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Uniqueness of Inverse Sturm-Liouville Problems with Two Delta Interaction Points

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Abstract: We establish various uniqueness results for inverse spectral problems of Sturm-Liouville equations with two δ-interaction points.

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1. Introduction

We consider inverse problems for the boundary value problem (BVP)

$$(L(q(x)), h, H, a_s, \alpha_s = 1, 2)$$

generated by the differential equation

$$ly := -y'' + q(x)y = \lambda y, \qquad x \in (0, a_1) \cup (a_1, a_2) \cup (a_2, \pi), \tag{1}$$

with the Robin boundary conditions

$$U(y) := y'(0) - hy(0) = 0, \quad V(y) := y'(\pi) + Hy(\pi) = 0, \tag{2}$$

and the transmission conditions at the points $x = a_s, \ s = 1, 2$

$$I(y) := \begin{cases} y(a_s + 0) = y(a_s - 0) = y(a_s), \\ y'(a_s + 0) - y'(a_s - 0) = \alpha_s y(a_s) \end{cases} \tag{3}$$

where $q(x)$ is a reel-valued function in $L_2(0, \pi); h, H$ and all α_s are real numbers, and λ is a spectral parameter.

Notice that, we can understand problem (1) and (3) as analyzing the equation

$$y'' + \left(\lambda^2 - \alpha_1 \delta(x - a_1) - \alpha_2 \delta(x - a_2) - q(x)\right) y = 0 \tag{4}$$

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where $\delta(x)$ is the Dirac function (see [1]).

There are two main branches which are called direct and inverse problems for spectral problems of differential operators. One of the types of problems, the direct problem consists of examining the spectral properties of an operator. Another type of problem, inverse problems, aim to reconstruct the operator using their spectral properties. Direct and inverse problems for the classical Sturm-Liouville operators have been comprehensively investigated ([6, 9, 14] and the references therein). Some classes of direct and inverse problems for discontinuous BVPs in various statements have been considered in [2, 7, 8, 12, 13, 15, 16]. Notice that, inverse spectral problems for non-selfadjoint Sturm-Liouville operators on a finite interval with discontinuity inside an interval have been investigated in [10, 11].

In this study, we determine various uniqueness results for inverse spectral problems of Sturm-Liouville equations with δ-interactions for two point.

2. Properties of the Spectral Characteristics of L

In this part, we present the spectral characteristics of L and present the relationship among these spectral characteristics. The technique used is analogous to those used in [6].

Let $y(x)$ and $z(x)$ be continuously differentiable functions on the intervals $(0, a_1)$, (a_1, a_2), (a_2, π). Denote $\langle y, z\rangle := yz' - y'z$. If $y(x)$ and $z(x)$ satisfy the conditions (3), then

$$\langle y, z\rangle_{x=a_s-0} = \langle y, z\rangle_{x=a_s+0}, \; s = 1, 2, \tag{5}$$

i.e., the function $\langle y, z\rangle$ is continuous on $(0, \pi)$.

Let $\varphi(x,\lambda)$, $\psi(x,\lambda)$, $C(x,\lambda)$, $S(x,\lambda)$ be solutions of (1) under the conditions

$$C(0,\lambda) = \varphi(0,\lambda) = S'(0,\lambda) = \psi(\pi,\lambda) = 1, \; C'(0,\lambda) = S(0,\lambda) = 0, \quad \varphi'(0,\lambda) = h, \quad \psi'(\pi,\lambda) = -H \tag{6}$$

and the conditions (3). Then $U(\varphi) = V(\psi) = 0$.

Denote $\Delta(\lambda) := \langle \varphi(x,\lambda), \psi(x,\lambda)\rangle$. Thanks to (5) and the Ostrogradskii-Liouville theorem (see [4]), $\Delta(\lambda)$ does not depend on x. The function $\Delta(\lambda)$ is called the characteristic function of L. Obviously,

$$\Delta(\lambda) = -V(\varphi) = U(\psi). \tag{7}$$

Clearly, the function $\Delta(\lambda)$ has at most a countable set of zeros $\{\lambda_n\}$ and it is entire in λ.

Theorem 2.1 *The eigenvalues $\{\lambda_n\}_{n\geq 1}$ of the BVP L coincide with the zeros of the characteristic function. The functions $\varphi(x,\lambda_n)$ and $\psi(x,\lambda_n)$ are eigenfunctions, and*

$$\psi(x,\lambda_n) = \beta_n \varphi(x,\lambda_n), \quad \beta_n \neq 0.$$

Proof Let $\Delta(\lambda_0) = 0$. Then by $\langle \varphi(x, \lambda_0), \ \psi(x, \lambda_0) \rangle = 0$, we get $\psi(x, \lambda_0) = \beta_n \varphi(x, \lambda_0)$, and the functions $\psi(x, \lambda_0)$, $\varphi(x, \lambda_0)$ satisfy the boundary conditions (2). Hence, λ_0 is eigenvalue, and $\psi(x, \lambda_0)$, $\varphi(x, \lambda_0)$ are eigenfunctions related to λ_0.

Conversely, let λ_0 be an eigenvalue of L. We need to show that $\Delta(\lambda_0) = 0$. Assuming the converse, suppose that $\Delta(\lambda_0) \neq 0$. Then the functions $\psi(x, \lambda_0)$ and $\varphi(x, \lambda_0)$ are linearly independent. In this case $y(x, \lambda_0) = c_1 \varphi(x, \lambda_0) + c_2 \psi(x, \lambda_0)$ is a general solution of the problem L. If $c_1 \neq 0$, we can write

$$\varphi(x, \lambda_0) = \frac{1}{c_1} y(x, \lambda_0) - \frac{c_2}{c_1} \psi(x, \lambda_0).$$

Then we have

$$\langle \varphi(x, \lambda_0), \ \psi(x, \lambda_0) \rangle = -\frac{1}{c_1} \left[y'(\pi, \lambda_0) + H y(\pi, \lambda_0) \right] = 0.$$

Thus, the expression we get is contradiction.

Since for each eigenvalue there is come into being only one eigenfunction, there is come into being sequence β_n such that $\psi(x, \lambda_n) = \beta_n \varphi(x, \lambda_n)$.

Denote

$$\gamma_n = \int_0^\pi \varphi^2(x, \lambda_n) dx. \tag{8}$$

The set $\{\lambda_n, \gamma_n\}_{n \geq 1}$ is called the spectral data of L. □

Lemma 2.2 *The equality*

$$\dot{\Delta}(\lambda_n) = -\beta_n \gamma_n \tag{9}$$

holds. Here $\dot{\Delta}(\lambda) = \frac{d}{d\lambda} \Delta(\lambda)$.

Proof Since

$$-\psi''(x, \lambda) + q(x)\psi(x, \lambda) = \lambda \varphi(x, \lambda), \quad -\varphi''(x, \lambda_n) + q(x)\varphi(x, \lambda_n) = \lambda \psi(x, \lambda_n),$$

we obtain

$$\frac{d}{dx} \langle \varphi(x, \lambda), \ \psi(x, \lambda_n) \rangle = (\lambda - \lambda_n) \psi(x, \lambda) \varphi(x, \lambda_n).$$

Integrating from 0 to π and by aid of the conditions (2), (3), we have

$$(\lambda - \lambda_n) \int_0^\pi \psi(x, \lambda) \varphi(x, \lambda_n) dx = -\Delta(\lambda).$$

Because $\lambda \to \lambda_n$ Lemma 2.2, this yields

$$\dot{\Delta}(\lambda_n) = -\beta_n \gamma_n.$$

□

Theorem 2.3 *The eigenvalues* $\{\lambda_n\}$ *and the eigenfunctions* $\varphi(x,\lambda_n)$, $\psi(x,\lambda_n)$ *are real. All zeros of* $\Delta(\lambda)$ *are simple, i.e.,* $\dot{\Delta}(\lambda_n)\neq 0$. *Eigenfunctions related to different eigenvalues are orthogonal in* $L_2(0,\pi)$.

Proof Let λ_n and λ_k $(\lambda_n\neq\lambda_k)$ be eigenvalues with eigenfunctions $y_n(x)$ and $y_k(x)$ respectively. Using the conditions (2), (3), then integration by parts yields

$$\int_0^\pi \ell y_n(x).y_k(x)dx=\int_0^\pi y_n(x).\ell y_k(x)dx$$

and hence

$$\int_0^\pi y_n(x)y_k(x)dx=0.$$

Further, let $\lambda^0=u+iv, v\neq 0$ be a non-real eigenvalue with an eigenfunction $y^0(x,\lambda)\neq 0$. Since $q(x),h$ and H are real, we get that $\overline{\lambda_0}=u-iv$ is also the eigenvalue with the eigenfunction $\overline{y_0(x)}$. Since $\lambda^0\neq\overline{\lambda_0}$, we derive as before

$$\left\|y^0\right\|_{L_2}^2=\int_0^\pi y^0(x)\overline{y_0(x)}dx=0$$

which is impossible. Thus, all eigenvalues $\{\lambda_n\}$ of L are real, and consequently the eigenfunctions $\varphi(x,\lambda_n)$ and $\psi(x,\lambda_n)$ are real too. Since $\gamma_n\neq 0,\beta_n\neq 0$, we get by virtue of (9) that $\dot{\Delta}(\lambda_n)\neq 0$. □

Now, consider the solution $\varphi(x,\lambda)$. Let $C_0(x,\lambda)$ and $S_0(x,\lambda)$ interval be smooth solutions of (1) on the interval $(0,\pi)$ under the initial conditions $C_0(0,\lambda)=S'(0,\lambda)=1$, $C_0'(0,\lambda)=S(0,\lambda)=0$. Then

$$C(x,\lambda)=C_0(x,\lambda),S(x,\lambda)=S_0(x,\lambda),0<x<a_1, \tag{10}$$

$$C(x,\lambda)=A_1C_0(x,\lambda)+B_1S_0(x,\lambda),S(x,\lambda)=A_2C_0(x,\lambda)+B_2S_0(x,\lambda),a_1<x<a_2, \tag{11}$$

$$C(x,\lambda)=A_3C_0(x,\lambda)+B_3S_0(x,\lambda),S(x,\lambda)=A_4C_0(x,\lambda)+B_4S_0(x,\lambda),a_2<x<\pi \tag{12}$$

where

$$\begin{cases} A_1 = 1 - \alpha_1 C_0(a_1,\lambda) S_0(a_1,\lambda),\ B_1 = \alpha_1 \left[C_0(a_1,\lambda)\right]^2, \\ A_2 = -\alpha_1 \left[S_0(a_1,\lambda)\right]^2,\ B_2 = 1 + \alpha_1 C_0(a_1,\lambda) S_0(a_1,\lambda), \\ A_3 = \left[1 - \alpha_1 C_0(a_1,\lambda) S_0(a_1,\lambda)\right].\left[1 - \alpha_2 C_0(a_2,\lambda) S_0(a_2,\lambda)\right] - \alpha_1\alpha_2 \left[C_0(a_1,\lambda)\right]^2 \left[S_0(a_2,\lambda)\right]^2, \\ B_3 = \alpha_1 \left[C_0(a_1,\lambda)\right]^2 \left[1 + \alpha_2 C_0(a_2,\lambda) S_0(a_2,\lambda)\right] + \alpha_2 \left[C_0(a_2,\lambda)\right]^2 \left[1 - \alpha_1 C_0(a_1,\lambda) S_0(a_1,\lambda)\right], \\ A_4 = -\alpha_1 \left[S_0(a_1,\lambda)\right]^2 \left[1 - \alpha_2 C_0(a_2,\lambda) S_0(a_2,\lambda)\right] - \alpha_2 \left[1 + \alpha_1 C_0(a_1,\lambda) S_0(a_1,\lambda)\right] \left[S_0(a_2,\lambda)\right]^2, \\ B_4 = \left[1 + \alpha_1 C_0(a_1,\lambda) S_0(a_1,\lambda)\right] \left[1 + \alpha_2 C_0(a_2,\lambda) S_0(a_2,\lambda)\right] - \alpha_1\alpha_2 \left[C_0(a_1,\lambda)\right]^2 \left[S_0(a_2,\lambda)\right]^2. \end{cases} \tag{13}$$

Let $\lambda = \rho^2, \rho = \sigma + i\tau$. It is easy to verify that the $C_0(x,\lambda)$ satisfies the following relations:

$$C_0(x,\lambda) = \cos\rho x + \frac{\sin\rho x}{2\rho}\int_0^x q(t)dt + \frac{1}{2\rho}\int_0^x q(t)\sin\rho(x-2t)dt + O(\frac{1}{\rho^2}\exp(|\tau|\,x)), \tag{14}$$

$$C_0'(x,\lambda) = -\rho\sin\rho x + \frac{\cos\rho x}{2}\int_0^x q(t)dt + \frac{1}{2}\int_0^x q(t)\cos\rho(x-2t)dt + O(\frac{1}{\rho}\exp(|\tau|\,x)). \tag{15}$$

Analogously,

$$S_0(x,\lambda) = \frac{\sin\rho x}{\rho} - \frac{\cos\rho x}{2\rho^2}\int_0^x q(t)dt + \frac{1}{2\rho^2}\int_0^x q(t)\cos\rho(x-2t)dt + O(\frac{1}{\rho^3}\exp(|\tau|\,x)), \tag{16}$$

$$S_0'(x,\lambda) = \cos\rho x + \frac{\sin\rho x}{2\rho}\int_0^x q(t)dt - \frac{1}{2\rho}\int_0^x q(t)\sin\rho(x-2t)dt + O(\frac{1}{\rho^2}\exp(|\tau|\,x)). \tag{17}$$

By virtue of (13) and (14)-(17),

$$A_1 = 1 - \tfrac{1}{2\rho}\alpha_1\sin 2\rho a_1 + O(\tfrac{1}{\rho^2}\exp(|\tau|\,x)),\quad B_1 = \tfrac{1}{2}\alpha_1(1+\cos 2\rho a_1) + O(\tfrac{1}{\rho}\exp(|\tau|\,x)),$$

$$A_2 = O(\tfrac{1}{\rho^2}\exp(|\tau|\,x)),\quad B_2 = 1 + O(\tfrac{1}{\rho}\exp(|\tau|\,x)),$$

$$A_3 = 1 - \tfrac{1}{2\rho}\left[\alpha_1\sin 2\rho a_1 + \alpha_2\sin 2\rho a_2\right] + O(\tfrac{1}{\rho^2}\exp(|\tau|\,x)),$$

$$B_3 = \tfrac{1}{2}\left[\alpha_1(1+\cos 2\rho a_1) + \alpha_2(1+\cos 2\rho a_2)\right] + O(\tfrac{1}{\rho}\exp(|\tau|\,x)),$$

$$A_4 = O(\tfrac{1}{\rho^2}\exp(|\tau|\,x)), B_4 = 1 + O(\tfrac{1}{\rho}\exp(|\tau|\,x)).$$

Since $\varphi(x,\lambda) = C(x,\lambda) + h.S(x,\lambda)$, we calculate using (10)-(17):

$$\varphi(x,\lambda) = \cos\rho x + \begin{cases} +(h + \frac{1}{2}\int_0^x q(t)dt)\frac{\sin\rho x}{\rho} + O(\frac{1}{\rho^2}\exp(|\tau|x)), 0 < x < a_1 \\ (h + \frac{1}{2}\alpha_1 + \frac{1}{2}\int_0^x q(t)dt)\frac{\sin\rho x}{\rho} - \\ -\frac{1}{2}\alpha_1\frac{\sin\rho(2a_1 - x)}{\rho} + O(\frac{1}{\rho^2}\exp(|\tau|x)), \quad a_1 < x < a_2 \\ (h + \frac{1}{2}(\alpha_1+\alpha_2) + \frac{1}{2}\int_0^x q(t)dt)\frac{\sin\rho x}{\rho} - \frac{1}{2}\alpha_1\frac{\sin\rho(2a_1-x)}{\rho} - \\ -\frac{1}{2}\alpha_2\frac{\sin\rho(2a_2-x)}{\rho} + O(\frac{1}{\rho^2}\exp(|\tau|x)), \ a_2 < x < \pi \end{cases} \tag{18}$$

and

$$\varphi'(x,\lambda) = -\rho\sin\rho x + \begin{cases} (h + \frac{1}{2}\int_0^x q(t)dt)\cos\rho x + O(\frac{1}{\rho}\exp(|\tau|x)), \ 0 < x < a_1 \\ (h + \frac{1}{2}\alpha_1 + \frac{1}{2}\int_0^x q(t)dt)\cos\rho x + \\ +\frac{1}{2}\alpha_1\cos\rho(2a_1 - x) + O(\frac{1}{\rho}\exp(|\tau|x)), \ a_1 < x < a_2 \\ (h + \frac{1}{2}(\alpha_1+\alpha_2) + \frac{1}{2}\int_0^x q(t)dt)\cos\rho x + \frac{1}{2}\alpha_1\cos\rho(2a_1 - x) + \\ +\frac{1}{2}\alpha_2\cos\rho(2a_2 - x) + O(\frac{1}{\rho}\exp(|\tau|x)), \ a_2 < x < \pi. \end{cases} \tag{19}$$

It follows from (7),(18) and (19) that

$$\Delta(\lambda) = \rho\sin\rho\pi - \omega\cos\rho\pi - \frac{1}{2}(\alpha_1\cos\rho(2a_1 - \pi) + \alpha_2\cos\rho(2a_2 - \pi)) + O(\frac{1}{\rho}\exp(|\tau|x)) \tag{20}$$

where

$$\omega = h + H + \frac{1}{2}(\alpha_1+\alpha_2) + \frac{1}{2}\int_0^\pi q(t)dt.$$

Using (20) and Rouche's theorem ([5]), by the well-known method [3] for $n \to \infty$

$$\rho_n = n + o(1).$$

Similarly, from Rouche's theorem one can prove that for sufficiently large values of n, every circle $\sigma_n(\delta) = \{\rho : (\rho - n) \le \delta\}$ contains exactly one zero $\Delta(\rho^2)$. Since $\delta > 0$ is arbitrary, we must have

$$\rho_n = n + \varepsilon_n, \ \varepsilon_n = o(1), \ n \to \infty. \tag{21}$$

Since ρ_n are zeros of $\Delta(\rho^2)$, from (20) we get

$$\begin{array}{c} n\sin\varepsilon_n\pi - \omega\cos\varepsilon_n\pi - \frac{1}{2}(\alpha_1\cos 2na_1.\cos 2a_1\varepsilon_n \\ +\alpha_2\cos 2na_2.\cos 2a_2\varepsilon_n).\cos\varepsilon_n\pi + \sigma_n = 0 \end{array} \tag{22}$$

where

$$\begin{array}{c} \sigma_n = \varepsilon_n\sin\varepsilon_n\pi - \frac{1}{2}(\alpha_1\sin 2(n+\varepsilon_n)a_1 \\ +\alpha_2\sin 2(n+\varepsilon_n)a_2)\sin\varepsilon_n\pi + o(\exp|\tau_n|\pi), \ \tau_n = \operatorname{Im}\rho_n. \end{array}$$

Hence $\sin \varepsilon_n \pi = O(\frac{1}{n})$, that is, $\varepsilon_n = O(\frac{1}{n})$. Using (22) we get more precisely

$$\varepsilon_n = \frac{1}{\pi n}(\omega + \frac{1}{2}\left(\alpha_1 \cos 2na_1 + \alpha_2 \cos 2na_2\right)) + O(\frac{1}{n}). \tag{23}$$

Substituting (23) into (21), we get

$$\rho_n = n + \frac{1}{\pi n}(\omega + \frac{1}{2}\alpha_1 \cos 2na_1 + \frac{1}{2}\alpha_2 \cos 2na_2) + O(\frac{1}{n}). \tag{24}$$

At last, using (8), (18), (19) and (24) one can calculate

$$\gamma_n = \frac{\pi}{2} + \frac{\omega_n}{n} + O(\frac{1}{n})$$

where

$$\omega_n = -\frac{1}{2}(a_2 - a_1)\alpha_1 \sin 2na_1 - \frac{1}{2}(\pi - a_2)(\alpha_1 \sin 2na_1 + \alpha_2 \sin 2na_2).$$

If $q(x)$ is a smooth function one can more precise asymptotics for the spectral data.

3. Formulation of the Inverse Problem and Uniqueness Theorems

In this part, we investigate three types inverse problems of rescuing L by using its spectral characteristics, namely

(i) from the Weyl function,

(ii) from the so-called spectral data,

(iii) from two spectra.

For each class of inverse problems we show the relation between the different spectral characteristics and prove the corresponding uniqueness theorems.

Let $\Phi(x,\lambda)$ be the solution of (4) under the conditions $U(\Phi) = 1$ and $V(\Phi) = 0$. We set $M(\lambda) := \Phi(0,\lambda)$. The functions $\Phi(x,\lambda)$ and $M(\lambda)$ are called the Weyl solution and the Weyl function for the BVP L, respectively. Clearly,

$$\Phi(x,\lambda) = \frac{\psi(x,\lambda)}{\Delta(\lambda)} = S(x,\lambda) + M(\lambda)\varphi(x,\lambda), \tag{25}$$

$$M(\lambda) = \frac{\Delta^1(\lambda)}{\Delta(\lambda)} \tag{26}$$

where $\Delta^1(\lambda) = \psi(0,\lambda) = V(S)$ is the characteristic function of the BVP L_1, which is equation (4) with the boundary conditions $U(y) = 0$, $y(\pi) = 0$ and $S(x,\lambda)$ is defined from the equality

$$\psi(x,\lambda) = \Delta^1(\lambda)\varphi(x,\lambda) + \Delta(\lambda)S(x,\lambda).$$

Let $\{\mu_n\}_{n\geq 1}$ be zeros of $\Delta^1(\lambda)$, i.e., the eigenvalues of L_1.

First, let us prove the uniqueness theorems for the solutions of the problems $(i)-(iii)$. For this purpose we agree that together with L we consider a BVP $\widetilde{L}$ of the same form but with different coefficients $\widetilde{q}(x), \widetilde{h}, \widetilde{H}, \widetilde{a}_s$ and $\widetilde{\alpha}_s$, $s=1,2$. Everywhere below if a certain symbol e denotes an object to L, then the corresponding symbol $\widetilde{e}$ with tilde denotes the analogous object related to $\widetilde{L}$.

Theorem 3.1 *If $M(\lambda)=\widetilde{M}(\lambda)$, then $L=\widetilde{L}$. Thus, the specification of the Weyl function $\widetilde{M}(\lambda)$, uniquely determines the operator L.*

Proof Let us define the matrix $P(x,\lambda)=[P_{jk}(x,\lambda)]_{j,k=1,2}$ by the formula

$$P(x,\lambda)=\begin{bmatrix}\widetilde{\varphi}(x,\lambda) & \widetilde{\Phi}(x,\lambda)\\ \widetilde{\varphi'}(x,\lambda) & \widetilde{\Phi'}(x,\lambda)\end{bmatrix}=\begin{bmatrix}\varphi(x,\lambda) & \Phi(x,\lambda)\\ \varphi'(x,\lambda) & \Phi'(x,\lambda)\end{bmatrix}. \tag{27}$$

By (25), we calculate

$$\left.\begin{array}{c} P_{j1}(x,\lambda)=\varphi^{(j-1)}(x,\lambda)\widetilde{\Phi'}(x,\lambda)-\Phi^{(j-1)}(x,\lambda)\widetilde{\varphi'}(x,\lambda)\\ P_{j2}(x,\lambda)=\Phi^{(j-1)}(x,\lambda)\widetilde{\varphi}(x,\lambda)-\varphi^{(j-1)}(x,\lambda)\widetilde{\Phi}(x,\lambda)\end{array}\right\} \tag{28}$$

and

$$\left.\begin{array}{c} \varphi(x,\lambda)=P_{11}(x,\lambda)\widetilde{\varphi}(x,\lambda)+P_{12}(x,\lambda)\widetilde{\varphi'}(x,\lambda)\\ \Phi(x,\lambda)=P_{11}(x,\lambda)\widetilde{\Phi}(x,\lambda)+P_{12}(x,\lambda)\widetilde{\Phi'}(x,\lambda)\end{array}\right\}. \tag{29}$$

According to (25) and (28), for each fixed x, the functions $P_{jk}(x,\lambda)$ are meromorphic in λ with simple poles in the points λ_n and $\widetilde{\lambda}_n$. Denote $G_\delta^0=G_\delta\cap\widetilde{G_\delta}$, where $G_\delta:=\{\lambda:|\lambda-n|\geq\delta\}$. By virtue of (18), (19) and (28) we get

$$|P_{11}(x,\lambda)-1|\leq C_\delta|\lambda|^{-1},\quad |P_{12}(x,\lambda)|\leq C_\delta|\lambda|^{-1},\ \lambda\in G_\delta^0 \tag{30}$$

where C_δ is a constant.

On the other hand according to (25) and (28),

$$P_{11}(x,\lambda)=\varphi(x,\lambda)\widetilde{S}(x,\lambda)+S(x,\lambda)\widetilde{\varphi}(x,\lambda)+(\widetilde{M}(\lambda)-M(\lambda))\varphi(x,\lambda)\widetilde{\varphi'}(x,\lambda),$$

$$P_{12}(x,\lambda)=S(x,\lambda)\widetilde{\varphi}(x,\lambda)+\varphi(x,\lambda)\widetilde{S}(x,\lambda)+(M(\lambda)-\widetilde{M}(\lambda))\varphi(x,\lambda)\widetilde{\varphi}(x,\lambda).$$

Since $M(\lambda)=\widetilde{M}(\lambda)$, it follows that for each fixed x, the functions $P_{1k}(x,\lambda), k=1,2$, are entire in λ. Together with (30) this yields $P_{11}(x,\lambda)=1$, $P_{12}(x,\lambda)=0$. Substituting into (29), we obtain $\varphi(x,\lambda)=\widetilde{\varphi}(x,\lambda)$, $\Phi(x,\lambda)=\widetilde{\Phi}(x,\lambda)$ for all $x\in(0,a_1)\cup(a_1,a_2)\cup(a_2,\pi)$ and λ. Taking this into account, from (1) we obtain $q(x,\lambda)=\widetilde{q}(x,\lambda)$ on $(0,\pi)$, from (6) we obtain $h=\widetilde{h}$, $H=\widetilde{H}$, and from (3) we conclude that $a_s=\widetilde{a_s}$, $\alpha_s=\widetilde{\alpha_s}$, $s=1,2$. Consequently, $L=\widetilde{L}$. □

Theorem 3.2 *If $\lambda_n=\widetilde{\lambda_n}$, $\gamma_n=\widetilde{\gamma_n}, n\geq 1$, then $L=\widetilde{L}$. Thus, the specification of the spectral data $\{\lambda_n,\gamma_n\}_{n\geq 1}$ uniquely determines the operator L.*

Proof It follows from (26) that the Weyl function $M(\lambda)$ ise meromorphic with simple poles at points. Using (26), Theorem 2.1 and equality $\dot{\Delta}(\lambda_n)=-\beta_n\gamma_n$, we have

$$\operatorname{Re}\mathrm{s}_{\lambda=\lambda_n}M(\lambda)=\frac{\psi(0,\lambda)}{\dot{\Delta}(\lambda_n)}=\frac{\beta_n}{\dot{\Delta}(\lambda_n)}=\frac{1}{\gamma_n}. \tag{31}$$

Since the Weyl function $M(\lambda)$ is regular for $\lambda\in\tau_n$, applying the Rouche theorem ([5]), we conclude that

$$M(\lambda)=\frac{1}{2\pi i}\int_{\tau_n}\frac{M(\mu)}{\lambda-\mu}d\mu,\ \lambda=int\tau_n$$

where the contour τ_n is assumed to have the counterclockwise circuit. Calculating this integral by the residue theorem and taking (31) into account we arrive at

$$M(\lambda)=\sum_{k=1}^{\infty}\frac{1}{\gamma_k(\lambda-\lambda_k)}. \tag{32}$$

Under the hypothesis of the theorem we obtain, in view of (32), that $M(\lambda)=\widetilde{M}(\lambda)$, and consequently by Theorem 3.1, $L=\widetilde{L}$. □

Theorem 3.3 *If $\lambda_n=\widetilde{\lambda}_n$ and $\mu_n=\widetilde{\mu}_n$, $n\geq 1$, then $L=\widetilde{L}$. Thus the specification of two spectra $\{\lambda_n,\mu_n\}_{n\geq 1}$ uniquely determines L.*

Proof It is obvious that characteristic functions $\Delta(\lambda)$ and $\Delta^1(\lambda)$ are uniquely determined by the sequences $\{\lambda_n\}_{n\geq 1}$ and $\{\mu_n\}_{n\geq 1}$, respectively. If $\lambda_n=\widetilde{\lambda}_n$, $\mu_n=\widetilde{\mu}_n$, $n\geq 1$, then $\Delta(\lambda)=\widetilde{\Delta}(\lambda)$, $\Delta^1(\lambda)=\widetilde{\Delta}^1(\lambda)$. Together with (26) this yields $M(\lambda)=\widetilde{M}(\lambda)$. By Theorem 3.1, we obtain $L=\widetilde{L}$. □

Remark 3.4 *By (26), the specification of two spectra $\{\lambda_n,\mu_n\}_{n\geq 1}$ is equivalent to the specification of the Weyl function $M(\lambda)$. On the other hand, if follows from (32) that the specification of the Weyl function $M(\lambda)$ is equivalent to the specification of the data $\{\lambda_n,\gamma_n\}_{n\geq 1}$. Consequently, three statements of the inverse problem of reconstruction of the problem L from the Weyl function, from the spectral data and from two spectra are equivalent.*

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