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Time Fractional Equation with Non-homogenous Dirichlet Boundary Conditions

Süleyman ÇETİNKAYA*[1], Ali DEMİR[2]

Abstract

In this research, we discuss the construction of analytic solution of non-homogenous initial boundary value problem including PDEs of fractional order. Since non-homogenous initial boundary value problem involves Caputo fractional order derivative, it has classical initial and boundary conditions. By means of separation of variables method and the inner product defined on $L^2[0,l]$, the solution is constructed in the form of a Fourier series with respect to the eigenfunctions of a corresponding Sturm-Liouville eigenvalue problem including fractional derivative in Caputo sense used in this study. Illustrative example presents the applicability and influence of separation of variables method on fractional mathematical problems.

Keywords: Caputo fractional derivative, Time-fractional diffusion equation, Mittag-Leffler function, Initial-boundary-value problems, Spectral method.

1. INTRODUCTION

Partial differential equations (PDEs) of fractional order turns out to be the best choice of modelling for the numerous processes and systems in various scientific research areas such as applied mathematics, industrial mathematics etc., Since PDEs of fractional order becomes an attractive research area, the mathematical knowledge and methods are used effectively to determine and analyze the solution of it. However further mathematical tools are necessary in view of the applications of mathematical models including fractional derivatives. This provides quite strong motivation and inspiration for scientists to make more research on it. This enriches the various branches of mathematics. Since mathematical models including PDEs of fractional order are suitable for the analysis of the behavior of the complex non-linear processes, it attracts increasing number of scientists.

The derivative in the sense of Caputo is one of the most common one since mathematical models with Caputo derivative gives better results compare to the analysis of ones including other fractional derivatives. In literature, increasing number of studies can be found supporting this conclusion [1-9]. Moreover, the Caputo derivative of constant is zero which is not hold by many fractional derivatives. The solutions of fractional PDEs and ordinary differential equations (ODEs) are determined in terms of Mittag-Leffler function.

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2. PRELIMINARY RESULTS

Some fundamental definitions and accomplished results of fractional derivative in Caputo sense are presented in this section.

Definition 2.1. The Caputo fractional derivative of $u(t)$ of order q where $n-1<q<n$ is given by the equation

$$D^q u(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{n-q-1} u^{(n)}(s) ds,$$
$$t \in [t_0, t_0+T] \quad (1)$$

where $u^{(n)}(t) = \frac{d^n u}{dt^n}$. Note that Caputo fractional derivative becomes the integer order derivative when q is an integer.

Definition 2.2. The fractional derivative of order q for $0<q<1$ in the Caputo sense is defined in the following form:

$$D^q u(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} u'(s) ds,$$
$$t \in [t_0, t_0+T] \quad (2)$$

Definition 2.3. The two parameter Mittag-Leffler function by which the solution of eigenvalue problem is denoted, defined in the following form:

$$E_{\alpha,\beta}(\lambda(t-t_0)^\alpha) = \sum_{k=0}^{\infty} \frac{(\lambda(t-t_0)^\alpha)^k}{\Gamma(\alpha k+\beta)}, \alpha, \beta > 0 \quad (3)$$

where λ is a constant. Especially, by taking $t_0 = 0$, $\alpha=\beta=q$ we get

$$E_{q,q}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+q)}, \quad q>0. \quad (4)$$

Moreover substituting $q=1$, in the equation (4) we have $E_{1,1}(\lambda t) = e^{\lambda t}$. For further reading see [10,11].

3. MAIN RESULTS

Let us consider the following initial boundary value problem including time fractional derivative in Caputo sense.

$$D_t^\alpha u(x,t) = \gamma^2 u_{xx}(x,t), \quad (8)$$

$$u(x,0) = f(x), \quad (9)$$

$$u(0,t) = u_0, \ u(l,t) = u_1 \quad (10)$$

where $0<\alpha<1$, $\gamma \in \mathbb{R}$, $0 \le x \le l, 0 \le t \le T$, u_0 and u_1 are constants.

Before investigating the solution of the problem (8)-(10), let us define the function $v(x,t)$ which homogenizes the boundary conditions (10) as follows:

$$v(x,t) = u(x,t) + \frac{x}{l}(u_0 - u_1) - u_0. \quad (11)$$

Via (10), the problem (8)-(10) turns into the following problem (12)-(14).

$$D_t^\alpha v(x,t) = \gamma^2 v_{xx}(x,t), \quad (12)$$

$$v(x,0) = f(x) + \frac{x}{l}(u_0 - u_1) - u_0 , \quad (13)$$

$$v(0,t) = 0, v(l,t) = 0 \quad (14)$$

where $0<\alpha<1$, $\gamma \in \mathbb{R}$, $0 \le x \le l, 0 \le t \le T$, u_0 and u_1 are constants.

By means of separation of variables method, the generalized solution of above problem is constructed in analytical form. Thus a solution of problem (12)-(14) has the following form:

$$v(x,t;\alpha) = X(x)\, T(t;\alpha) \quad (15)$$

where $0 \le x \le l$, $0 \le t \le T$.

Plugging (15) into (12) and arranging it, we have

$$\frac{D_t^\alpha(T(t;\alpha))}{T(t;\alpha)} = \gamma^2 \frac{X''(x)}{X(x)} = -\lambda^2. \quad (16)$$

Equation (16) produces a fractional differential equation with respect to time and an ordinary differential equation with respect to space. The first ordinary differential equation is obtained by taking the equation on the right hand side of Eq. (16). Hence with boundary conditions (14), we have the following problem:

$$X''(x) + \lambda^2 X(x) = 0, \quad (17)$$

$$X(0) = X(l) = 0. \tag{18}$$

The solution of eigenvalue problem (17)-(18) is accomplished by making use of the exponential function of the following form:

$$X(x) = e^{rx}. \tag{19}$$

Hence the characteristic equation is computed as follows:

$$r^2 + \lambda^2 = 0. \tag{20}$$

Case 1: If $\lambda = 0$, then the characteristic equation have coincident solutions $r_{1,2} = 0$, which leads to the general solution of the eigenvalue problem (17)-(18) have the following form:

$$X(x) = k_1 x + k_2.$$

By making use of the first boundary condition, we have

$$X(0) = k_2 = 0 \Rightarrow k_2 = 0. \tag{21}$$

Hence the solution becomes

$$X(x) = k_1 x. \tag{22}$$

Similarly second boundary condition leads to

$$X(l) = k_1 l \ = 0 \Rightarrow k_1 = 0. \tag{23}$$

which implies that

$$X(x) = 0. \tag{24}$$

As a result, the characteristic equation (20) can not have the solution for $\lambda = 0$.

Case 2: If $\lambda \neq 0$, then the characteristic equation have the solutions

$$r_{1,2} = \mp i\lambda \tag{25}$$

which leads to the general solution of the eigenvalue problem (17)-(18) have the following form:

$$X(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x). \tag{26}$$

By making use of the first boundary condition, we have

$$X(0) = c_1 = 0 \Rightarrow c_1 = 0. \tag{27}$$

Hence the solution becomes

$$X(x) = c_2 \sin(\lambda x). \tag{28}$$

Similarly last boundary condition leads to

$$X(l) = c_2 \sin(\lambda l) = 0 \tag{29}$$

which implies that

$$\sin(\lambda l) = 0. \tag{30}$$

Let $w_n = \sqrt{\lambda} l$. The solutions of (30) can be denoted by means of $w_n = n\pi$ which are eigenvalues of the problem (17)-(18). Moreover we have

$$\lambda_n = \frac{w_n^2}{l^2}, 0 < w_1 < w_2 < w_3 < \cdots \tag{31}$$

As a result

$$X_n(x) = c_n \sin\left(w_n\left(\frac{x}{l}\right)\right) = \sin\left(w_n\left(\frac{x}{l}\right)\right),$$

$$n = 1,2,3, \dots \tag{32}$$

represent the solutions of the eigenvalue problem.

From equation (16) for each eigenvalue λ_n, we have the following differential equation:

$$\frac{D_t^\alpha(T(t;\alpha))}{T(t;\alpha)} = -\gamma^2\lambda^2 \tag{33}$$

which has the following solutions

$$T_n(t;\alpha) = k_1 E_{\alpha,1}(-\gamma^2\lambda_n^2 t^\alpha) = E_{\alpha,1}\left(-\gamma^2\frac{w_n^2}{l^2}t^\alpha\right), n = 1,2,3, \dots \tag{34}$$

As a result, the specific solutions of problem (12)-(14) can be written as

$$v_n(x,t;\alpha) = X_n(x)T_n(t;\alpha) = E_{\alpha,1}\left(-\gamma^2\frac{w_n^2}{l^2}t^\alpha\right)\sin\left(w_n\left(\frac{x}{l}\right)\right), n = 1,2,3, \dots \tag{35}$$

which leads to following general solution of problem (12)-(14)

$$v(x,t;\alpha) = \sum_{n=1}^{\infty} d_n \sin\left(w_n\left(\frac{x}{l}\right)\right) E_{\alpha,1}\left(-\gamma^2 \frac{w_n^2}{l^2} t^{\alpha}\right) \quad (36)$$

Note that the general solution (36) satisfies both boundary conditions (14) and the fractional equation (12). By making use of the inner product defined on $L^2[0,l]$, we determine the coefficients d_n in such a way that the general solution (36) satisfies the initial condition (13). Plugging $t=0$ into the general solution (36) and making equal to the initial condition (13), we have

$$v(x,0) = f(x) + \frac{x}{l}(u_0 - u_1) - u_0 = \sum_{n=1}^{\infty} d_n \sin\left(w_n\left(\frac{x}{l}\right)\right). \quad (37)$$

By means of the inner product on $L^2[0,l]$, the coefficients d_n for $n = 1,2,3,\ldots$ are obtained as follows:

$$d_n = \frac{2}{l}\left[\int_0^l \sin\left(\frac{k\pi x}{l}\right) f(x)dx + (u_0 - u_1)\int_0^l \sin\left(\frac{k\pi x}{l}\right)\frac{x}{l}dx - u_0\int_0^l \sin\left(\frac{k\pi x}{l}\right)dx\right]. \quad (38)$$

Substituting (38) in (36) leads to the solution of the problem (12)-(14). By making use of (11) and this solution, we obtain the general solution of the problem (8)-(10).

4. ILLUSTRATIVE EXAMPLE

In this section, we first consider the following nonhomogenous initial boundary value problem:

$$u_t(x,t) = u_{xx}(x,t)\ , 0 \le x \le 2, 0 \le t \le T$$

$$u(0,t) = 1\ , u(2,t) = 1, 0 \le t \le T$$

$$u(x,0) = -\sin(\pi x) + 1, 0 \le x \le 2 \quad (39)$$

which has the solution in the following form:

$$u(x,t) = -\sin(\pi x)\, e^{-\pi^2 t} + 1. \quad (40)$$

Now let us take the following fractional heat-like problem into consideration:

$$D_t^{\alpha} u(x,t) = u_{xx}(x,t),\ \ 0 < \alpha < 1,\ \ 0 \le x \le 2, 0 \le t \le T \quad (41)$$

$$u(x,0) = -\sin(\pi x) + 1, 0 \le x \le 2 \quad (42)$$

$$u(0,t) = 1, u(2,t) = 1, 0 \le t \le T \quad (43)$$

To make the boundary conditions (43) homogenous, we apply the transformation

$$v(x,t) = u(x,t) - 1 \quad (44)$$

to the above problem which leads to the following fractional heat-like problem

$$D_t^{\alpha} v(x,t) = v_{xx}(x,t), \quad (45)$$

$$v(0,t) = 0\ , v(2,t) = 0, \quad (46)$$

$$v(x,0) = -\sin(\pi x) \quad (47)$$

where $0 < \alpha < 1, 0 \le x \le 2, 0 \le t \le T$.

By means of (36), the solution of problem (45)-(47) is represented in the following form:

$$v(x,t;\alpha) = \sum_{n=1}^{\infty} d_n \sin\left(w_n\left(\frac{x}{2}\right)\right) E_{\alpha,1}\left(-\frac{w_n^2}{2^2} t^{\alpha}\right). \quad (48)$$

The coefficients d_n in (48) are obtained by means of the equation (38) as follows:

$$\Rightarrow d_n = \int_0^2 -\sin\left(\frac{k\pi x}{2}\right)\sin(\pi x)\,dx\ .$$

$d_n = 0$ for $n \neq 2$. For $n = 2$, d_2 is obtained as follows:

$$\Rightarrow d_2 = -\int_0^2 \sin^2(\pi x)\,dx = -\frac{1}{2}\left(x + \frac{\sin(2\pi x)}{4\pi}\right)\Big|_{x=0}^{x=2} = -1. \quad (49)$$

Substituting (49) in (48) leads to the solution of the problem (45)-(47).

$$v(x,t;\alpha) = -\sin\left(w_2 \frac{x}{2}\right) E_{\alpha,1}\left(-\frac{w_2^2}{2^2} t^{\alpha}\right). \quad (50)$$

By making use of (44) and the solution (50), we obtain the general solution of the problem (41)-(43) as follows:

$$u(x,t;\alpha) = -\sin(\pi x)\,E_{\alpha,1}(-\pi^2 t^\alpha) + 1. \quad (51)$$

5. CONCLUSION

In this study, we determine the analytic solution of one dimensional time fractional initial boundary value problem with non-homogenous Dirichlet boundary conditions. By making use of seperation of variables, the solution is constructed in the form of a Fourier series in terms of the eigenfunctions of a corresponding Sturm-Liouville eigenvalue problem.

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