



A New Numerical Scheme for Singularly Perturbed Reaction-Diffusion Problems

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Highlights

- In the paper, we propose the numerical solution for the singular perturbation problem.
- We demonstrate that the method is first-order uniformly convergent.
- The example is solved by using the Thomas algorithm.

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Abstract

This study is related to a novel numerical technique for solving the singularly perturbed reaction-diffusion boundary value problems. First, explicit boundaries for the solution of the problem are established. Then, a finite difference scheme is established on a uniform mesh supported by the method of integral identities using the remainder term in integral form and the exponential rules with weight. The uniform convergence and stability of these schemes are investigated concerning the perturbation parameter in the discrete maximum norm. At last, the numerical results that provide theoretical results are presented.

1. INTRODUCTION

In the present study, we deal with the singularly perturbed reaction-diffusion problem in the form of a boundary value as follows:

$$Lu \equiv -\varepsilon u'' + a(x)u = f(x), \quad 0 < x < l, \quad (1)$$

$$u(0) = A, \quad u(l) = B. \quad (2)$$

Here, $0 < \varepsilon \ll 1$ is a small positive parameter, A and B represent constants. Further, we suppose that $a(x) \geq \alpha > 0$ and $f(x)$ are smooth enough functions in $[0, l]$. Under these conditions obtained, the reaction-diffusion problem has a unique solution $u(x)$, which can show two boundary layers near $x = 0$ and $x = l$ for small ε values.

The problems with singular perturbation properties for differential equations are mathematically known as problems in which the coefficients of the terms containing the highest-order derivative are a positive small parameter. The solution to such problems has a very rapid change in some parts of layer regions. In other words, the solutions change fast in thin transition layers called boundary layers and change regularly and slowly in other regions. These problems emerge in various fields of mathematics, including Navier–Stokes equation of fluid flow at higher Reynolds numbers, fluid dynamics, chemical reactions, oceanography, meteorology, quantum mechanics, reaction-diffusion status, theory of magneto-hydrodynamics duct problems, elasticity, etc [1-8].

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These problems are known to be caused by a positive small parameter ε for which the solution has a multiscale property. Therefore, these types of problems have thin transitional layers where solutions change very quickly for small values, and these solutions which are moving away from layers appear to change regularly and slowly. Thus, exploring the properties of the solution within boundary layers or near corners is the main difficulty as it requires a fine enough grid to result in a numerical solution [9-13].

Almost most of the traditional numerical techniques are not appropriate for singularly perturbed boundary value problems. Therefore, we require to find uniform convergent techniques to solve these problems, whose precision and accuracy of which do not depend on the values that the parameter will take. Some of the most useful and the easiest methods to produce these methods consist of fitted finite difference and finite element methods that use private elements like exponential elements and fitted mesh methods. Here the finite difference method mentioned is applicable very readily to a uniform mesh. In recent years, a great deal of research has been studied on numerical methods to solve singular perturbation problems [14-24]. The method we use has many advantages because fewer smooth conditions are required for the estimate of our problem. Therefore, the method is more useful and gives good results. Because of these results, we can also use our method for different types of singular perturbation boundary value problems.

We will show the structure of the article as follows:

In section 2, we present some features of the solution of the problem (1)-(2). In Section 3, we have created the finite difference scheme and the mesh for the singularly perturbed reaction-diffusion boundary value problems. In section 4, the stability analysis, error evaluations and uniform convergence of the singular perturbation reaction-diffusion boundary value problems have been proven according to the perturbation parameter at the maximum norm. Lastly, the algorithm is presented for the singularly perturbed problem, and numerical results are given in tables and graphs.

2. PROPERTIES OF THE EXACT SOLUTION AND ITS DERIVATIVE

This section has presented some features of solutions to the problem (1)-(2). These properties are used in the next sections for the analysis of the convenient numerical solutions. Here, we will use the following notations.

Lemma 2.1. We suppose that $a(x), f(x) \in C^1[0, l]$. Then the following estimates provide the solution $u(x)$ of the singular perturbation problem (1)-(2)

$$\|u\| \leq C_0, \quad (3)$$

and

$$|u'(x)| \leq C \left\{ 1 + \frac{1}{\sqrt{\varepsilon}} \left(e^{-\frac{\sqrt{\alpha}x}{\sqrt{\varepsilon}}} + e^{-\frac{\sqrt{\alpha}(l-x)}{\sqrt{\varepsilon}}} \right) \right\}, \quad 0 \leq x \leq l. \quad (4)$$

Here, we take

$$C_0 = |A| + |B| + \alpha^{-1} \max_{x \in [0, l]} |f(x)|.$$

Proof. Firstly, let us demonstrate the accuracy of (3). Here we will apply the maximum principle for the singularly perturbed problem (1)-(2).

Assume that L be the differential operator in the the Equation (1) and $v(x) \in C^1[0, l]$. If $v(0) \geq 0$, $v(l) \geq 0$ and $Lv(x) \geq 0$, afterward $0 < x < l$, then $v(x) \geq 0$ for all $0 \leq x \leq l$.

Next, consider a Barrier function as follows

$$\Psi(x) = \pm u(x) + |A| + |B| + \alpha^{-1} \max_{x \in [0, l]} |f(x)|.$$

From the maximum principle, we find the relation as follows

$$L\Psi(x) = \pm f(x) + a(x)(|A| + |B|) + a(x)\alpha^{-1} \max_{x \in [0, l]} |f(x)|, \quad (a(x) \geq \alpha > 0)$$

$$\geq \pm f(x) + |A| + |B| + \max_{x \in [0, l]} |f(x)|$$

$$\geq \pm f(x) + \max_{x \in [0, l]} |f(x)| \geq 0,$$

$$\Psi(0) = \pm u(0) + |A| + |B| + \alpha^{-1} \max_{x \in [0, l]} |f(0)| = \pm A + |A| + |B| + \alpha^{-1} \max_{x \in [0, l]} |f(0)| \geq 0,$$

$$\Psi(l) = \pm B + |B| + |A| + \alpha^{-1} \max_{x \in [0, l]} |f(l)| \geq 0,$$

and

$$\Psi(x) = \pm u(x) + |A| + |B| + \alpha^{-1} \max_{x \in [0, l]} |f(x)| \geq 0.$$

By applying the barrier function obtained after these processes to the conditions in Lemma 2.1, we obtain the following expression

$$|u(x)| \leq |A| + |B| + \alpha^{-1} \max_{x \in [0, l]} |f(x)|. \quad (5)$$

It gives the proof of (3).

Now, let us give the proof of the inequality (4). From (1)-(2), we can write as follows

$$|u''(x)| \leq \frac{|f(x) - a(x)u(x)|}{\varepsilon} \leq \frac{|f| + |a||u|}{\varepsilon}.$$

From the inequality (3), we obtain

$$|u''(x)| \leq \frac{C}{\varepsilon}, \quad x \in [0, l]. \quad (6)$$

Then, we need to get relations for $|u'(0)|$ and $|u'(l)|$. Here we will use the following relationship for $g \in C^2$ and $\gamma \neq \beta$,

$$g'(x) = \frac{g(\gamma) - g(\beta)}{\gamma - \beta} - \int_{\beta}^{\gamma} \left[\frac{\gamma - \xi}{\gamma - \beta} - T_0(x - \xi) \right] g''(\xi) d\xi, \quad \gamma \leq x \leq \beta, \quad (7)$$

where

$$T_0(\lambda) = \begin{cases} 1, & \lambda \geq 0, \\ 0, & \lambda < 0. \end{cases}$$

First, we evaluate $u'(0)$. Using the values $\beta = 0$, $\gamma = \sqrt{\varepsilon}$, $x = 0$, and $g(x) \equiv u(x)$ in the Equation (7), we get

$$|u'(0)| \leq \frac{|u(\sqrt{\varepsilon})|}{\sqrt{\varepsilon}} + \int_0^{\sqrt{\varepsilon}} \frac{\sqrt{\varepsilon} - \xi}{\sqrt{\varepsilon}} |u''(\xi)| d\xi.$$

Considering the relations (3) and (6) in this inequality, we have

$$|u'(0)| \leq \frac{C}{\sqrt{\varepsilon}}. \quad (8)$$

Now we take a review for $|u'(l)|$. Using the values $\beta = l - \sqrt{\varepsilon}$, $\gamma = l$, and $g(x) \equiv u(x)$, $x = l$ in Equation (7), we take the following inequality

$$|u'(l)| \leq \frac{C}{\sqrt{\varepsilon}}. \quad (9)$$

If we take derivative from Equation (1), we have

$$-\varepsilon v'' + a(x)v = \Phi(x). \quad (10)$$

Here we get

$$v(x) = u'(x), \quad \Phi(x) = f'(x) - a'(x)u(x).$$

From the relations (8) and (9), we have

$$|v(0)| = O\left(\frac{1}{\sqrt{\varepsilon}}\right), \quad |v(l)| = O\left(\frac{1}{\sqrt{\varepsilon}}\right). \quad (11)$$

We estimate the function $\Phi(x)$ given below

$$|\Phi(x)| \leq |f'(x)| + |a'(x)||u(x)| \leq C.$$

The solution to the problem (10)-(11) is shown as follows

$$v(x) = v_1(x) + v_2(x).$$

The functions $v_1(x)$ and $v_2(x)$ are given as solutions of the following problems, respectively:

$$Lv_1(x) = \Phi(x), \quad (12)$$

$$v_1(0) = v_1(l) = 0, \quad (13)$$

and

$$Lv_2(x) = 0, \quad (14)$$

$$v_2(0) = v(0), \quad v_2(l) = v(l). \quad (15)$$

The solution to the problem (12)-(13) is written according to the maximum principle as follows

$$|v_1(x)| \leq \alpha^{-1} \max_{x \in [0, l]} |\Phi(x)|. \quad (16)$$

Because the function $\Phi(x)$ is uniformly restricted according to ε , we have

$$|v_1(x)| \leq C, \quad 0 \leq x \leq l. \quad (17)$$

The solution to the problem (14)-(15) is written according to the maximum principle as follows

$$|v_2(x)| \leq \theta(x). \quad (18)$$

From these inequalities, the function $\theta(x)$ is demonstrated to be a solution of the problem in the form below:

$$-\varepsilon\theta''(x) + \alpha\theta(x) = 0, \quad (19)$$

$$\theta(0) = |v_2(0)|, \quad \theta(l) = |v_2(l)|. \quad (20)$$

From problem (19)-(20), we obtain

$$\theta(x) = \theta(0) \frac{\sinh\left(\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}(l-x)\right)}{\sinh\left(\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}l\right)} + \theta(l) \frac{\sinh\left(\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}x\right)}{\sinh\left(\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}l\right)}.$$

According to the boundary conditions of the problem (17), we have

$$|\theta(x)| \leq \frac{c}{\sqrt{\varepsilon}} \left\{ e^{-\frac{\sqrt{\alpha}x}{\sqrt{\varepsilon}}} + e^{-\frac{\sqrt{\alpha}(l-x)}{\sqrt{\varepsilon}}} \right\}. \quad (21)$$

From the inequalities (17), (18), and (21), we can write

$$|u'(x)| \leq |v_1(x)| + |v_2(x)|,$$

which arrive at the proof of (4).

Thus, we arrive at the proof of Lemma 2.1.

3. DISCRETIZATION

Here, the continuous problem (1)-(2) is constructed by the finite difference scheme. Then, ω_h is described with a uniform mesh on the range $[0, l]$ as follows:

$$\omega_h = \left\{ x_i = ih, \quad i = 1, \dots, N-1; \quad h = \frac{l}{N} \right\},$$

and

$$\bar{\omega}_h = \omega_h \cup \{x = 0 \text{ and } x = l\}.$$

To create the difference scheme, we begin with the following identity:

$$\chi_i^{-1} h^{-1} \int_{x_{i-1}}^{x_{i+1}} Lu(x)\varphi_i(x)dx = \chi_i^{-1} h^{-1} \int_{x_{i-1}}^{x_{i+1}} f(x)\varphi_i(x)dx, \quad i = 1, \dots, N-1. \quad (22)$$

Here $\varphi_i(x)$ are basis functions defined in the following format

$$\varphi_i(x) = \begin{cases} \varphi_i^{(1)}(x) = \frac{\sinh(\gamma_i(x - x_{i-1}))}{\sinh(\gamma_i h)}, & x_{i-1} < x < x_i, \\ \varphi_i^{(2)}(x) = \frac{\sinh(\gamma_i(x_{i+1} - x))}{\sinh(\gamma_i h)}, & x_i < x < x_{i+1}, \\ 0, & x \notin (x_{i-1}, x_{i+1}), \end{cases}$$

$$\gamma_i = \sqrt{\frac{a(x_i)}{\varepsilon}},$$

$$\chi_i = h^{-1} \int_{x_{i-1}}^{x_i} \varphi_i^{(1)}(x) dx + h^{-1} \int_{x_i}^{x_{i+1}} \varphi_i^{(2)}(x) dx = \frac{2 \tanh(\gamma_i h/2)}{\gamma_i h}.$$

It can easily be seen that the basis functions $\varphi_i^{(1)}(x)$ and $\varphi_i^{(2)}(x)$ are given as solutions of the problems in the form below, respectively:

$$-\varepsilon \varphi''(x) + a_i \varphi(x) = 0, \quad x_{i-1} < x < x_i, \quad (23)$$

$$\varphi(x_{i-1}) = 0, \quad \varphi(x_i) = 1, \quad (24)$$

and

$$-\varepsilon \varphi'' + a_i \varphi = 0, \quad x_i < x < x_{i+1}, \quad (25)$$

$$\varphi(x_i) = 1, \quad \varphi(x_{i+1}) = 0. \quad (26)$$

If the necessary arrangements are made and partial integration is applied in Equations (19)-(20), we obtain

$$\chi_i^{-1} h^{-1} \varepsilon \int_{x_{i-1}}^{x_{i+1}} \varphi_i'(x) u'(x) dx + \chi_i^{-1} h^{-1} a_i \int_{x_{i-1}}^{x_{i+1}} u(x) \varphi_i(x) dx = f_i - R_i, \quad i = 1, \dots, N-1. \quad (27)$$

Here the remaining term R_i is defined as follows

$$R_i = \chi_i^{-1} h^{-1} \int_{x_{i-1}}^{x_{i+1}} [a(x) - a(x_i)] u(x) \varphi_i(x) dx + \chi_i^{-1} h^{-1} \int_{x_{i-1}}^{x_{i+1}} [f(x_i) - f(x)] \varphi_i(x) dx. \quad (28)$$

If the quadrature formulas (2.1) and (2.2) in the study [24] are applied to the Equation (27) on each of the intervals (x_{i-1}, x_i) and (x_i, x_{i+1}) , we find the exact relation below

$$\begin{aligned} & \chi_i^{-1} h^{-1} \varepsilon \int_{x_{i-1}}^{x_{i+1}} \varphi_i'(x) u'(x) dx + \chi_i^{-1} h^{-1} a_i \int_{x_{i-1}}^{x_{i+1}} u(x) \varphi_i(x) dx \\ &= -\varepsilon \chi_i^{-1} \left\{ 1 + a_i \varepsilon^{-1} \int_{x_{i-1}}^{x_i} \varphi_i'(x) (x - x_i) dx \right\} u_{\bar{x}, i} \\ & \quad + a_i \chi_i^{-1} \left\{ h^{-1} \int_{x_{i-1}}^{x_i} \varphi_i^{(1)}(x) dx + h^{-1} \int_{x_i}^{x_{i+1}} \varphi_i^{(2)}(x) dx \right\} u_i \\ &= -\varepsilon \theta_i u_{\bar{x}, i} + a_i u_i. \end{aligned} \quad (29)$$

Here θ_i is taken as follows

$$\begin{aligned} \theta_i &= \chi_i^{-1} \left\{ 1 + a_i \varepsilon^{-1} \int_{x_{i-1}}^{x_i} \varphi_i^{(1)}(x) (x - x_i) dx \right\} \\ &\equiv \chi_i^{-1} \left\{ 1 - a_i \varepsilon^{-1} \int_{x_i}^{x_{i+1}} \varphi_i^{(2)}(x) (x - x_i) dx \right\}. \end{aligned}$$

After doing simple operations on the coefficient θ_i , we obtain

$$\theta_i = \frac{\rho^2 a_i}{4 \sinh^2(\sqrt{a_i} \rho / 2)}, \quad \rho = \frac{h}{\sqrt{\varepsilon}}. \quad (30)$$

If we go back to (29) and consider the relation (27), we can write a difference scheme for the Equation (1) as follows,

$$lu_i \equiv -\varepsilon \theta_i u_{\bar{x}x,i} + a_i u_i = f_i - R_i, \quad i = 1, \dots, N-1. \quad (31)$$

If the remaining term R_i is omitted in (32), to approach the problem (1)-(2) we receive a difference scheme in the form below.

$$ly_i \equiv -\varepsilon \theta_i y_{\bar{x}x,i} + a_i y_i = f_i, \quad i = 1, \dots, N-1, \quad (32)$$

$$y_0 = A, \quad y_N = B. \quad (33)$$

Here the coefficient θ_i is given by the formula (30).

4. STABILITY BOUND AND UNIFORM CONVERGENCE

To research the convergence of the difference scheme, notice that the error function $z_i = y_i - u_i$, $i = 1, \dots, N-1$ is a solution of the discrete singularly perturbed problem:

$$lz_i \equiv -\varepsilon \theta_i z_{\bar{x}x,i} + a_i z_i = R_i, \quad i = 1, \dots, N-1, \quad (34)$$

$$z_0 = 0, \quad z_N = 0, \quad (35)$$

where the truncation error R_i and the coefficient θ_i are given by the relations (28) and (30), respectively.

Lemma 4.1. Under the conditions of $a(x)$, $f(x) \in C^1[0, l]$, the remaining term R_i provides the following inequality

$$\|R\|_{\infty, \omega_h} \leq Ch. \quad (36)$$

Proof. From the correlation (28), we can write

$$R_i = R_{a,i} + R_{f,i}. \quad (37)$$

Here we get

$$R_{a,i} = \chi_i^{-1} h^{-1} \int_{x_{i-1}}^{x_{i+1}} u(x) [a(x_i) - a(x)] dx, \quad (38)$$

and

$$R_{f,i} = \chi_i^{-1} h^{-1} \int_{x_{i-1}}^{x_{i+1}} \varphi_i(x) [f(x) - f(x_i)] dx. \quad (39)$$

Firstly, we show the remaining term $R_{a,i}$. If we apply the mean value theorem in the relation (38), the following expression is obtained

$$|a(x_i) - a(x)| = |a'(\xi_i)(x - x_i)| \leq \max_{[0,l]} |a'(x)| |x - x_i| \leq C_0 h. \quad (40)$$

If we write this inequality in (33), we obtain

$$|R_{a,i}| \leq C_0 C_1 h \chi_i^{-1} h^{-1} \int_{x_{i-1}}^{x_{i+1}} \varphi_i(x) dx \leq C_0 C_1 h \leq Ch.$$

Here, we get

$$|R_{a,i}| \leq Ch. \quad (41)$$

Now, let us show the remaining term $R_{f,i}$. If we apply the mean-value theorem in the relation (39), we obtain

$$|f(x) - f(x_i)| = |f'(\eta_i)(x - x_i)| \leq \max_{[0,l]} |f'(x)| |x - x_i| \leq C_1 h \quad \eta_i \in (x, x_i). \quad (42)$$

If we write this inequality in (40), we get

$$|R_{f,i}| \leq C_1 h \chi_i^{-1} h^{-1} \int_{x_{i-1}}^{x_{i+1}} \varphi_i(x) dx \leq Ch.$$

From this inequality, we have

$$|R_{f,i}| \leq Ch. \quad (43)$$

Hence, taking into consideration the inequalities (41) and (43) in (37), we obtain

$$\|R\|_{\infty, \omega_h} \leq \|R_a\|_{\infty, \omega_h} + \|R_f\|_{\infty, \omega_h} \leq Ch.$$

Thus, we can express the convergence result of the problem (1)-(2). Therefore, we show the validity of Lemma 4.1.

Lemma 4.2. Suppose that the error function $z_i = y_i - u_i$, $i = 1, \dots, N$ is the solution to the difference problem (34)-(35). The following inequality holds:

$$\|z\|_{\infty, \bar{\omega}_h} \leq \alpha^{-1} \|R\|_{\infty, \omega_h}. \quad (44)$$

Proof. We use the discrete maximum principle. We can write the discrete maximum principle as follows

$$lv_i \geq 0, \quad v_0 \geq 0, \quad v_N \geq 0, \quad v_i \geq 0, \quad i = 1, \dots, N - 1.$$

Applying the maximum principle to the problem (34)-(35), we have

$$\|z\|_{\infty, \bar{\omega}_h} \leq \alpha^{-1} \|R\|_{\infty, \omega_h}. \quad (45)$$

Also, from the formula (7) for R_i , it is clear that

$$\|R\|_{\infty, \omega_h} \leq Ch. \quad (46)$$

So, the proof of Lemma 4.2 is completed from inequalities (45) and (46).

We now can express the result of the convergence of the problem (3)-(4).

Theorem 4.1. Under the conditions $a(x), f(x) \in C^1[0, l]$, the solution to the singularly perturbed difference problem (32)-(33) is uniform convergence to the solution of the singularly perturbed problem (1)-(2) in $\bar{\omega}_h$ with respect to ε . The following evaluation satisfies:

$$\|y - u\|_{C(\bar{\omega}_h)} \leq Ch. \quad (47)$$

Proof. Here, the proof of the Theorem 4.1 emerges from the two lemmas we explained earlier.

5. NUMERICAL EXAMPLES

In this chapter, to verify the constructed theoretical results we give numerical results obtained using the difference scheme (32)-(33).

Example 5.1. Let us take the problem

$$-\varepsilon u'' + u = -(\cos^2 \pi x + 2(\varepsilon \pi)^2 \cos 2\pi x), \quad 0 < x < l,$$

$$u(0) = 0, \quad u(1) = 0.$$

We find the exact solution to the problem as follows

$$u_\varepsilon(x) = \frac{e^{\frac{-x}{\sqrt{\varepsilon}}} + e^{\frac{-(1-x)}{\sqrt{\varepsilon}}}}{1 + e^{\frac{-1}{\sqrt{\varepsilon}}}} - \cos^2 \pi x.$$

Thomas algorithm is used in this section. Because of a three-diagonal matrix in the setup of the difference scheme, this matrix is solved with the help of the Thomas algorithm. For the Thomas algorithm, $\alpha_1 = 0$ and $\beta_1 = 0$ is taken in the boundary term

$$y_0 = k_1 y_1 + \mu_1 (y_1 \neq 0).$$

Let us organize the recurrence relation of

$$y_i = y_{i+1} \alpha_{i+1} + \beta_{i+1},$$

by writing $A_i \neq 0$ and $B_i \neq 0$ in the open writing of difference problems

$$A_i y_{i-1} - C_i y_i + B_i y_{i+1} = -F_i, \quad i = 1, \dots, N-1,$$

$$y_0 = y_N = 0.$$

We obtain the relations

$$\alpha_{i+1} = \frac{B_i}{C_i - \alpha_i A_i}, \alpha_1 = 0, \quad i = 1, \dots, N-1,$$

$$\beta_{i+1} = \frac{F_i + A_i \beta_i}{C_i - \alpha_i A_i}, \beta_1 = 0, \quad i = 1, \dots, N-1,$$

with $C_i - \alpha_i A_i \neq 0$. Now we can calculate all values of y_i from the recurrence formula

$$y_i = y_{i+1} \alpha_{i+1} + \beta_{i+1}, \quad i = N-1, \dots, 0.$$

Therefore, by relations

$$y_N = k_2 y_{N-1} + \mu_2, \text{ and } y_{N-1} = y_1 + \alpha_N y_N + \beta_N.$$

We write

$$y_N = \frac{\mu_2 + k_2\beta_N}{1 - \alpha_N k_2},$$

with $1 - \alpha_N k_2 \neq 0$ and it can be taken $\mu_2 = k_2 = 0$ in boundary condition $y_N = 0$.

In our example we get

$$a(x) = 1, f(x) = -(\cos^2\pi x + 2(\varepsilon\pi)^2 \cos 2\pi x).$$

Hence, the following expressions are obtained for the difference scheme with exponential coefficients

$$A = -\varepsilon\theta_i, B = A, C = a_i h^2 - 2\varepsilon\theta_i, \text{ and } F = h^2 f_i.$$

Here we take the following term for absolute errors,

$$e_\varepsilon^N = \max_i |u_\varepsilon(x_i) - y_i|.$$

Table 1. The exact solution, approximate solution, and error results for $\varepsilon = 10^{-5}$ and $N = 2^4$ values of Example 5.1

x	$u_\varepsilon(x)$	y_i	e_ε^N
0.00000	0.00000000	0.00000000	0.00000000
0.12500	-0.85355339	-0.85363922	0.00008583
0.18750	-0.69134172	-0.69141161	0.00006989
0.25000	-0.50000000	-0.50005109	0.00005109
⋮	⋮	⋮	⋮
0.93750	-0.96193976	-0.96201071	0.00007095
1.00000	0.00000000	0.00000000	0.00000000

Table 2. The exact solution, approximate solution, and error results for $\varepsilon = 10^{-6}$ and $N = 2^5$ values of Example 5.1

x	$u_\varepsilon(x)$	y_i	e_ε^N
0.00000	0.00000000	0.00000000	0.00000000
0.03125	-0.99039264	-0.99039288	0.00000024
0.06250	-0.96193977	-0.96194008	0.00000031
0.09375	-0.91573481	-0.91573510	0.00000030
⋮	⋮	⋮	⋮
0.96875	-0.99039264	-0.99039288	0.00000024
1.00000	0.00000000	0.00000000	0.00000000

Table 3. The exact solution, approximate solution, and error results for $\varepsilon = 10^{-8}$ and $N = 2^{10}$ values of Example 5.1

x	$u_\varepsilon(x)$	y_i	e_ε^N
0.00000	0.00000000	0.00000000	0.00000000
0.00098	-0.99993320	-1.01149996	0.01156676
0.00195	-0.99996235	-1.01533225	0.01536990
0.00293	-0.99991529	-1.01529920	0.01538391
\vdots	\vdots	\vdots	\vdots
0.99902	-0.99993320	-1.01149996	0.01156675
1.00000	0.00000000	0.00000000	0.00000000

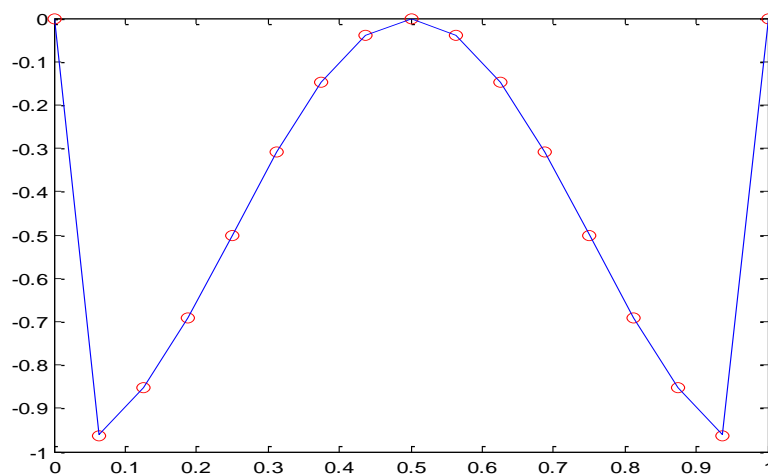


Figure 1. Graph curve of the exact solution and approximate solution for $\varepsilon = 10^{-5}$ and $N = 2^4$ values of Example 5.1

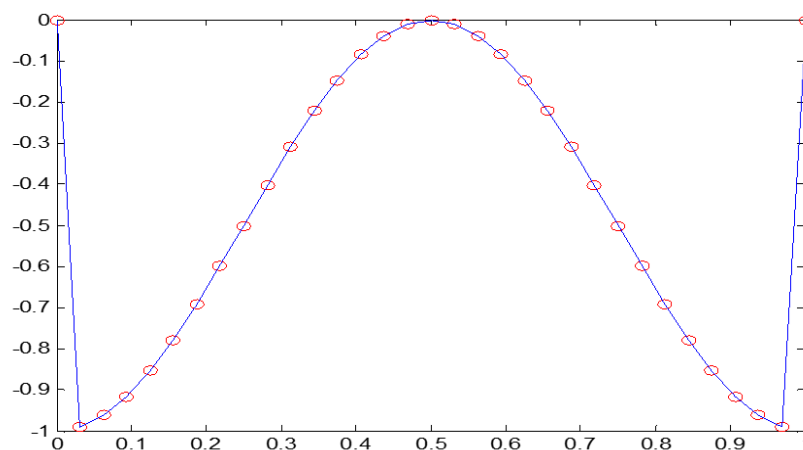


Figure 2. Graph curve of the exact solution and approximate solution for $\varepsilon = 10^{-6}$ and $N = 2^5$ values of Example 5.1

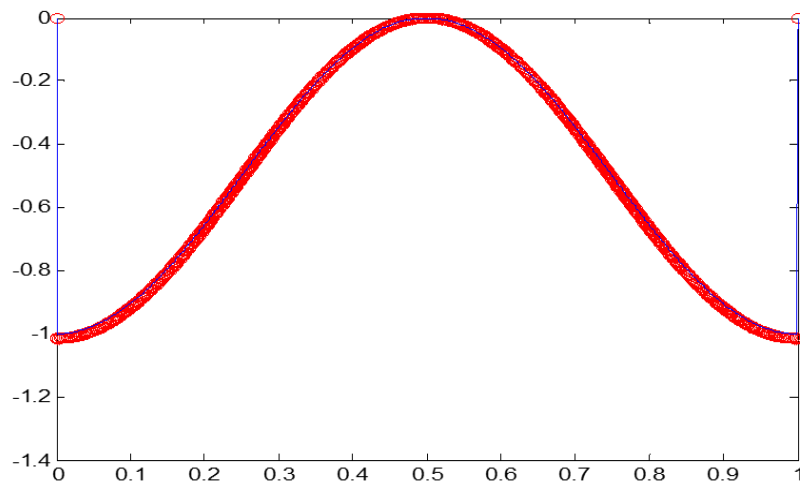


Figure 3. Graph curve of the exact solution and approximate solution for $\varepsilon = 10^{-8}$ and $N = 2^{10}$ values of Example 5.1

In Tables 1-3, it is seen that there is a convergence ratio with uniform convergence speed when values of N increasing and ε decreasing are taken. We see that the maximum error occurs in the boundary stratum areas. Also, we know that y is the numerical approximation to u for different worths of ε and N . In the given mesh, the approximate and exact solution for different values of ε and N are plotted in Figures 1-3. We compare the approximate and exact solution for different worths of N and ε in Tables 1-3. The consequences obtained here are that the discrete solution converges smoothly concerning the perturbation parameter ε , and the errors converge smoothly with almost unity ratios as calculated according to our theoretical research.

6. CONCLUSION

In this work, we suggest a uniform numerical technique to resolve the singularly perturbed problem of reaction-diffusion form. This technique is based on the boundary layer using the asymptotic estimation technique, interpolated quadrature forms, exponential basis functions, weights, and remainder terms. Also, the numerical results for different ε values and different numbers of mesh spacing N are given in Tables 1-3. It is demonstrated that the finite difference method has ε -uniform convergence regarding the perturbation parameter at the discrete maximum norm. The technique suitable for the standard test problem is applied. The biggest benefit of our method is that it provides suitable solutions. We applied the present method to test problems. In conclusion, numerical results show us how accurate and reliable analytical techniques we offer for singular perturbation of reaction-diffusion boundary value problems.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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