xi(n-r)}^{(r)} + U_{2r+\xi(n-r)}^{(r)}}{\prod_{\substack{1 \le k \le r+1 \\ k \neq i}} (\alpha_i - \alpha_k)} \alpha_i^{\lfloor (n-r)/2 \rfloor} \end{aligned}$$

$$=\sum_{i=1}^{r+1}\frac{\sum_{j=1}^{r}(-1)^j\tilde{\sigma}_j^i\ U^{(r)}_{2r-2j+\xi(n+1)}+U^{(r)}_{2r+\xi(n+1)}}{\prod_{\substack{1\leq k\leq r+1\\ k\neq i}}(\alpha_i-\alpha_k)}\left(\alpha_i^{\lfloor (n+1)/2\rfloor}+sd\alpha_i^{\lfloor (n-r)/2\rfloor}\right),$$

and using relation (2.6) in Theorem 2.5 for even r, we get

$$\begin{aligned}
V_n^{(r,s)} =&U^{(r)}_{n+1}+scbU^{(r)}_{n-r-1}+scdU^{(r)}_{n-2r-1}\\
=&\sum_{i=1}^{r+1}\frac{\sum_{j=1}^{r}(-1)^j\tilde{\sigma}_j^i\ U^{(r)}_{2r-2j+\xi(n+1)}+U^{(r)}_{2r+\xi(n+1)}}{\prod_{\substack{1\leq k\leq r+1\\ k\neq i}}(\alpha_i-\alpha_k)}\alpha_i^{\lfloor (n+1)/2\rfloor}\\
&+scb\sum_{i=1}^{r+1}\frac{\sum_{j=1}^{r}(-1)^j\tilde{\sigma}_j^i\ U^{(r)}_{2r-2j+\xi(n-r-1)}+U^{(r)}_{2r+\xi(n-r-1)}}{\prod_{\substack{1\leq k\leq r+1\\ k\neq i}}(\alpha_i-\alpha_k)}\alpha_i^{\lfloor (n-r-1)/2\rfloor}\\
&+scd\sum_{i=1}^{r+1}\frac{\sum_{j=1}^{r}(-1)^j\tilde{\sigma}_j^i\ U^{(r)}_{2r-2j+\xi(n-2r-1)}+U^{(r)}_{2r+\xi(n-2r-1)}}{\prod_{\substack{1\leq k\leq r+1\\ k\neq i}}(\alpha_i-\alpha_k)}\alpha_i^{\lfloor (n-2r-1)/2\rfloor}\\
=&\sum_{i=1}^{r+1}\frac{\sum_{j=1}^{r}(-1)^j\tilde{\sigma}_j^i\ U^{(r)}_{2r-2j+\xi(n+1)}+U^{(r)}_{2r+\xi(n+1)}}{\prod_{\substack{1\leq k\leq r+1\\ k\neq i}}(\alpha_i-\alpha_k)}\\
&\times\left(\alpha_i^{\lfloor (n+1)/2\rfloor}+scb\alpha_i^{\lfloor (n-r-1)/2\rfloor}+scd\alpha_i^{\lfloor (n-2r-1)/2\rfloor}\right).
\end{aligned}$$

□

Remark 5.3. If we take $c=d=1$, we obtain the Binet form for the sequence $(f_n)_n$ given by Yazlik et *al.* [12].

Equivalently, we can express the Binet forms of $(U_n^{(r)})_n$ and $(V_n^{(r,s)})_n$ as follows.

Theorem 5.4. *For any integer $r\geq 1$, we have*

$$U_n^{(r)}=\sum_{i=1}^{r+1}A_i^{(n)}\alpha_i^{\lfloor n/2\rfloor},$$

and

$$V_n^{(r,s)}=\begin{cases}\sum_{i=1}^{r+1}A_i^{(n+1)}\left(\alpha_i^{\lfloor (n+1)/2\rfloor}+sd\alpha_i^{\lfloor (n-r)/2\rfloor}\right), & \text{if } r \text{ is odd},\\ \sum_{i=1}^{r+1}A_i^{(n+1)}\left(\alpha_i^{\lfloor (n+1)/2\rfloor}+scb\alpha_i^{\lfloor (n-r-1)/2\rfloor}+scd\alpha_i^{\lfloor (n-2r-1)/2\rfloor}\right), & \text{if } r \text{ is even},\end{cases}$$

where

$$A_i^{(n)}=$$

$$\frac{\sum_{j=1}^{r}-\alpha_i^{j-1}(ab-\alpha_i)U^{(r)}_{2r-2j+\xi(n)}+\sum_{j=\lfloor (r+2)/2\rfloor}^{r}-\alpha_i^{j-\lfloor (r+2)/2\rfloor}(a^{\xi(r+1)}d+b^{\xi(r+1)}c)U^{(r)}_{2r-2j+\xi(n)}+U^{(r)}_{2r+\xi(n)}}{\alpha_i^{\lfloor\frac{r-1}{2}\rfloor}\left((r+1)\alpha_i^{\lfloor\frac{r+2}{2}\rfloor}-rab\alpha_i^{\lfloor\frac{r}{2}\rfloor}-\lfloor\frac{r+1}{2}\rfloor(a^{\xi(r+1)}d+b^{\xi(r+1)}c)\right)},$$

with initial conditions

$$U_0^{(r)}=0, U_1^{(r)}=1, U_2^{(r)}=a, \ldots, U_r^{(r)}=a^{\lfloor r/2\rfloor}b^{\lfloor (r-1)/2\rfloor},$$
$$V_0^{(r,s)}=s+1, V_1^{(r,s)}=a, V_2^{(r,s)}=ab, \ldots, V_r^{(r,s)}=a^{\lfloor (r+1)/2\rfloor}b^{\lfloor r/2\rfloor}.$$

Proof. Considering $P(x)=x^{r+1}-abx^r-(a^{\xi(r+1)}d+b^{\xi(r+1)}c)x^{\lfloor\frac{r+1}{2}\rfloor}-(-1)^{\xi(r)}cd=(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{r+1})$, then for $1\leq i\leq r+1$, we get

$$\begin{aligned} P'(\alpha_i) &= (r+1)\alpha_i^r - rab\alpha_i^{r-1} - \left\lfloor\frac{r+1}{2}\right\rfloor(a^{\xi(r+1)}d+b^{\xi(r+1)}c)\alpha_i^{\lfloor\frac{r-1}{2}\rfloor} \\ &= \alpha_i^{\lfloor\frac{r-1}{2}\rfloor}\left((r+1)\alpha_i^{\lfloor\frac{r+2}{2}\rfloor} - rab\alpha_i^{\lfloor\frac{r}{2}\rfloor} - \left\lfloor\frac{r+1}{2}\right\rfloor(a^{\xi(r+1)}d+b^{\xi(r+1)}c)\right) \\ &= \prod_{\substack{1\leq k\leq r+1\\ k\neq i}}(\alpha_i-\alpha_k). \end{aligned}$$

On the other hand, using Lemma 5.1 and formula (5.2) for odd r, we get

$$\begin{aligned} \sigma_1 &= \sum_{1\leq i_1\leq r+1}\alpha_{i_1} = -a_r = ab, \\ \sigma_2 &= \sum_{1\leq i_1<i_2\leq r+1}\alpha_{i_1}\alpha_{i_2} = -a_{r-1} = 0, \\ &\vdots \\ \sigma_{(r-1)/2} &= \sum_{1\leq i_1<i_2<\cdots<i_{(r-1)/2}\leq r+1}\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{(r-1)/2}} = (-1)^{(r-1)/2}a_{r-1} = 0, \\ \sigma_{(r+1)/2} &= \sum_{1\leq i_1<i_2<\cdots<i_{(r+1)/2}\leq r+1}\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{(r+1)/2}} = (-1)^{(r+1)/2}a_{(r+1)/2} \\ &= (-1)^{(r+1)/2+1}(c+d), \\ &\vdots \\ \sigma_r &= \sum_{1\leq i_1<i_2<\cdots<i_r\leq r+1}\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_r} = (-1)^r a_1 = 0, \\ \sigma_{r+1} &= \prod_{1\leq i_1<i_2<\cdots<i_{r+1}\leq r+1}\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{r+1}} = (-1)^{(r+1)}a_0 = (-1)^{\xi(r+1)+r+1}cd = cd. \end{aligned}$$

Then

$$\begin{aligned} \tilde{\sigma}_1^i &= \sum_{\substack{1\leq i_1\leq r+1\\ i_1\neq i}}\alpha_{i_1}, \\ \tilde{\sigma}_2^i &= \sum_{\substack{1\leq i_1<i_2\leq r+1\\ i_1,i_2\neq i}}\alpha_{i_1}\alpha_{i_2}, \\ &\vdots \\ \tilde{\sigma}_{(r-1)/2}^i &= \sum_{\substack{1\leq i_1<i_2<\cdots<i_{(r-1)/2}\leq r+1\\ i_1,i_2,\ldots,i_{(r-1)/2}\neq i}}\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{(r-1)/2}}, \\ \tilde{\sigma}_{(r+1)/2}^i &= \sum_{\substack{1\leq i_1<i_2<\cdots<i_{(r+1)/2}\leq r+1\\ i_1,i_2,\ldots,i_{(r+1)/2}\neq i}}\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{(r+1)/2}}, \\ &\vdots \\ \tilde{\sigma}_r^i &= \prod_{\substack{1\leq i_1<i_2<\cdots<i_r\leq r+1\\ i_1,i_2,\ldots,i_r\neq i}}\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_r}, \end{aligned}$$

thus

$$\begin{array}{lll}
\tilde{\sigma}_1^i & = & ab-\alpha_i,\\
\tilde{\sigma}_2^i & = & (-\alpha_i)\tilde{\sigma}_1^i=(-\alpha_i)(ab-\alpha_i),\\
 & \vdots & \\
\tilde{\sigma}_j^i & = & (-\alpha_i)^{j-1}(ab-\alpha_i),\\
 & \vdots & \\
\tilde{\sigma}_{(r-1)/2}^i & = & (-\alpha_i)^{(r-1)/2-1}(ab-\alpha_i),\\
\tilde{\sigma}_{(r+1)/2}^i & = & (-1)^{(r+1)/2+1}(c+d)+(-\alpha_i)^{(r+1)/2-1}(ab-\alpha_i),\\
\tilde{\sigma}_{(r+1)/2+1}^i & = & (-\alpha_i)(-1)^{(r+1)/2+1}(c+d)+(-\alpha_i)(-\alpha_i)^{(r+1)/2-1}(ab-\alpha_i),\\
\tilde{\sigma}_{(r+1)/2+2}^i & = & (-\alpha_i)^2(-1)^{(r+1)/2+1}(c+d)+(-\alpha_i)^2(-\alpha_i)^{(r+1)/2-1}(ab-\alpha_i),\\
\tilde{\sigma}_{(r+1)/2+3}^i & = & (-\alpha_i)^3(-1)^{(r+1)/2+1}(c+d)+(-\alpha_i)^3(-\alpha_i)^{(r+1)/2-1}(ab-\alpha_i),\\
 & \vdots & \\
\tilde{\sigma}_r^i & = & {\alpha_i}^{\frac{r-1}{2}}(-1)^{r+1}(c+d)+(-\alpha_i)^{r-1}(ab-\alpha_i).
\end{array}$$

Using Lemma 5.1 and formula (5.2) for even r, we get

$$\begin{array}{lll}
\sigma_1 & = & \sum\limits_{1\le i_1\le r+1}\alpha_{i_1}=-a_r=ab,\\
\sigma_2 & = & \sum\limits_{1\le i_1<i_2\le r+1}\alpha_{i_1}\alpha_{i_2}=-a_{r-1}=0,\\
 & \vdots & \\
\sigma_{r/2} & = & \sum\limits_{1\le i_1<i_2<\cdots<i_{r/2}\le r+1}\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{r/2}}=(-1)^{r/2}a_{r/2+1}=0,\\
\sigma_{(r+2)/2} & = & \sum\limits_{1\le i_1<i_2<\cdots<i_{(r+2)/2}\le r+1}\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{(r+1)/2}}=(-1)^{(r+2)/2}a_{r/2}\\
 & = & (-1)^{(r+2)/2+1}(ad+bc),\\
 & \vdots & \\
\sigma_r & = & \sum\limits_{1\le i_1<i_2<\cdots<i_r\le r+1}\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_r}=(-1)^r a_1=0,\\
\sigma_{r+1} & = & \prod\limits_{1\le i_1<i_2<\cdots<i_{r+1}\le r+1}\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{r+1}}=(-1)^{(r+1)}a_0=(-1)^{\xi(r+1)+r+1}cd=cd,
\end{array}$$

then

$$\begin{array}{lll}
\tilde{\sigma}_1^i & = & ab-\alpha_i,\\
\tilde{\sigma}_2^i & = & (-\alpha_i)\tilde{\sigma}_1^i=(-\alpha_i)(ab-\alpha_i),\\
 & \vdots & \\
\tilde{\sigma}_j^i & = & (-\alpha_i)^{j-1}(ab-\alpha_i),\\
 & \vdots & \\
\tilde{\sigma}_{r/2}^i & = & (-\alpha_i)^{r/2-1}(ab-\alpha_i),\\
\tilde{\sigma}_{(r+2)/2}^i & = & (-1)^{(r+2)/2+1}(ad+bc)+(-\alpha_i)^{r/2}(ab-\alpha_i),\\
\tilde{\sigma}_{(r+2)/2+1}^i & = & (-\alpha_i)(-1)^{(r+2)/2+1}(ad+bc)+(-\alpha_i)^{r/2+1}(ab-\alpha_i),\\
\tilde{\sigma}_{(r+2)/2+2}^i & = & (-\alpha_i)^2(-1)^{(r+2)/2+1}(ad+bc)+(-\alpha_i)^{r/2+2}(ab-\alpha_i),\\
\tilde{\sigma}_{(r+2)/2+3}^i & = & (-\alpha_i)^3(-1)^{(r+2)/2+1}(ad+bc)+(-\alpha_i)^{r/2+3}(ab-\alpha_i),\\
 & \vdots & \\
\tilde{\sigma}_r^i & = & (-\alpha_i)^{(r-2)/2}(-1)^{(r+2)/2+1}(ad+bc)+(-\alpha_i)^{r-1}(ab-\alpha_i).
\end{array}$$

□

Considering that $r \geq 2$ and $\alpha_1, \alpha_2, \ldots, \alpha_{r+1}$ are nonzero roots, the Binet forms of the sequences $(U_n^{(r)})_n$ and $(V_n^{(r,s)})_n$ have two equivalent expressions given in the following corollaries.

Corollary 5.5. *For any integer $r \geq 2$, we have*

$$U_n^{(r)} = \sum_{i=1}^{r+1} B_i^{(n)} \alpha_i^{\lfloor n/2 \rfloor},$$

and

$$V_n^{(r,s)} = \begin{cases} \sum\limits_{i=1}^{r+1} B_i^{(n+1)} \left(\alpha_i^{\lfloor (n+1)/2 \rfloor} + sd\alpha_i^{\lfloor (n-r)/2 \rfloor}\right), & \text{if } r \text{ is odd,} \\ \sum\limits_{i=1}^{r+1} B_i^{(n+1)} \left(\alpha_i^{\lfloor (n+1)/2 \rfloor} + scb\alpha_i^{\lfloor (n-r-1)/2 \rfloor} + scd\alpha_i^{\lfloor (n-2r-1)/2 \rfloor}\right), & \text{if } r \text{ is even,} \end{cases}$$

where

$$B_i^{(n)} = \frac{\sum\limits_{j=1}^{\lfloor r/2 \rfloor} -\alpha_i^{j-1}(ab-\alpha_i) U_{2r-2j+\xi(n)}^{(r)} + \sum\limits_{j=\lfloor (r+2)/2 \rfloor}^{r} (-1)^j \frac{cd}{\alpha_i(-\alpha_i)^{r-j}} \; U_{2r-2j+\xi(n)}^{(r)} + U_{2r+\xi(n)}^{(r)}}{\alpha_i^{\left\lfloor \frac{r-1}{2} \right\rfloor} \left((r+1)\alpha_i^{\left\lfloor \frac{r+2}{2} \right\rfloor} - rab\alpha_i^{\left\lfloor \frac{r}{2} \right\rfloor} - \left\lfloor \frac{r+1}{2} \right\rfloor (a^{\xi(r+1)}d + b^{\xi(r+1)}c) \right)},$$

with initial conditions

$$U_0^{(r)} = 0, U_1^{(r)} = 1, U_2^{(r)} = a, \ldots, U_r^{(r)} = a^{\lfloor r/2 \rfloor} b^{\lfloor (r-1)/2 \rfloor},$$
$$V_0^{(r,s)} = s+1, V_1^{(r,s)} = a, V_2^{(r,s)} = ab, \ldots, V_r^{(r,s)} = a^{\lfloor (r+1)/2 \rfloor} b^{\lfloor r/2 \rfloor}.$$

Proof. Assume that r is odd, then

$$lcl\tilde{\sigma}_1^i = \sum_{\substack{1 \leq i_1 \leq r+1 \\ i_1 \neq i}} \alpha_{i_1} = ab - \alpha_i,$$

$$\tilde{\sigma}_2^i = \sum_{\substack{1 \leq i_1 < i_2 \leq r+1 \\ i_1, i_2 \neq i}} \alpha_{i_1}\alpha_{i_2} = (-\alpha_i)(ab-\alpha_i),$$

$$\vdots$$

$$\tilde{\sigma}_j^i = \sum_{\substack{1 \leq i_1 < i_2 < \cdots < i_j \leq r+1 \\ i_1, i_2, \ldots, i_j \neq i}} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_j} = (-\alpha_i)^{j-1}(ab-\alpha_i),$$

$$\vdots$$

$$\tilde{\sigma}_{(r-1)/2}^i = \sum_{\substack{1 \leq i_1 < i_2 < \cdots < i_{(r-1)/2} \leq r+1 \\ i_1, i_2, \ldots, i_{(r-1)/2} \neq i}} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{(r-1)/2}} = (-\alpha_i)^{(r-1)/2-1}(ab-\alpha_i),$$

$$\tilde{\sigma}_{(r+1)/2}^i = \sum_{\substack{1 \leq i_1 < i_2 < \cdots < i_{(r+1)/2} \leq r+1 \\ i_1, i_2, \ldots, i_{(r+1)/2} \neq i}} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{(r+1)/2}} = (-1)^{(r+1)/2+1}(c+d)$$

$$+ (-\alpha_i)^{(r+1)/2-1}(ab-\alpha_i) = \frac{cd}{\alpha_i} \frac{1}{(-\alpha_i)^{\frac{r-1}{2}}},$$

$$\vdots$$

$$\tilde{\sigma}_{r-t}^{i} = \sum_{\substack{1\le i_1<i_2<\cdots<i_{(r-t)}\le r+1\\ i_1,i_2,\ldots,i_{(r-t)}\neq i}} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{(r-t)}} = \frac{cd}{\alpha_i}\frac{1}{(-\alpha_i)^t},$$

$$\vdots$$

$$\tilde{\sigma}_{r-2}^{i} = \sum_{\substack{1\le i_1<i_2<\cdots<i_{(r-2)}\le r+1\\ i_1,i_2,\ldots,i_{(r-2)}\neq i}} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{(r-2)}} = \frac{cd}{\alpha_i}\frac{1}{(-\alpha_i)^2},$$

$$\tilde{\sigma}_{r-1}^{i} = \sum_{\substack{1\le i_1<i_2<\cdots<i_{(r-1)}\le r+1\\ i_1,i_2,\ldots,i_{(r-1)}\neq i}} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{(r-1)}} = \frac{cd}{\alpha_i}\frac{1}{(-\alpha_i)},$$

$$\tilde{\sigma}_{r}^{i} = \prod_{\substack{1\le i_1<i_2<\cdots<i_r\le r+1\\ i_1,i_2,\ldots,i_r\neq i}} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_r} = \frac{cd}{\alpha_i}.$$

Assume that r is even, then

$$\begin{aligned}
\tilde{\sigma}_{1}^{i} &= \sum_{\substack{1\le i_1\le r+1\\ i_1\neq i}} \alpha_{i_1} = ab-\alpha_i,\\
\tilde{\sigma}_{2}^{i} &= \sum_{\substack{1\le i_1<i_2\le r+1\\ i_1,i_2\neq i}} \alpha_{i_1}\alpha_{i_2} = (-\alpha_i)(ab-\alpha_i),\\
&\vdots\\
\tilde{\sigma}_{j}^{i} &= \sum_{\substack{1\le i_1<i_2<\cdots<i_j\le r+1\\ i_1,i_2,\ldots,i_j\neq i}} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_j} = (-\alpha_i)^{j-1}(ab-\alpha_i),\\
&\vdots\\
\tilde{\sigma}_{r/2}^{i} &= \sum_{\substack{1\le i_1<i_2<\cdots<i_{r/2}\le r+1\\ i_1,i_2,\ldots,i_{r/2}\neq i}} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{r/2}} = (-\alpha_i)^{r/2-1}(ab-\alpha_i),
\end{aligned}$$

$$\begin{aligned}
\tilde{\sigma}_{(r+2)/2}^{i} &= \sum_{\substack{1\le i_1<i_2<\cdots<i_{(r+2)/2}\le r+1\\ i_1,i_2,\ldots,i_{(r+2)/2}\neq i}} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{(r+2)/2}} = (-1)^{(r+2)/2+1}(ad+bc)\\
&+ (-\alpha_i)^{(r+2)/2-1}(ab-\alpha_i) = \tfrac{cd}{\alpha_i}\tfrac{1}{(-\alpha_i)^{(r+2)/2}},\\
&\vdots\\
\tilde{\sigma}_{r-t}^{i} &= \sum_{\substack{1\le i_1<i_2<\cdots<i_{(r-t)}\le r+1\\ i_1,i_2,\ldots,i_{(r-t)}\neq i}} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{(r-t)}} = \tfrac{cd}{\alpha_i}\tfrac{1}{(-\alpha_i)^t},\\
&\vdots\\
\tilde{\sigma}_{r-2}^{i} &= \sum_{\substack{1\le i_1<i_2<\cdots<i_{(r-2)}\le r+1\\ i_1,i_2,\ldots,i_{(r-2)}\neq i}} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{(r-2)}} = \tfrac{cd}{\alpha_i}\tfrac{1}{(-\alpha_i)^2},\\
\tilde{\sigma}_{r-1}^{i} &= \sum_{\substack{1\le i_1<i_2<\cdots<i_{(r-1)}\le r+1\\ i_1,i_2,\ldots,i_{(r-1)}\neq i}} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_{(r-1)}} = \tfrac{cd}{\alpha_i}\tfrac{1}{(-\alpha_i)},\\
\tilde{\sigma}_{r}^{i} &= \prod_{\substack{1\le i_1<i_2<\cdots<i_r\le r+1\\ i_1,i_2,\ldots,i_r\neq i}} \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_r} = \tfrac{cd}{\alpha_i}.
\end{aligned}$$

□

Corollary 5.6. *For any integer $r \geq 2$, we have*

$$U_n^{(r)} = \sum_{i=1}^{r+1} C_i^{(n)} \alpha_i^{\lfloor n/2 \rfloor},$$

and

$$V_n^{(r,s)} = \begin{cases} \sum_{i=1}^{r+1} C_i^{(n+1)} \left(\alpha_i^{\lfloor (n+1)/2 \rfloor} + sd\alpha_i^{\lfloor (n-r)/2 \rfloor}\right), & \text{if } r \text{ is odd,} \\ \sum_{i=1}^{r+1} C_i^{(n+1)} \left(\alpha_i^{\lfloor (n+1)/2 \rfloor} + scb\alpha_i^{\lfloor (n-r-1)/2 \rfloor} + scd\alpha_i^{\lfloor (n-2r-1)/2 \rfloor}\right), & \text{if } r \text{ is even,} \end{cases}$$

where

$$C_i^{(n)} = \frac{\sum_{j=1}^{r} (-1)^j \frac{cd}{\alpha_i(-\alpha_i)^{r-j}} U_{2r-2j+\xi(n)}^{(r)} + \sum_{j=1}^{\lfloor r/2 \rfloor} (-1)^{j+\lfloor r/2 \rfloor} \frac{(a^{\xi(r+1)}d + b^{\xi(r+1)}c)}{\alpha_i(-\alpha_i)^{\lfloor r/2 \rfloor - j}} U_{2r-2j+\xi(n)}^{(r)} + U_{2r+\xi(n)}^{(r)}}{\alpha_i^{\lfloor \frac{r-1}{2} \rfloor}\left((r+1)\alpha_i^{\lfloor \frac{r+2}{2} \rfloor} - rab\alpha_i^{\lfloor \frac{r}{2} \rfloor} - \left\lfloor \frac{r+1}{2} \right\rfloor (a^{\xi(r+1)}d + b^{\xi(r+1)}c)\right)},$$

with initial conditions

$$U_0^{(r)} = 0, U_1^{(r)} = 1, U_2^{(r)} = a, \ldots, U_r^{(r)} = a^{\lfloor r/2 \rfloor} b^{\lfloor (r-1)/2 \rfloor},$$
$$V_0^{(r,s)} = s+1, V_1^{(r,s)} = a, V_2^{(r,s)} = ab, \ldots, V_r^{(r,s)} = a^{\lfloor (r+1)/2 \rfloor} b^{\lfloor r/2 \rfloor}.$$

6. Examples

In this section, we present some numerical results, for specific values of r and s.

(1) For $s = r = 1$, we derive the bi-periodic 1-Fibonacci sequence $(U_n^{(1)})_n$ and its companion sequence, the bi-periodic 1-Lucas sequence of type 1, $(V_n^{(1,1)})_n$

$$U_n^{(1)} = \begin{cases} aU_{n-1}^{(1)} + cU_{n-2}^{(1)}, & if \quad n \equiv 0 \pmod 2, \\ bU_{n-1}^{(1)} + dU_{n-2}^{(1)}, & if \quad n \equiv 1 \pmod 2, \end{cases}$$

for $n \geq 2$ with $U_0^{(1)} = 0, \quad U_1^{(1)} = 1$ and nonzero real numbers a, b, c and d.
Its linear recurrence relation is given by

$$U_n^{(1)} = (ab + c + d)U_{n-2}^{(1)} - cdU_{n-4}^{(1)},$$

for $n \geq 4$ with $U_0^{(1)} = 0, U_1^{(1)} = 1, U_2^{(1)} = a, U_3^{(1)} = ab + d.$
Its generating function is

$$G(x) = \frac{x + ax^2 - cx^3}{1 - (ab + c + d)x^2 + cdx^4}.$$

Its Binet form is

$$U_n^{(1)} = \left(\frac{U_{2+\xi(n)}^{(1)} + (\alpha - ab - d - c)U_{\xi(n)}^{(1)}}{2\alpha - ab - d - c}\right)\alpha^{\lfloor n/2 \rfloor} + \left(\frac{U_{2+\xi(n)}^{(1)} + (\beta - ab - d - c)U_{\xi(n)}^{(1)}}{2\beta - ab - d - c}\right)\beta^{\lfloor n/2 \rfloor},$$

with

$$\begin{cases} U_{\xi(n)}^{(1)} = 0, \quad U_{2+\xi(n)}^{(1)} = a, & if \quad n \equiv 0 \pmod 2, \\ U_{\xi(n)}^{(1)} = 1, \quad U_{2+\xi(n)}^{(1)} = ab + d, & if \quad n \equiv 1 \pmod 2, \end{cases}$$

where α and β are the roots of the quadratic equation $x^2 - (ab + c + d)x + cd = 0$.
The bi-periodic 1-Lucas sequence of type 1, $(V_n^{(1,1)})_n$

$$V_n^{(1,1)} = \begin{cases} bV_{n-1}^{(1,1)} + dV_{n-2}^{(1,1)}, & if \quad n \equiv 0 \pmod 2, \\ aV_{n-1}^{(1,1)} + cV_{n-2}^{(1,1)}, & if \quad n \equiv 1 \pmod 2, \end{cases}$$

for $n \geq 2$ with $V_0^{(1,1)} = 2, V_1^{(1,1)} = a$.
Its linear recurrence relation is given by

$$V_n^{(1,1)} = (ab + c + d)V_{n-2}^{(1,1)} - cdV_{n-4}^{(1,1)},$$

for $n \geq 4$ with $V_0^{(1,1)} = 2, V_1^{(1,1)} = a, V_2^{(1,1)} = ab + 2d, V_3^{(1,1)} = a^2b + 2ad + ac$.
The link between $U_n^{(1)}$ and $V_n^{(1,1)}$ is

$$V_n^{(1,1)} = U_{n+1}^{(1)} + dU_{n-1}^{(1)}, \qquad n \geq 1.$$

Its generating function is given by

$$H(x) = \frac{2 + ax - (ab + 2c)x^2 + adx^3}{1 - (ab + c + d)x^2 + cdx^4}.$$

Its Binet form is

$$V_n^{(1,1)} = \left(\frac{V_{2+\xi(n)}^{(1,1)} + (\alpha - ab - d - c)V_{\xi(n)}^{(1,1)}}{2\alpha - ab - d - c}\right)\alpha^{\lfloor n/2\rfloor} + \left(\frac{V_{2+\xi(n)}^{(1,1)} + (\beta - ab - d - c)V_{\xi(n)}^{(1,1)}}{2\beta - ab - d - c}\right)\beta^{\lfloor n/2\rfloor},$$

with

$$\begin{cases} V_{\xi(n)}^{(1,1)} = 2, \quad V_{2+\xi(n)}^{(1,1)} = ab + 2d, & if \quad n \equiv 0 \pmod 2, \\ V_{\xi(n)}^{(1,1)} = a, \quad V_{2+\xi(n)}^{(1,1)} = a^2b + 2ad + ac, & if \quad n \equiv 1 \pmod 2. \end{cases}$$

We can also write

$$V_n^{(1,1)} = \left(\frac{U_{2+\xi(n+1)}^{(1)} + (\alpha - ab - d - c)U_{\xi(n+1)}^{(1)}}{2\alpha - ab - d - c}\right)\left(\alpha^{\lfloor (n+1)/2\rfloor} + d\alpha^{\lfloor (n-1)/2\rfloor}\right) + \left(\frac{U_{2+\xi(n+1)}^{(1)} + (\beta - ab - d - c)U_{\xi(n+1)}^{(1)}}{2\beta - ab - d - c}\right)\left(\beta^{\lfloor (n+1)/2\rfloor} + d\beta^{\lfloor (n-1)/2\rfloor}\right).$$

An explicit formula of $(U_n^{(1)})_n$ is given by

$$U_{n+1}^{(1)} = \sum_{2i+2t+2k=n} \binom{i+t}{t}\binom{t}{k}(ab)^i(c+d)^{t-k}(-cd)^k,$$

and an explicit formula of $(V_n^{(1,1)})_n$ is given by

$$V_n^{(1,1)} = \sum_{2i+2t+2k=n} \binom{i+t}{t}\binom{t}{k}(ab)^i(c+d)^{t-k}(-cd)^k + sd\sum_{2i+2t+2k=n-2} \binom{i+t}{t}\binom{t}{k}(ab)^i(c+d)^{t-k}(-cd)^k.$$

(2) For $r = 2$, we derive the bi-periodic 2-Fibonacci sequence $(U_n^{(2)})_n$ and its two companion sequences, the bi-periodic 2-Lucas sequence of type s, $(V_n^{(2,s)})_n$ with $s \in \{1, 2\}$

$$U_n^{(2)} = \begin{cases} aU_{n-1}^{(2)} + cU_{n-3}^{(2)}, & if \quad n \equiv 0 \pmod 2, \\ bU_{n-1}^{(2)} + dU_{n-3}^{(2)}, & if \quad n \equiv 1 \pmod 2, \end{cases}$$

for $n \geq 3$ with $U_0^{(2)} = 0, U_1^{(2)} = 1, U_2^{(2)} = a$ and nonzero real numbers a, b, c and d.
Its linear recurrence relation is

$$U_n^{(2)} = abU_{n-2}^{(2)} + (ad + bc)U_{n-4}^{(2)} - cdU_{n-6}^{(2)},$$

for $n \geq 6$ with $U_0^{(2)} = 0, U_1^{(2)} = 1, U_2^{(2)} = a, U_3^{(2)} = ab, U_4^{(2)} = a^2b + c, U_5^{(2)} = a^2b^2 + bc + ad$.
Its generating function is

$$G(x) = \frac{x + ax^2 + cx^4}{1 - abx^2 - (ad + bc)x^4 - cdx^6}.$$

Its Binet form is

$$\begin{aligned} U_n^{(2)} = &\left(\frac{U_{4+\xi(n)}^{(2)} - (ab - \alpha)U_{2+\xi(n)}^{(2)} + (\alpha^2 - \alpha ab - ad - bc)U_{\xi(n)}^{(2)}}{3\alpha^2 - 2ab\alpha - ad - bc} \right) \alpha^{\lfloor n/2 \rfloor} \\ &+ \left(\frac{U_{4+\xi(n)}^{(2)} - (ab - \beta)U_{2+\xi(n)}^{(2)} + (\beta^2 - \beta ab - ad - bc)U_{\xi(n)}^{(2)}}{3\beta^2 - 2ab\beta - ad - bc} \right) \beta^{\lfloor n/2 \rfloor} \\ &+ \left(\frac{U_{4+\xi(n)}^{(2)} - (ab - \gamma)U_{2+\xi(n)}^{(2)} + (\gamma^2 - \gamma ab - ad - bc)U_{\xi(n)}^{(2)}}{3\gamma^2 - 2ab\gamma - ad - bc} \right) \gamma^{\lfloor n/2 \rfloor}, \end{aligned}$$

with

$$\begin{cases} U_{\xi(n)}^{(2)} = 0, \quad U_{2+\xi(n)}^{(2)} = a, \quad U_{4+\xi(n)}^{(2)} = a^2b + c, & if \quad n \equiv 0 \pmod 2, \\ U_{\xi(n)}^{(2)} = 1, \quad U_{2+\xi(n)}^{(2)} = ab, \quad U_{4+\xi(n)}^{(2)} = a^2b^2 + bc + ad, & if \quad n \equiv 1 \pmod 2, \end{cases}$$

where α, β and γ are the roots of the equation $x^3 - abx^2 - (ad + bc)x - cd = 0$.

If the roots are nonzero, we can write

$$\begin{aligned} U_n^{(2)} = &\left(\frac{U_{4+\xi(n)}^{(2)} - (ab - \alpha)U_{2+\xi(n)}^{(2)} + \frac{cd}{\alpha}U_{\xi(n)}^{(2)}}{3\alpha^2 - 2ab\alpha - ad - bc} \right) \alpha^{\lfloor n/2 \rfloor} \\ &+ \left(\frac{U_{4+\xi(n)}^{(2)} - (ab - \beta)U_{2+\xi(n)}^{(2)} + \frac{cd}{\beta}U_{\xi(n)}^{(2)}}{3\beta^2 - 2ab\beta - ad - bc} \right) \beta^{\lfloor n/2 \rfloor} \\ &+ \left(\frac{U_{4+\xi(n)}^{(2)} - (ab - \gamma)U_{2+\xi(n)}^{(2)} + \frac{cd}{\gamma}U_{\xi(n)}^{(2)}}{3\gamma^2 - 2ab\gamma - ad - bc} \right) \gamma^{\lfloor n/2 \rfloor}. \end{aligned}$$

The bi-periodic 2-Lucas sequence of type s, $(V_n^{(2,s)})_n$ is defined by

$$V_n^{(2,s)} = \begin{cases} bV_{n-1}^{(2,s)} + dV_{n-3}^{(2,s)}, & if \quad n \equiv 0 \pmod 2, \\ aV_{n-1}^{(2,s)} + cV_{n-3}^{(2,s)}, & if \quad n \equiv 1 \pmod 2, \end{cases}$$

for $n \geq 3$ with $V_0^{(2,s)} = s + 1, V_1^{(2,s)} = a, V_2^{(2,s)} = ab$.
Its linear recurrence relation is given by

$$V_n^{(2,s)} = abV_{n-2}^{(2,s)} + (ad + bc)V_{n-4}^{(2,s)} + cdV_{n-6}^{(2,s)},$$

for $n \geq 6$ with $V_0^{(2,s)} = s+1, V_1^{(2,s)} = a, V_2^{(2,s)} = ab, V_3^{(2,s)} = a^2b + (s+1)c, V_4^{(2,s)} = a^2b^2 + (s + 1)bc + ad, V_5^{(2,s)} = a^3b^2 + (s + 2)abc + a^2d$.
The link between $U_n^{(2)}$ and $V_n^{(2,s)}$ is

$$V_n^{(2,s)} = U_{n+1}^{(2)} + scbU_{n-3}^{(2)} + scdU_{n-5}^{(2)}, \qquad n \geq 5.$$

Its generating function is

$$H(x)=\frac{(s+1)+ax-absx^2+(s+1)cx^3-adsx^4}{1-abx^2-(ad+bc)x^4-cdx^6}.$$

Its Binet form is

$$\begin{aligned}V_n^{(2,s)}=&\left(\frac{V_{4+\xi(n)}^{(2,s)}-(ab-\alpha)V_{2+\xi(n)}^{(2,s)}+(\alpha^2-\alpha ab-ad-bc)V_{\xi(n)}^{(2,s)}}{3\alpha^2-2ab\alpha-ad-bc}\right)\alpha^{\lfloor n/2\rfloor}\\&+\left(\frac{V_{4+\xi(n)}^{(2,s)}-(ab-\beta)V_{2+\xi(n)}^{(2,s)}+(\beta^2-\beta ab-ad-bc)V_{\xi(n)}^{(2,s)}}{3\beta^2-2ab\beta-ad-bc}\right)\beta^{\lfloor n/2\rfloor}\\&+\left(\frac{V_{4+\xi(n)}^{(2,s)}-(ab-\gamma)V_{2+\xi(n)}^{(2,s)}+(\gamma^2-\gamma ab-ad-bc)V_{\xi(n)}^{(2,s)}}{3\gamma^2-2ab\gamma-ad-bc}\right)\gamma^{\lfloor n/2\rfloor},\end{aligned}$$

with

$$\begin{cases}V_{\xi(n)}^{(2,s)}=s+1,\ V_{2+\xi(n)}^{(2,s)}=ab,\ V_{4+\xi(n)}^{(2,s)}=a^2b^2+(s+1)bc+ad, & \text{if } n\equiv 0 \pmod 2,\\ V_{\xi(n)}^{(2,s)}=a,\ V_{2+\xi(n)}^{(2,s)}=a^2b+(s+1)c,\ V_{4+\xi(n)}^{(2,s)}=a^3b^2+(s+2)abc+a^2d, & \text{if } n\equiv 1 \pmod 2.\end{cases}$$

If the roots are nonzero, we can also write

$$\begin{aligned}V_n^{(2,s)}=&\left(\frac{U_{4+\xi(n+1)}^{(2)}-(ab-\alpha)U_{2+\xi(n+1)}^{(2)}+(\alpha^2-\alpha ab-ad-bc)U_{\xi(n+1)}^{(2)}}{3\alpha^2-2ab\alpha-ad-bc}\right)\\&\times\left(\alpha^{\lfloor (n+1)/2\rfloor}+scb\alpha^{\lfloor (n-3)/2\rfloor}+scd\alpha^{\lfloor (n-5)/2\rfloor}\right)\\&+\left(\frac{U_{4+\xi(n+1)}^{(2)}-(ab-\beta)U_{2+\xi(n+1)}^{(2)}+(\beta^2-\beta ab-ad-bc)U_{\xi(n+1)}^{(2)}}{3\beta^2-2ab\beta-ad-bc}\right)\\&\times\left(\beta^{\lfloor (n+1)/2\rfloor}+scb\beta^{\lfloor (n-3)/2\rfloor}+scd\beta^{\lfloor (n-5)/2\rfloor}\right)\\&+\left(\frac{U_{4+\xi(n+1)}^{(2)}-(ab-\gamma)U_{2+\xi(n+1)}^{(2)}+(\gamma^2-\gamma ab-ad-bc)U_{\xi(n+1)}^{(2)}}{3\gamma^2-2ab\gamma-ad-bc}\right)\\&\times\left(\gamma^{\lfloor (n+1)/2\rfloor}+scb\gamma^{\lfloor (n-3)/2\rfloor}+scd\gamma^{\lfloor (n-5)/2\rfloor}\right).\end{aligned}$$

An explicit formula of $(U_n^{(2)})_n$ is given by

$$U_n^{(2)}=\sum_{2i+4t+2k=n}\binom{i+t}{t}\binom{t}{k}(ab)^i(ad+bc)^{t-k}(cd)^k,$$

and an explicit formula of $(V_n^{(2,s)})_n$ is given by

$$\begin{aligned}V_n^{(2,s)}=&\sum_{2i+4t+2k=n}\binom{i+t}{t}\binom{t}{k}(ab)^i(ad+bc)^{t-k}(cd)^k\\&+sbc\sum_{2i+4t+2k=n-4}\binom{i+t}{t}\binom{t}{k}(ab)^i(ad+bc)^{t-k}(cd)^k\\&+scd\sum_{2i+4t+2k=n-6}\binom{i+t}{t}\binom{t}{k}(ab)^i(ad+bc)^{t-k}(cd)^k.\end{aligned}$$

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References

[1] S. Abbad, H. Belbachir and B. Benzaghou, *Companion sequences associated to the r-Fibonacci sequence: algebraic and combinatorial properties*, Turk. J. Math. **43** (3), 1095-1114, 2019.

[2] H. Belbachir, *A combinatorial contribution to the multinomial Chu-Vandermonde convolution*, Les Annales RECITS **1**, 27-32, 2014.

[3] H. Belbachir and F. Bencherif, *Linear recurrent sequences and powers of a square matrix*, Integers **6**, A12, 2006.

[4] G. Bilgici, *Two generalizations of Lucas sequence*, Appl. Math. Comput. **245**, 526-538, 2014.

[5] L. Cerlienco, M. Mignotte and F. Piras, *Suites récurrentes linéaires, propriétés algébriques et arithmétiques*, Enseignement Mathématiques **33**, 67-108, 1987.

[6] M. Edson and O. Yayenie, *A new generalization of Fibonacci sequences and extended Binet's Formula*, Integers, **9**, 639-654, 2009.

[7] D. Kalman, *Generalized Fibonacci Numbers by matrix methods*, Fibonacci Quart. **20** (1), 73-76, 1982.

[8] J.A. Raab *A generalization of the connection between the Fibonacci sequence and Pascal's triangle*, Fibonacci Quart. **1**, 21-31, 1963.

[9] M. Sahin, *The Gelin-Cesàro identity in some conditional sequences*, Hacet. J. Math. Stat. **40** (6), 855-861, 2011.

[10] E. Tan and A.B. Ekin, *Bi-periodic Incomplete Lucas Sequences*, Ars Combin. **123**, 371-380, 2015.

[11] O. Yayenie, *A note on generalized Fibonacci sequence*, Appl. Math. Comput. **217**, 5603-5611, 2011.

[12] Y. Yazlik, C. Köme and V. Madhusudanan, *A new generalization of Fibonacci and Lucas p-numbers*, J. Comput. Anal. Appl. **25** (4), 657-669, 2018.