Maltepe Journal of Mathematics
ISSN:2667-7660, URL:https://dergipark.org.tr/tr/pub/mjm
Volume III Issue 1 (2021), Pages 1-5. Doi:https://doi.org/10.47087/mjm.826165

COMPACTIFICATIONS OF A FIXED SET

RAMKUMAR SOLAI* AND C. GANESA MOORTHY**
*DEPARTMENT OF BASIC ENGINEERING, GOVERNMENT POLYTECHNIC COLLEGE, GANDHARVAKOTTAI, PUDUKKOTTAI DISTRICT - 613 301, TAMIL NADU, INDIA , PHONE:+91-9952159815, ORCID:HTTPS://ORCID.ORG/0000-0001-5011-774X
**DEPARTMENT OF MATHEMATICS, ALAGAPPA UNIVERSITY, KARAIKUDI, SIVAGANGAI DISTRICT - 630 003, TAMIL NADU, INDIA, PHONE: +91-9840653866, ORCID:HTTPS://ORCID.ORG/0000-0003-3119-7531

Abstract. If a Tychonoff space is fixed, then we may consider all possible Hausdorff compactifications of the space. If an infinite set is fixed, then we may vary Tychonoff topologies on the set and the compactifications may also be varied.Magills construction for compactifications of a fixed Tychonoff space through partitions is applied to derive compactifications of various Tychonoff spaces (X, τ), with a fixed set X and with a variation in Tychonoff topologies τ. The structure of required partitions is also analyzed. When topologies are varied, some possible extensions of mappings are obtained in this regard.

1. Introduction

Compactification of a space X is a compact space containing X as a dense subspace. If a Tychonoff space is fixed, then we may consider all possible Hausdorff compactifications of the space. If an infinite set is fixed, then we may vary Tychonoff topologies on the set and the compactifications may also be varied. Magill's [10] construction of compactifications through partitions is improved in the second section of this article, when topologies are also varied. The structure of required partitions is also analyzed in the second section. In a compact extension of a topological group, the inverse operation should be extendable homeomorphically from the base topological group (See:[1]). The third section of this article is to study such extensions of mappings, when topologies are also varied. The authors have also contributed a classical work for compactifications including order relations (See: [11], [13], [14], [15]). Recent works are also available in literature regarding compactifications and lattice structure of a collection of compactifications (See: [2], [3], [7]). The major application of Hausdorff compactifications is obtaining completeness under all uniformities inducing same topologies, apart from

2020 *Mathematics Subject Classification.* Primary: 54D35 ; Secondaries: 54C20; 54A10 .
Key words and phrases. Hausdorff compactifications; extension of mappings; Complete upper semi lattices.

Submitted on November 15th, 2020. Published on April 30th, 2021. Communicated by Pratulananda Das.

other applications (See: [4], [6], [8] and [9]). All definitions which are not defined here are followed from [12].

2. Set Fixation

Let us fix an infinite set X. We consider the collection of all (Hausdorff) Tychonoff topologies on X. If τ_1 and τ_2 are two Tychonoff topologies on X, then we write $\tau_1 \leq \tau_2$ if $\tau_1 \subseteq \tau_2$. The supremum of any collection of Tychonoff topologies does exist and it is also a Tychonoff space. Let (Y_1, τ_1') and (Y_2, τ_2') be two Hausdorff compactifications of (X, τ_1) and (X, τ_2) respectively, when X is fixed. Then we write $(Y_1, \tau_1') \geq (Y_2, \tau_2')$ if there is a continuous function f from (Y_1, τ_1') onto (Y_2, τ_2') such that $f(x) = x$, for all $x \in X$. In this case, $\{f^{-1}(y) : y \in Y_2\}$ form a partition for Y_1 by compact subsets of Y_1. Moreover, each $x \in X$ is in at most one partitioning set $f^{-1}(y)$. That is, $f^{-1}(y) \cap X$ is either an empty set or a singleton set, for every $y \in Y_2$.

On the other hand, let us consider a partition π for Y_1 by compact subsets of a compactification (Y_1, τ_1') of (X, τ_1) such that the following is true: To each $A \in \pi$, $A \cap X$ is either an empty set or a singleton set. Define $Y_2 = Y_1/\pi$, and let $f : Y_1 \to Y_1/\pi = Y_2$ be the natural quotient mapping. Endow Y_2 with the quotient topology τ_2' corresponding to the quotient mapping f. Then (Y_2, τ_2') is a compact space, which may not be Hausdorff. However we have the following Result 2.1 on Hausdorffness. A variation of the Theorem 2.1 may be found in [12, Problem 4Q]. Note that if $A \in \pi$ is such that $A \cap X$ is a singleton set $\{x\}$, say, then x is identified with $f(A)$ as an element of Y_2. In this way, X is considered as a dense subset of (Y_2, τ_2').

Theorem 2.1. *(Y_2, τ_2') is a Hausdorff compactification of (X, τ_2) (for some τ_2), if and only if for a given τ_1'-open subset U containing a given $A \in \pi$, there is a τ_1'-open set V, which is a union of members of π, such that $A \subseteq V \subseteq U$.*

Proof. Suppose (Y_2, τ_2') is Hausdorff. Let $A \in \pi$ and U be a τ_1'-open set containing A. Then $f(Y_1 \backslash U)$ is a τ_2'-compact subset of Y_2, because f is continuous. It is a τ_2'-closed set, because (Y_2, τ_2') is Hausdorff. Then $f(A) \in Y_2 \backslash (f(Y_1 \backslash U))$ or $A \subseteq f^{-1}(Y_2 \backslash (f(Y_1 \backslash U))) \subseteq U$, where $V = f^{-1}(Y_2 \backslash (f(Y_1 \backslash U)))$ is a τ_1'-open set, which is a union of members of π.

To prove the converse part, consider two distinct members $A, B \in \pi$. Since (Y_1, τ_1') is normal, there are disjoint τ_1'-open sets U_1 and V_1 such that $A \subseteq U_1$ and $B \subseteq V_1$. Then there are τ_1'-open sets U_2 and V_2, which are unions of members of π such that $A \subseteq U_2 \subseteq U_1$ and $B \subseteq V_2 \subseteq V_1$. Then $f(U_2)$ and $f(V_2)$ are two disjoint τ_2'-open sets of Y_2 such that $f(A) \in f(U_2)$ and $f(B) \in f(V_2)$. This proves the Hausdorffness of (Y_2, τ_2'). □

Let us now give a sufficient condition for a partition to obtain a Hausdorff compactification.

Theorem 2.2. *If the subfamily of all non singleton members of π is a locally finite family in (Y_1, τ_1'), then (Y_2, τ_2') is a Hausdorff compactification of (X, τ_2), for some Hausdorff topology τ_2 in X .*

Proof. To prove the Hausdorffness of (Y_2, τ_2'), consider two distinct elements y_1, y_2 in Y_2. Then there are $A, B \in \pi$ such that $A = f^{-1}(y_1)$ and $B = f^{-1}(y_2)$, respectively. For any $x \in A$, there is a τ_1'-open set U_x of x, which intersects only a finite number of non singleton members $C_1, C_2 \cdots C_n$ of π such that $\overline{U_x} \cap B = \phi$ and $C_i \neq A$, for every i. Define a τ_1'-open set $V_x = U_x \backslash (\bigcup_{i=1}^{n} C_i)$ containing x. Then $\{V_x : x \in A\}$ is an open cover of A and this cover has a finite subcover $\{V_{x_1}, V_{x_2}, \cdots V_{x_m}\}$, say. Let $U = \bigcup_{i=1}^{m} V_{x_i}$. Then U is a τ_1'-open set such that $A \subseteq U$; $\overline{U} \cap B = \phi$, and such that U is a union of members of π. Similarly, we can find a τ_1'-open set V such that $B \subseteq V$, $\overline{U} \cap \overline{V} = \phi$, and such that V is a union of members of π. Then $f(U)$ and $f(V)$ are disjoint τ_2'-open sets in Y_2 such that $f(A) \in f(U)$ and $f(B) \in f(V)$. This proves the Hausdorffness of (Y_2, τ_2'). □

This Theorem 2.2 generalizes Lemma 2.1 in [10].

If we fix an infinite set X, vary Tychonoff topologies τ on X and vary (Hausdorff) compactifications (Y, τ') of (X, τ), then we obtain a complete upper semi-lattice $\mathcal{L}(X)$ under the relation " $\geq$ " defined above, that relates two compactifications. The largest element of this semi-lattice is the Stone-Čech compactification of X endowed with the discrete topology.

If a Tychonoff topology τ is fixed in X, then the collection $\mathcal{L}(X, \tau)$ of all compactifications of (X, τ) is a complete upper semi sublattice of $\mathcal{L}(X)$.

If $((X, \tau_i))_{i \in I}$ is a collection of Tychonoff topologies on an infinite set X, τ^* is the supremum of $(\tau_i)_{i \in I}$, and (Y_i, τ_i') is a compactification of (X, τ_i), for every $i \in I$, then the supremum of $(Y_i, \tau_i')_{i \in I}$ is of the form $(Y, \tau^{*'})$, where (X, τ^*) is a topological dense subspace of $(Y, \tau^{*'})$. Here $(Y, \tau^{*'})$ is the closure of the natural embedding of X into the Cartesian product $\prod_{i \in I} Y_i$, with the product topology. So, the mapping f from $\mathcal{L}(X)$ onto the complete upper semi-lattice of Tychonoff topologies on X, defined by $f((Y, \tau))$ = the subspace topology of τ on X, is an order preserving mapping and a join preserving mapping. This discussion leads to a convex structure of $\mathcal{L}(X, \tau)$ and a congruence relation through f (See: [5, p.17 and p.20]).

3. Self Extendable Mappings

Theorem 3.1. *Let (X, τ) be a locally compact Hausdorff space and (Y, τ') be its one point compactification, where $Y = X \cup \{\infty\}$, say. Let $h : (X, \tau) \to (X, \tau)$ be an onto homeomorphism. Then h has a unique homeomorphic extension $h' : (Y, \tau') \to (Y, \tau')$, and in this case $h'(\infty) = \infty$.*

Proof. Define $h'(\infty) = \infty$ and $h'(x) = h(x)$, for all $x \in X$. Fix a compact subset K of X. Then $h(K)$ and $h^{-1}(K)$ are compact subsets of X, and $h(X \backslash K)$ and $h^{-1}(X \backslash K)$ are open subsets of X. So h' and h'^{-1} are continuous at ∞. The continuity of h' and h'^{-1} at any point of X follows from the fact that X is open in (Y, τ'). This completes the proof. □

Theorem 3.2. *Let $((X, \tau_i))_{i \in I}$ be a collection of Tychonoff spaces and $((Y_i, \tau_i'))_{i \in I}$ be a collection such that*

(i) *Each (Y_i, τ_i') is a compactification of (X, τ_i).*
(ii) *For any continuous mapping $h_i : (X, \tau_i) \to (X, \tau_i)$, there is a continuous extension $h_i' : (Y_i, \tau_i') \to (Y_i, \tau_i')$.*

Let $h : X \to X$ be a mapping such that $h : (X, \tau_i) \to (X, \tau_i)$ is continuous, for every $i \in I$. Then there is a continuous mapping $h' : (Y, \tau^{*'}) \to (Y, \tau^{*'})$, that is an *extension of h, where* τ^* *is the supremum of* $(\tau_i)_{i\in I}$ *and* $(Y, \tau^{*'})$ *is the supremum of* $((Y_i, \tau_i'))_{i\in I}$.

Proof. Let $h_i' : (Y_i, \tau_i') \to (Y_i, \tau_i')$ be the continuous extension of $h : (X, \tau_i) \to (X, \tau_i)$. Define $H : \prod_{i\in I}(Y_i, \tau_i') \to \prod_{i\in I}(Y_i, \tau_i')$ by $H((y_i)_{i\in I}) = (h_i'(y_i))_{i\in I}$. Then H is continuous. Then the required $h' : (Y, \tau^{*'}) \to (Y, \tau^{*'})$ is the restriction of H to $(Y, \tau^{*'})$, where $(Y, \tau^{*'})$ is considered as a subspace of $\prod_{i\in I}(Y_i, \tau_i')$ as in Section 2. □

Remark. Suppose (ii) in Proposition 3.2 is replaced by

(ii)' For any surjective homeomorphism $h_i : (X, \tau_i) \to (X, \tau_i)$, there is a unique homeomorphic (or continuous) extension $h_i' : (Y_i, \tau_i') \to (Y_i, \tau_i')$.

Assume that $h : X \to X$ is a one to one and onto mapping such that $h_i : (X, \tau_i) \to (X, \tau_i)$ is an onto homeomorphism, for every $i \in I$. Then there is a homeomorphic (or continuous) mapping $h' : (Y, \tau^{*'}) \to (Y, \tau^{*'})$, that is an extension of h, for $(Y, \tau^{*'})$ given in Proposition 3.2.

Proof. If each h_i' is a homeomorphism, then H defined in the proof of the Proposition 3.2 is a homeomorphism. □

4. Conclusion

For a fixed infinite set, we may vary Tychonoff topologies on the set and the compactifications may also be varied. Magill's [10] construction of compactifications through partitions is improved and the structure of required partitions is also analyzed . In a compact extension of a topological group, the inverse operation should be extendable homeomorphically from the base topological group (See:[1]). Finally mappings are extended homeomorphically from topological space to its compact extension, when topologies are also varied.

Acknowledgments. The authors are grateful to the anonymous reviewers and the editor for their valuable suggestions and useful comments to improve the manuscript.

References

[1] A. V. Arhangel'skii, General topology II, Springer, Berlin, 1996.

[2] T. Alaste, *Function lattices and compactifications*, Appl. Gen. Topol., **15**(2), 2014, 183–202.

[3] G. Bezhanishvili, P. J. Morandi and B. Olberding, An extension of de Vries duality to completely regular spaces and compactifications, Topology Appl., **257**, (2019), 85 - 105.

[4] A. A. Chekeev and T. J. Kasymova, Ultrafilter-completeness on zero-sets of uniformly continuous functions, Topology Appl., **252**, (2019), 27 - 41.

[5] G. Grätzer, General lattice theory, Academic press, New York, (1978).

[6] A. W. Gutiĕrrez, On the metric compactification of infinite-dimensional $\mathcal{L}_p$ spaces, arXiv preprint arXiv:1802.04710, (2018).

[7] W. He and Z. Xiao, Lattice of compactifications of a topological group, Categories and General Algebraic Structures with Applications, **10**(1), (2019), 39 - 50.

[8] A. T. Junhan, A Report on Hausdorff Compactifications of **R**, arXiv preprint arXiv:1901.08167, (2019).

[9] M. Krukowski, Arzela-Ascoli theorem via the Wallman compactification, Quaestiones Mathematicae, **41**(3), (2018), 349 - 357.

[10] K.D. Magill, jr, The lattice of compactificatons of a locally compact space, Proc. London math. soc., **18**, (1968), 231 - 244.

[11] C. G. Moorthy, K. T. Rajambal, K. S. Lakshmi, R. Parvatham and H. M. Srivastava, A general compactification for Hausdorff spaces, Proceedings of the international conference on Analysis and its applications, (eds. KS Lakshmi, R. Parvatham, and HM Srivastava, Allied publishers Ltd), (2001), 67 - 71.

[12] J. R. Porter and R. Grandwoods, Extensions and absolutes of Hausdorff spaces, Springer-Verlag, Newyork, (1988).

[13] S. Ramkumar and C. G. Moorthy, A construction for topological extensions, Asian-Eur. J. Math., **4**(3), (2011), 481 - 494.

[14] S. Ramkumar and C. G. Moorthy, A construction for semi-compactifications, Asian-Eur. J. Math., **5**(1), (2012), 1250008.

[15] S. Ramkumar and C. Ganesa Moorthy, Extendability of semi-metrics to compactifications, Asian-Eur. J. Math., **6**(1), (2013), 1350015.

Ramkumar Solai,
Department of Basic Engineering, Government Polytechnic College, Gandharvakottai, Pudukkottai District - 613 301, Tamil nadu, India, phone:+91-9952159815, ORCID:https://orcid.org/0000-0001-5011-774X,
Email address: ramkumarsolai@gmail.com

C. Ganesa Moorthy,
Department of Mathematics,, Alagappa University, Karaikudi, Sivagangai District - 630 003, Tamil nadu, India, phone: +91-9840653866, ORCID:https://orcid.org/0000-0003-3119-7531,
Email address: ganesamoorthyc@gmail.com